Singularities of germs and vanishing homology

Roberto Giménez Conejero

Supervised by

Juan José Nuño Ballesteros

Facultat de Ciències Matemàtiques
Universitat de València

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A Güeli

PODER PODER

Soñaré que te sueño
y querer despertar;
porque, si no, no puedo.

Pensaré que te pienso
y así soñar parar;
porque, si no, no quiero.

Ahora quiero quererte,
poder en ti soñar,
pensar poder tenerte.

R.G.C.

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Resumen

Esta tesis cubre dos artículos conjuntos con Nuño-Ballesteros ([GCNB21, GCNB20]), un artículo conjunto con Nuño-Ballesteros y Lê Dũng Tràng ([GCTNB21]) y un trabajo en desarrollo con Mond. Estos tres trabajos delimitan las tres partes principales del texto.

Como se ha mencionado, el texto está dividido en tres partes. La primera de ellas trata el estudio de singularidades de gérmenes de aplicaciones holomorfas en el contexto de la teoría de Thom-Mather, i.e., módulo $\mathscr{A}$-equivalencia. En particular, nos centramos en gérmenes de corrango uno de $\mathbb{C}^n$ en $\mathbb{C}^{n+1}$, pero también desarrollamos la teoría para gérmenes de $\mathbb{C}^n$ en $\mathbb{C}^p$, con $n < p$, y gérmenes con una intersección completa con singularidad aislada (comunmente conocidos como ICIS) en el dominio.

El principal objetivo de la primera parte del texto es encontrar una *buena* caracterización de la equisingularidad de Whitney para familias a un parámetro de gérmenes $f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ $\mathscr{A}$-finitos de corrango uno. Una caracterización de la equisingularidad de Whitney ya fue dada por Gaffney en [Gaf93]: una familia de gérmenes $f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ es Whitney equisingular si, y solo si, es excelente y todas las multiplicidades polares en el dominio y codominio son constantes a lo largo de la familia. No obstante, esta caracterización tiene el inconveniente de necesitar una gran cantidad de invariantes para asegurar la equisingularidad.

Se han hecho algunos avances desde el resultado de Gaffney, por ejemplo, Jorge Pérez y Saia redujeron el número de invariantes en [JPS06], necesitando todavía una gran cantidad de ellos. Además, Houston tiene un artículo no publicado basado en una prepublicación inédita de Gaffney en el que trata esta cuestión (véase [Hou08]).

Nuestra contribución ha sido, en primer lugar, encontrar una condición para deshacerse de la hipótesis de *excelencia*. Más concretamente, Houston conjeturó en [Hou10] que una familia de gérmenes de corrango uno era excelente si el número de Milnor en la imagen era constante a lo largo de la familia. Hemos resuelto esta conjetura en el par de dimensiones $(n, n+1)$, ergo usamos el invariante μ_I para asegurar la excelencia de la familia.

Por otro lado, nos inspiramos en el trabajo de Teissier para hipersuperficies con singularidad aislada en [Tei82] y en el de Gaffney para ICIS en [Gaf96]: caracterizar la equisingularidad de Whitney en términos de una secuencia μ^* o, lo que es lo mismo, estudiar los *números de Milnor* de secciones genéricas con codimensión creciente. Así, probamos un resultado similar para gérmenes de aplicaciones usando esta filosofía, usando el teorema de Gaffney y reduciendo el número de invariantes que necesitamos. Pese a ello, en el caso de gérmenes de aplicaciones, necesitamos condiciones que controlen el dominio y el codominio por separado, y es por ello que usamos la secuencia $\mu_I^*(f_t)$ para el codominio, usando el número de Milnor en la imagen usual, y la secuencia $\mu_I^*\big(D^2(f_t),\pi\big)$ para el dominio, usando una definición análoga del número de Milnor en la imagen para gérmenes de aplicaciones con un ICIS en el dominio. Esta última secuencia fue la motivación para desarrollar la teoría de gérmenes de aplicaciones desde un ICIS.

Por último, cabe destacar que hemos probado algunos resultados interesantes a lo largo de nuestro trabajo en esta dirección. El primero de ellos es el principio de conservación del número de Milnor en la imagen, así como su semicontinuidad superior. Entre otros, también es de excepcional interés una versión débil de la conjetura de Mond. Recordamos que la conjetura de Mond afirma que el número de Milnor en la imagen de un germen $\mathscr{A}$-finito es mayor o igual que su $\mathscr{A}_e$-codimensión (con igualdad en el caso homogéneo con pesos, llamado también casi homogéneo). Así pues, hemos probado que el número de Milnor en la imagen es cero si, y solo si, el gérmen es estable (o, equivalentemente, que su $\mathscr{A}_e$-codimensión es cero).

La segunda parte de este texto trata la monodromía geométrica local de las fibraciones de Milnor-Lê: probamos que una monodromía geométrica local de un germen $f:(X,x)\to(\mathbb{C},0)$ no fija ningún punto si $f\in\mathfrak{m}_{X,x}^2$. Esto es una generalización de un teorema de Lê Dũng Tràng en [Trá75] enunciado para gérmenes con dominio suave, $f:(\mathbb{C}^{n+1},x)\to(\mathbb{C},0)$. Además, Tibar enunció este resultado en su tesis doctoral y en un artículo (véase [Tib92, Tib93]).

Para probar esta generalización usamos una técnica desarrollada por Lê Dũng Tràng llamada *el carrusel*: un campo vectorial con propiedades adecuadas. La idea principal de la prueba es levantar este campo vectorial a X y tomar su flujo para tener una monodromía geométrica, por lo tanto, también usamos las técnicas mostradas en [GWdPL76] para probar los lemas de isotopía de Thom-Mather.

Este teorema, así como su versión original dada por Lê Dũng Tràng, tiene aplicaciones interesantes. Mediante un teorema clásico de Lefschetz, el teorema que demostramos implica que el número de Lefschetz de una monodromía geométrica local es cero. Esto también es un resultado de A'Campo en [A'C73], que da una versión más general usando maquinaria matemática pesada y cuya prueba, en la versión más general, atribuye a Deligne. Como corolario de esta aplicación, podemos probar que el hecho de ser suave es un invariante topológico de gérmenes de hipersuperficies (X,x), usando también un teorema de Lê Dũng Tràng en [Trá73a]. Este corolario puede ser probado, también, con otro teorema de A'Campo en [A'C73], que es, a su vez, consecuencia de nuestro resultado principal.

Finalmente, mostramos un teorema de no *coalescencia* en un contexto general. Esto quiere decir que, bajo ciertas condiciones, una familia de *singularidades*, en algún sentido, no puede escindirse a lo largo de una familia si se conservan ciertos invariantes, como el número de Milnor. Por ejemplo, en [Trá73b], Lê Dũng Tràng demostró que una familia de hipersuperficies con singularidades aisladas no tiene coalescencia si la suma de los números de Milnor es constante a lo largo de la familia (véase también el trabajo de Bey en [Bey72] y el de Lazzeri en [Laz73a]). Otro ejemplo de no coalescencia fue dado en [CNnBOOT21] por Carvalho, Nuño-Ballesteros, Oréfice-Okamoto y Tomazella para familias de ICIS con número de Milnor total constante. Más precisamente, por contexto general nos referimos a una familia de hipersuperficies $f_t^{-1}(0)$ dadas por funciones $f_t : X_t \to \mathbb{C}$ con puntos críticos aislados en cierto espacio ambiente $(\mathfrak{X}, x_0)$ que además es un *espacio de Milnor* con ciertas hipótesis en la fibración y la familia.

Finalmente, la tercera parte es una nueva forma de encarar el estudio de inestabilidades de gérmenes de aplicaciones $\mathscr{A}$-fnitos f de $\mathbb{C}^n$ en $\mathbb{C}^p$, con $p > n$. Como el lector verá, la principal herramienta que usamos para controlar el número de Milnor en la imagen en la primera parte del texto son los espacios de puntos múltiples de los gérmenes. Pese a ello, no los usamos como un todo, sino que solo nos preocupamos por la parte alternada de su homología porque usamos una *secuencia espectral que calcula la imagen* (ICSS) para calcular los números de Milnor en la imagen. Esta es la razón para intentar usar toda la simetría de los espacios de puntos múltiples en lugar de solo su homología alternada. Esto lo hacemos utilizando la estructura de los espacios de puntos múltiples (de momento, en corrango uno) y teoría de representaciones.

La filosofía de esta nueva forma de aproximarse a problemas de inestabilidades de gérmenes es intentar transformarlos en problemas de álgebra lineal. De hecho, tenemos éxito cuando tratamos de relacionar la constancia de μ_I y la de μ_D en familias, donde μ_D es el *número de Milnor en los puntos dobles* dado en la primera parte del texto como $\mu_I\big(D^2(f_t), \pi\big)$. En particular, probamos que la constancia de μ_I implica la de μ_D en familias de monogérmenes de corrango uno.

Además, probamos que cualquier espacio de puntos múltiples de un monogermen de corrango uno que tenga una singularidad dará homología alternada cuando tomemos su fibra de Milnor. Esto es también una generalización de un teorema dado en la primera parte del texto para familias de gérmenes que admiten un *desdoblamiento estable a un parámetro*.

Methodology

The research procedure for this thesis has been the usual in the field of mathematics. We started looking for adequate bibliographical resources, both general and specific of the subject of our studies, and have extended these materials as needed for our goals. For the computations we have made use of the software SINGULAR, [DGPS21], and Mathematica, [Inc], implementing some algorithms specially adapted to our purposes.

Introduction

Short comment to read this thesis

This text is especially made to be read with a computer, as it is enriched with many hyperlinks along the texts. For example, if a concept appears for the first time in a chapter and it was defined in another chapter or in the appendix, there is a hyperlink taking the reader to the definition. These hyperlinks have a dark blue color, while the hyperlinks of the references are green and the hyperlinks for URL, outside the text, are light blue.

Also, the reader should be aware of the usual tools of a PDF viewer: if a hyperlink takes the reader to other part of the text, the PDF viewer can take the reader back to the page where the hyperlink was. Usually, in viewers such as *Adobe Acrobat Reader* or *SumatraPDF*, one can do this with the combination alt + ← in *Windows* and *Linux* and cmd + ← in *macOS*.

Outline of the thesis

This thesis covers two joint papers with Nuño-Ballesteros ([GCNB21, GCNB20]), a joint paper with Nuño-Ballesteros and Lê Dũng Tràng ([GCTNB21]) and a joint work in development with Mond. These three works delimit the three main parts of the text.

As we were saying, the text is divided intro three parts. The first of them is devoted to the study of singularities of holomorphic map germs in the context of the Thom-Mather theory, i.e., modulo $\mathscr{A}$-equivalence. In particular, we focus on corank one germs from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$, but we also develop the theory for germs from $\mathbb{C}^n$ to $\mathbb{C}^p$, with $n < p$, and germs with an isolated complete intersection singularity (ICIS) in the source.

The main goal of the first part of the text is finding a *nice* characterization of the Whitney equisingularity for one-parameter families of $\mathscr{A}$-finite germs $f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ of corank one. A characterization of the Whitney equisingularity was already given by Gaffney in [Gaf93]: a family of germs $f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is Whitney equisingular if, and only if, it is excellent and all the polar multiplicities in the source and target are constant along the family. However, this characterization has the inconvenient of needing a huge number of invariants to assure the equisingularity.

Some developments have been made since Gaffney's result, for example Jorge Pérez and Saia in [JPS06] reduced the number of invariants, still being a huge amount of invariants. Also, Houston has an unpublished paper based on a preprint of Gaffney addressing this issue (see [Hou08]).

Our contribution was, first of all, finding a condition to avoid the *excellency* hypothesis. To be more specific, Houston conjectured in [Hou10] that a family of corank one germs is excellent if the image Milnor number μ_I is constant along the family. We solve this conjecture for the dimensions $(n, n+1)$, so use the invariant μ_I to assure the excellency of the family.

On the other hand, we were inspired by the work of Teissier in [Tei82] for isolated singularities of hypersurfaces and by the work of Gaffney in [Gaf96] for ICIS: they characterized Whitney equisingularity in terms of μ^* sequences. This means studying the *Milnor numbers* of generic sections with increasingly codimension. Hence, we proved a similar result for map germs using this approach, using Gaffney's theorem and reducing the number of invariants we need. However, in the case of map germs, one needs control conditions on the source and target separately, so we used the sequence $\mu_I^*(f_t)$ for the target, with the usual image Milnor number, and a sequence $\mu_I^*\big(D^2(f_t), \pi\big)$ for the source, using an analogous definition of the image Milnor number defined for map germs with an ICIS in the source. This last sequence also motivated us to develop the theory of map germs on ICIS.

Finally, we want to remark that we have proven some interesting results while going on this direction. The first one is the conservation principle of the image Milnor number, as well as its upper semi-continuity. Among other developments, it is of exceptional interest a weak version of Mond's conjecture. Recall that Mond's conjecture states that the image Milnor number of an $\mathscr{A}$-finite germ is greater than, or equal to, its $\mathscr{A}_e$-codimension (with equality in the weighted homogeneous case). Hence, we proved that the image Milnor number is zero if, and only if, the germ is stable (or, equivalently, the $\mathscr{A}_e$-codimension is zero).

The second part of this text is about the local geometric monodromy of Milnor-Lê fibrations: we prove that a geometric local monodromy of a germ $f:(X,x)\to(\mathbb{C},0)$ does not have any fixed point if $f\in\mathfrak{m}_{X,x}^2$. This is a generalization of a theorem of Lê Dũng Tràng in [Trá75] stated for germs on smooth source, $f:(\mathbb{C}^{n+1},x)\to(\mathbb{C},0)$. Furthermore, Tibar stated this result in his PhD thesis and in one paper (see [Tib92, Tib93]).

In order to prove this generalization, we use a technique developed by Lê Dũng Tràng called the *carousel*, which is a vector field with convenient properties. The main idea of the proof is lifting this vector field to X and take its flow to have a geometric monodromy, hence, we also used the techniques shown in [GWdPL76] to prove Thom-Mather isotopy lemmas.

This theorem, as its original version given by Lê Dũng Tràng, has interesting applications. By a classical result of Lefschetz, the theorem we prove implies that the Lefschetz number of the local geometric monodromy is equal to zero. This is also a result of A'Campo in [A'C73], which has a more general version using heavy mathematical machinery and whose proof, in the most general version, is attributed to Deligne. As corollary of this application, we can prove that being smooth is a topological invariant of germs of hypersurfaces (X, x), using also a theorem of Lê Dũng Tràng in [Trá73a]. This corollary can also be proven using another theorem of A'Campo in [A'C73], which is, in turn, consequence of our main theorem.

Finally, we show a theorem of no *coalescence* in a general context. This means that, in some conditions, a family of *singularities*, in some sense, cannot split along the family provided the conservation of some invariants, such as the Milnor number. For example, in [Trá73b], Lê Dũng Tràng showed that a family of hypersurfaces with isolated singularities does not have coalescence provided the sum of the Milnor numbers is constant along the family (see also Bey's work in [Bey72] and Lazzeri's work in [Laz73a]). Another example of no coalescence is given in [CNnBOOT21] by Carvalho, Nuño-Ballesteros, Oréfice-Okamoto, and Tomazella for families of ICIS with constant total Milnor number. To be more precise, by general context we mean a family of hypersurfaces $f_t^{-1}(0)$ given by functions $f_t : X_t \to \mathbb{C}$ with isolated critical points inside an ambient space $(\mathfrak{X}, x_0)$ that is a *Milnor space* with some hypothesis on the fibration and the family.

Finally, the third part is a new approach to study instabilities of $\mathscr{A}$-finite map germs f from $\mathbb{C}^n$ to $\mathbb{C}^p$, with $p > n$. As the reader will see, the main tool we use to control the image Milnor number in the first part of the text is the multiple point spaces of the germs. However, we do not use them as a whole, but we only care about the alternating part of their homologies because we use an image-computing spectral sequence (ICSS) to compute the image Milnor numbers. This is a reason to try to use all the symmetric structure of the multiple point spaces instead of only their alternating homology. We do this using the structure of the multiple point spaces (at the moment, in corank one) and representation theory.

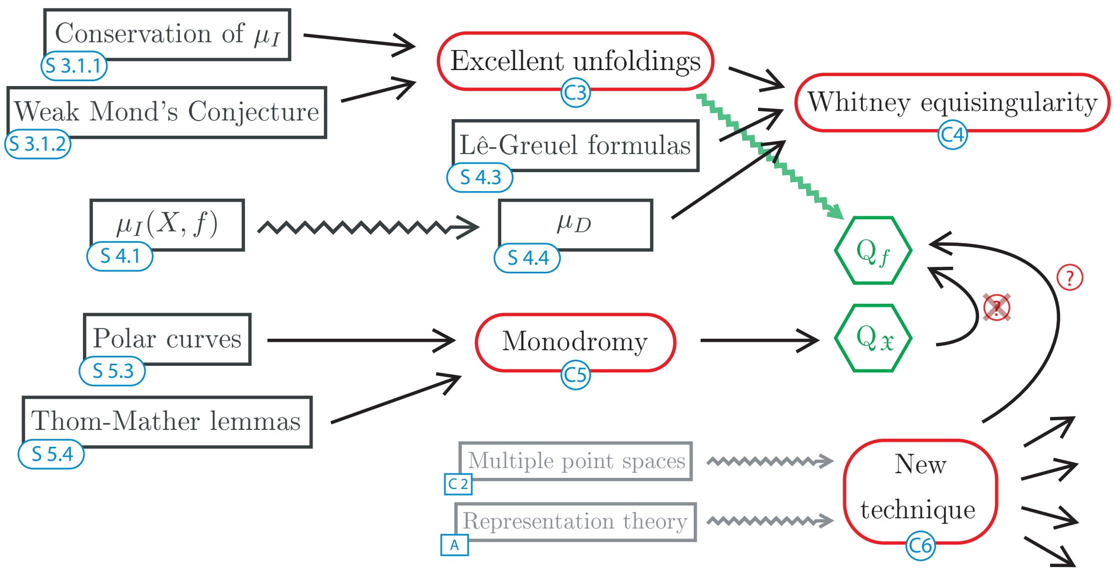
The spirit of this new approach is trying to translate problems of instabilities of germs into problems of linear algebra. In fact, this attempt is successful when we try to relate the constancy of μ_I and the constancy of μ_D in families, where μ_D is the *double point Milnor number* as given in the first part of the text by $\mu_I\big(D^2(f_t), \pi\big)$. In particular, we prove that the constancy of μ_I implies the constancy of μ_D in families of corank one mono-germs.

Furthermore, we prove that any multiple point space of a mono-germ of corank one that has a singularity will provide alternating homology when we take its Milnor fiber. This is also a generalization of a theorem given in the first part of the text for families of germs that admit a *one-parameter stable unfolding*.

Structure of the thesis

In Chapter 1 we introduce our main topic of research. For example, the main concepts of singularities of holomorphic map germs are presented in Section 1.2, with a particular interest in the image Milnor number and the $\mathscr{A}$-codimension of $\mathscr{A}$-finite germs. After that, as our objects are usually stratified manifolds, we introduce the concepts related with stratifications we use along the text, such as Whitney stratifications, Whitney equisingularity or topological triviality.

Chapter 2 is a short introduction to the multiple point spaces and the ICSS, with some examples and the main results we are going to use along the text.



Chapters 3 and 4 are conceived with a final objective: to characterize Whitney equisingularity of corank one germs from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ in a simple way. The first step in this direction is made in Chapter 3, solving Houston's conjecture on excellent unfoldings. To do so, we need to show some fundamental properties of the image Milnor number (see Section 3.1). Among other results, we show in Section 3.1.1 that the image Milnor number is conservative and in Section 3.1.2 that a corank one germ has image Milnor number equal to zero if, and only if, it is stable (this is called *weak Mond's conjecture* in the text). We end the chapter using these results to solve Houston's conjecture in Section 3.2.

Recall from the outline of the text that we use a theorem of Gaffney given in [Gaf93] to give a characterization of being Whitney equisingular with a few invariants. Hence, as the reader already knows, in Chapter 4 we use Houston's conjecture to drop the hypothesis on excellency.

Our next step is to give invariants that contain enough information to control the equisingularity. However, we need to make a distinction between the source and the target. To control the source, we use the image Milnor numbers of the sections of projections of the double point space of f (denoted as $D^2(f)$) and, as they are ICIS, we start the chapter introducing the topic of germs on ICIS in Section 4.1. This is, to some extent, an introduction to the work of Mond and Montaldi in [MM94].

We also need to recover the basic building we already have in the smooth setting, such as the multiple point spaces of germs on ICIS. Recall that we need to work with an analogous definition of the image Milnor number, but for germs that have an ICIS in the source. For this reason, we need to be able to use the ICSS easily. All this is covered in Section 4.2. Also, we take the chance to extend the work of Mond and Montaldi in [MM94] for germs $f:(X,S)\to(\mathbb{C}^p,0)$ with X an ICIS of dimension n and $p>n$, in general.

We study many properties of the projection of the double point space and the new invariant μ_D on the source (called *double point Milnor number*) in Section 4.4. For example, we study the relation between μ_D and μ_I, proving the weak Mond's conjecture for μ_D.

Another preparatory section we have skipped is Section 4.3, where we give a version of the Lê-Greuel formula for map germs on ICIS. This is also a generalization of the work of Pallarés-Torres and Nuño-Ballesteros in [NBPT19].

We finish the chapter giving our desired characterization of the equisingularity in Section 4.5.

Houston's conjecture raises a question.

Q_f: If the *total* image Milnor number is constant, do the instabilities coalesce?

In other words, can the excellency of a family fail because we have new instabilities along the family but *without* having homology in middle dimension?

We already know that this is not true for families of hypersurfaces with isolated singularities: if the family of hypersurfaces $g_t^{-1}(0)=H_t$ has isolated singularities and the total Milnor number is constant on t, then there is only one singularity along the family. A known proof involves working with some local geometric monodromy (see [Trá73b, Theorems A and B]). Furthermore, we can reproduce the part of having a local geometric monodromy in the setting of map germs using a stabilisation of the germ. Hence, it could happen that a generalization of the argument for hypersurfaces with isolated singularities includes the images of map germs as parametrized hypersurfaces.

$\mathrm{Q}_{\mathfrak{X}}$: If the *total Milnor number* is constant along a family $\mathfrak{X}=\{(X_t,t)\}_t$, do the *singularities* coalesce?

This is our motivation to begin with Chapter 5, and it is well explained in Section 5.1. In this chapter, we introduce the concepts of local monodromy and give some results that led to the proof of a general theorem we were talking about, from Section 5.2 to Section 5.8. However, we show in an example of Section 5.7 that the general theorem cannot work with the setting of map germs.

The last chapter with original results is Chapter 6. We introduce the main idea of a new technique in Section 6.1, which consist of using all the symmetric structure of the multiple point spaces by means of representation theory, the Marar-Mond criterion and an equation that relates the character of a group acting on a simplicial complex and the fixed points of the action. As the technique is new, we give several examples in Section 6.2 with all the details. Finally, we conclude the chapter with two new theorems whose proof involves this technique in Sections 6.3 and 6.4. The first one gives a useful characterization of having alternating homology in the Milnor fiber of a multiple point space (with some conditions): the multiple point is non-smooth if, and only if, it has alternating homology in its Milnor fiber. The second one states that the constancy of μ_I implies the constancy of μ_D in families, in some cases.

Chapter 7 contains a list of open problems we have faced along our work, open questions we asked ourselves and a general review of the text.

Finally, regarding the appendix, Chapter 6 uses fluently representation theory in all the arguments, so we advise the reader to see Appendix A. This appendix was made with the intention of writing a small course of representation theory, because it is not a common area singularists see. Furthermore, we can also see the basics of spectral sequences in Appendix B, one can find there beautiful well-known examples and the fundamental concepts of a spectral sequence.

Chapter 1

Preliminaries

The Dwarf sees farther than the Giant, when he has the Giant's shoulders to mount on.

Samuel Taylor Coleridge
The Friend

In this chapter we introduce the main objects we are going to use regarding map germs and stratifications.

1.1. Conventions and notation

Throughout this text we shall use some convenient conventions, if the context is not misleading, that we list here. We also follow some conventions of the main modern references, such as [MNB20].

- If a set is formed by one element we may refer to the set writing only the element (e.g., x instead of $\{x\}$).
- All the neighbourhoods are open neighbourhoods.
- The set germs are denoted as (X, S) where S is the set where the germ is defined, and they are also called simply *germs*.
- The map germs are denoted as $f : (X, S) \to Y$, or $f : (X, S) \to (Y, R)$ if $f(S) \subseteq R$; S will have always a finite number of points; and they are also called simply *germs*.
- A *mono-germ* is a map germ where S has only one point, otherwise it is a *multi-germ*.
- All the mappings, or germs, are holomorphic until otherwise stated.
- *Finite* maps are proper and finite-to-one maps.

- Usually, in our context, the germs have to be *finitely determined*, but we may not mention this detail in some explanations for the sake of the narrative.

Furthermore, we follow the standard notation to denote some objects:

- $\mathcal{O}_{X,S}$ is the ring of holomorphic map germs $f:(X,S)\to\mathbb{C}$. It is also written simply as $\mathcal{O}_n$ if $(X,S)=(\mathbb{C}^n,S)$ and the context is not misleading.
- $\mathfrak{m}_{X,S}$ is the ideal of $\mathcal{O}_{X,S}$ of functions that vanish on S, also written as $\mathfrak{m}_n$ if $(X,S)=(\mathbb{C}^n,S)$.
- $\theta_{\mathbb{C}^n,S}$ is the $\mathcal{O}_n$-module of germs of vector fields on $(\mathbb{C}^n,S)$, also written as θ_n.
- For a map germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$, $\theta(f)$ is the $\mathcal{O}_n$-module of vector fields along f.
- $\Sigma(f)$ is the critical set of a map f, or a set germ if f is a germ.
- The *discriminant* is the image of the critical set, $\Delta(f):=f\big(\Sigma(f)\big)$.
- The *corank* of a mapping at a point is the difference between the maximum rank and the rank at that point.

1.2. Map germs

In this section we give a quick review of map germs and $\mathscr{A}$-equivalence. For more details and further explanation we recommend [GWdPL76] and we specially recommend the more up-to-date reference [MNB20], which we follow.

1.2.1. $\mathscr{A}$-equivalence

Studying mappings between manifolds is a self-motivating topic of research considering its broad context. A holomorphic structure provides a mapping with many interesting properties, therefore it is a good class of mappings to study. Finally, if we face a tough problem, a good strategy is to break it down into simpler problems we may solve. This leads us to study map germs of holomorphic functions.

If we are in the *holomorphic world* it makes sense to work with map germs modulo biholomorphisms, i.e., it makes sense to define things modulo biholomorphisms in a holomorphic category.

Definition 1.2.1. We say that two germs, f and g, are *$\mathscr{A}$-equivalent*, or *left-right equivalent*, if there are germs of biholomorphisms, ϕ and ψ, such that the following diagram commutes

$$\begin{array}{ccc}(\mathbb{C}^n,S) & \xrightarrow{f} & (\mathbb{C}^p,0)\\ \phi\big\downarrow\wr & & \wr\big\downarrow\psi\\ (\mathbb{C}^n,S) & \xrightarrow{g} & (\mathbb{C}^p,0)\end{array}.$$

There is a kind of map germs where the question of whether or not two maps are $\mathscr{A}$-equivalent is simpler than in the general case.

Definition 1.2.2. A map germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ is *k-determined* if it is $\mathscr{A}$-equivalent to g whenever their Taylor polynomial of order k at S coincide. If a map germ is k-determined for some k we say that it is *finitely determined* or *$\mathscr{A}$-finite*.

After this last definition, knowing when a germ is finitely determined becomes a central problem. Moreover, as we shall see, finitely determined germs have more interesting properties.

Note that $\mathscr{A}$-equivalence is defined by means of the group $\mathrm{Bihol}(\mathbb{C}^n,S)\times\mathrm{Bihol}(\mathbb{C}^p,0)$ acting in a certain way on map germs, where $\mathrm{Bihol}(\mathbb{C}^m,R)$ denotes the group of germs, defined at the set R, of biholomorphisms[1], such as $\phi:(\mathbb{C}^m,R)\to(\mathbb{C}^m,R)$. This is a similar situation to what Thom and Mather studied in the 1960s, in the series of papers [Mat68a, Mat69a, Mat68b, Mat69b, Mat70, Mat71], but they studied the *smooth global* case. Indeed, the theory we introduce here is usually called *Thom-Mather theory*. This is justified, many of our definitions are directly inspired by their work.

Let us delve into it: a smooth map $f:M\to N$ is *stable* if its orbit under the natural action of $\mathrm{Diff}(M)\times\mathrm{Diff}(N)$ is open in the space of smooth functions $C^\infty(M,N)$ with respect to the Whitney topology (see [GG73, Definition II.3.1]), where $\mathrm{Diff}(P)$ is the group of diffeomorphisms of the manifold P to itself. Mather developed a way to determine if a proper map is stable, following the strategy of *divide et impera* we commented above, and a considerable part of his six papers of stability of C^∞ mappings is devoted to this: stability of a proper map is equivalent to *local stability* of its germs and it is also equivalent to *infinitesimal stability* (see, particularly, [Mat69a, pp. 266–268]).

Keep in mind Mather's work and let us go back to the holomorphic, and local, case. Consider a map germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$.

Definition 1.2.3. *(i)* A *d-parameter unfolding* of f is another map germ

$$\begin{aligned} F:(\mathbb{C}^n\times\mathbb{C}^d,S\times 0)&\to(\mathbb{C}^p\times\mathbb{C}^d,0)\\ (x,t)&\mapsto(f_t(x),t)\end{aligned}$$

such that $f_0=f$. Once we take a representative $F:U\to V\times T$ of F, the *maps* $f_t:U_t\to V$ are called *perturbations* of f, where $U_t\times\{t\}=U\cap F^{-1}(V\times\{t\})$. Also, $\mathbb{C}^d$ is called the *parameter space* and usually we omit it if it is clear from the context.

(ii) Two d-parameter unfoldings of f, F and G, are *equivalent as unfoldings*, or simply *equivalent*, if there are two germs of biholomorphisms, Φ and Ψ, which are themselves unfoldings of the identity in $\mathbb{C}^n$ and $\mathbb{C}^p$, respectively, such that the following

[1]This is a glimpse of an important topic regarding singularity theory, see [Wal81, MNB20] among others.

diagram commutes

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^d, S \times 0) & \xrightarrow{F} & (\mathbb{C}^p \times \mathbb{C}^d, 0) \\ \Phi \downarrow \wr & & \wr \downarrow \Psi \\ (\mathbb{C}^n \times \mathbb{C}^d, S \times 0) & \xrightarrow{G} & (\mathbb{C}^p \times \mathbb{C}^d, 0) \end{array} .$$

(iii) An unfolding is *trivial* if it is equivalent to the unfolding $f \times \mathrm{id}$, i.e., the unfolding that maps (x, t) into $\big(f(x), t\big)$.

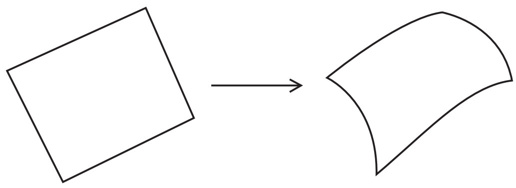
Remark 1.2.4. Essentially, an unfolding of f is *revealing* what is nearby f or, in other words, what are the possible *deformations* of f. If every unfolding of a map germ f is trivial then we say that f is *stable*, meaning that every little *perturbation* of f will not change anything modulo $\mathscr{A}$-equivalence. Otherwise we say that f has an *instability*, or it is *unstable*.

Example 1.2.5. Immersions and submersions are stable (see [MNB20, Exercise 3.2.4]).

Example 1.2.6. By Whitney's classification of stable mono-germs from $\mathbb{C}^n$ to $\mathbb{C}^{2n-1}$ in [Whi44], the stable mono-germs from $\mathbb{C}^2$ to $\mathbb{C}^3$ are:

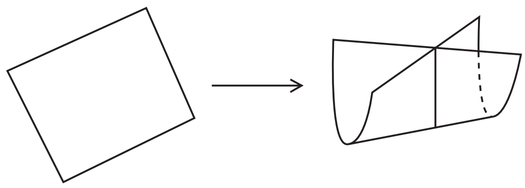
- immersions

$$(x, y) \mapsto (x, y, 0),$$



- and *Whitney umbrella*

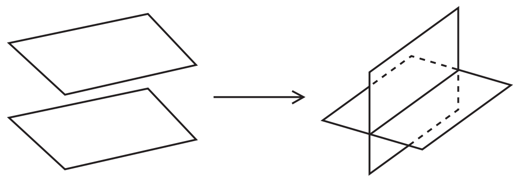
$$(x, y) \mapsto (x, y^2, xy).$$



Moreover, the stable multi-germs from $\mathbb{C}^2$ to $\mathbb{C}^3$, by [MNB20, Theorem 3.3], are:

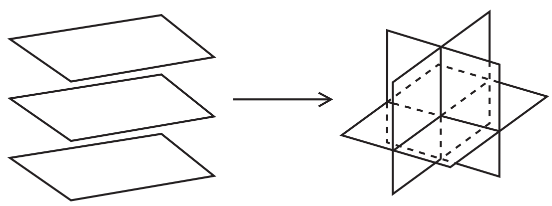
- transverse double points

$$\begin{cases} (x, y) \mapsto (x, y, 0) \\ (x, y) \mapsto (x, 0, y) \end{cases}$$



- and transverse triple points

$$\begin{cases} (x, y) \mapsto (x, y, 0) \\ (x, y) \mapsto (x, 0, y) \\ (x, y) \mapsto (0, x, y) \end{cases}$$

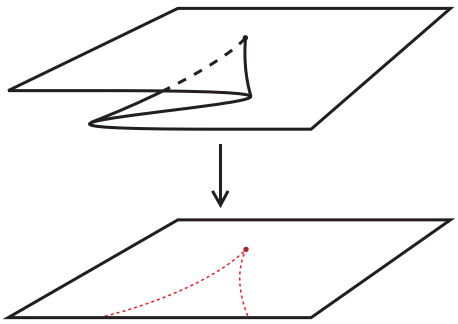


Example 1.2.7.

The map germ

$$f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$$
$$(x, y) \mapsto (x, xy + y^3)$$

is called the *Whitney cusp map* and it is stable.



If a germ f is stable then every unfolding is trivial but, if f is not stable, not every unfolding of f carries the same information. For example a trivial unfolding of f will give us no information. Within this sort of *partial order* of carrying information there are maximal unfoldings, the so-called *versal unfoldings*.

Definition 1.2.8. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a germ and consider the unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, S \times 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$. Then:

(i) If we consider a germ $h : (\mathbb{C}^a, 0) \to (\mathbb{C}^d, 0)$, the *pull-back of F by h* is the unfolding

$$h^*F : (\mathbb{C}^n \times \mathbb{C}^a, S \times 0) \to (\mathbb{C}^p \times \mathbb{C}^a, 0)$$
$$(x, t) \mapsto F\big(x, h(t)\big),$$

and it is denoted by h^*F.

(ii) The unfolding F is *versal* if any unfolding G of f is equivalent to h^*F for some germ $h : (\mathbb{C}^a, 0) \to (\mathbb{C}^d, 0)$, where $\mathbb{C}^d$ and $\mathbb{C}^a$ are the parameter spaces of F and G respectively.

(iii) The unfolding F is a *miniversal unfolding* if it is a versal unfolding with minimal dimension on the parameter space.

The intuitions one should have regarding these last definitions are:

- the pull-back of an unfolding simply carries part of the information of the original unfolding to another unfolding with other parameter space,
- a versal unfolding contains all the possible information of f, and
- any versal unfolding is equivalent to a constant unfolding of a miniversal unfolding (see [MNB20, Exercise 5.1.5]).

Example 1.2.9. The map germ

$$f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$$
$$x \mapsto (x^2, x^5)$$

has $F(x,a,b) = (x^2, x^5 + ax^3 + bx, a, b)$ as miniversal unfolding.

In Figure 1.1, there are represented in the parameter space of F the different perturbations of f (see [MNB20, Example 5.3] for all the computations and a similar representation, but taking into account the different *real representations*).

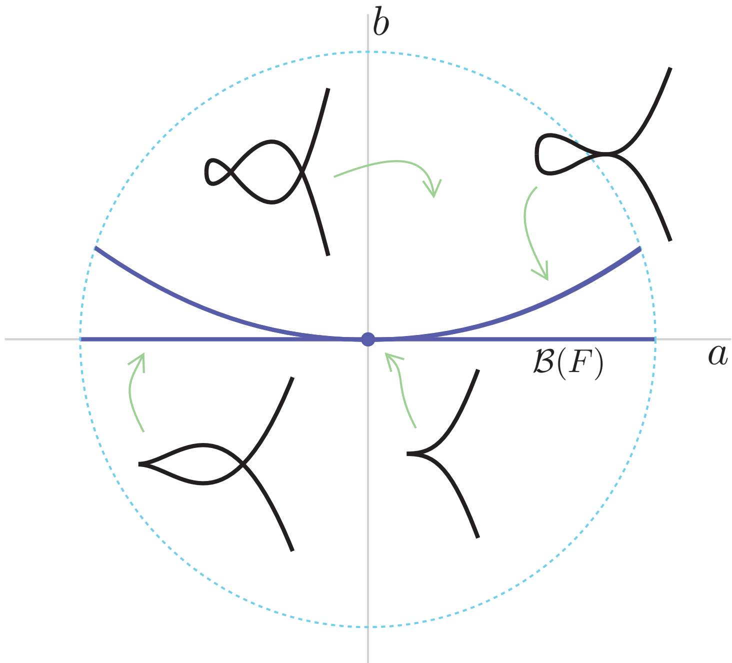


Figure 1.1: Representation of the parameter space of the miniversal unfolding $F(x,a,b) = (x^2, x^5 + ax^3 + bx, a, b)$ of $f(x) = (x^2, x^5)$.

An interesting mathematical object arises from the previous example. We can observe in Figure 1.1 that there is a set, denoted as $\mathcal{B}(F)$, where the deformation presents instabilities, and we see it because adjacent perturbations have different $\mathscr{A}$-classes since they even have different topological type. Furthermore, it seems that it has some geometric properties. This set is the *bifurcation set* and in order to study it we need to work with maps instead of map germs, considering that we have to take a representative of the unfolding to define it. Therefore, we need some notion of *stability of a map*.

Definition 1.2.10. A map $f : X \to Y$ is *locally stable* if the restriction to the critical set is finite and the germs $(f)_y : \big(X, f^{-1}(y) \cap \Sigma(f)\big) \to (Y, y)$ induced from f are stable for every $y \in Y$.

Remark 1.2.11. This notion is very related to Mather's notion of ∞-*structurally stable*, see [Mat69a, Theorem 3].

As we were anticipating:

Definition 1.2.12. The *bifurcation set* of a representative of an unfolding F, say $F : \mathcal{X} \to Y \times U$, is the set $\mathcal{B}(F)$ in the parameter space where the mappings $f_u : \mathcal{X} \cap F^{-1}\big(Y \times \{u\}\big) \to Y$ are not locally stable, i.e., f_u is not locally stable if $u \in \mathcal{B}(F)$.

There are some technicalities to assure that the representative of the unfolding is well chosen, they can be found in [MNB20, Section 5.4]. Furthermore, we can take the set germ $\big(\mathcal{B}(F), 0\big)$ and forget about taking representatives.

Remark 1.2.13. There is an important piece of notation regarding the bifurcation set: the *stable perturbations*. If we have a representative of an unfolding of f as before, any parameter u outside the bifurcation set gives us a *stable perturbation*, i.e., a perturbation of f that is locally stable. We always omit the word *locally* because it is clear that a perturbation is a map, not a map germ.

Let us stop for a moment to introduce a very useful concept (see Figure 1.2).

Definition 1.2.14. A *stabilisation* of an unstable germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a one-parameter unfolding $F(x,t) = \big(f_t(x), t\big)$ such that there is a representative where f_t is locally stable for every $t \neq 0$.

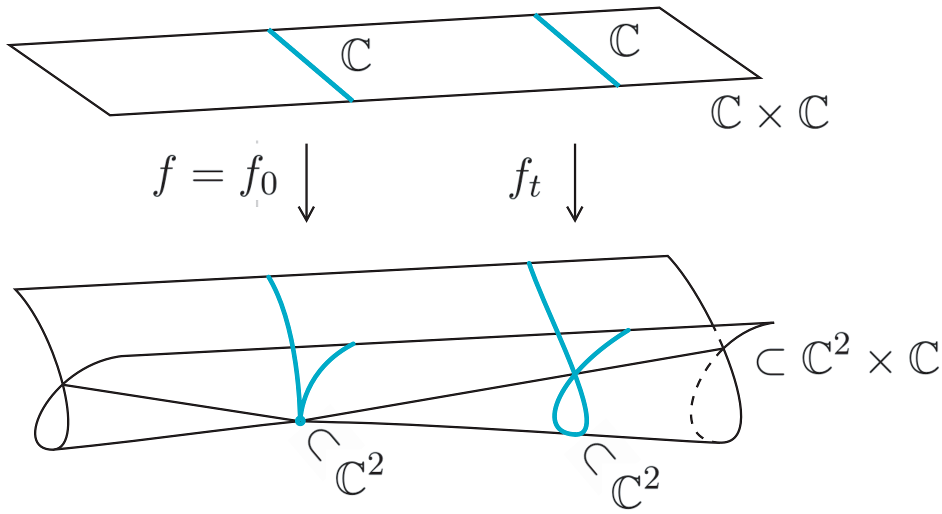


Figure 1.2: Real representation of the stabilisation $f_t(x) = (x^2, x^3 + tx)$.

This last definition is equivalent to saying that the unfolding F has $\big(\mathcal{B}(F), 0\big) = \big(\{0\}, 0\big)$. Furthermore, if we consider a versal unfolding and take a line in the parameter space such that it intersects $\mathcal{B}(\mathcal{F})$ only at the origin, then the induced unfolding by this line will be a stabilisation. To be more precise, if the versal unfolding is $\mathcal{F}$ and the line is parametrized by L, then $L^*\mathcal{F}$ is a stabilisation.

Remark 1.2.15. This situation does not always happen. For example, the bifurcation set of the unfolding could be the whole parameter space. Although, there are some dimensions where we can guarantee that any finitely determined map has versal unfolding with a null bifurcation set: the *nice dimensions* in Mather sense (see [Mat71, p. 208] and [MNB20, Sections 5.2,5.3 and 5.4] for further explanations and properties). Outside these dimensions there are examples where we cannot find a stable perturbation nearby. However, if the germ is of corank 1 and finitely determined, then the same result holds in general (see [MNB20, Proposition 5.6])[2].

Now that we have introduced the concept of stability in some depth, it is interesting to have in our set of tools some way of determine if a germ is stable or even how far from being stable a map germ is. For this purpose, we will *translate* many properties of the map germ into properties of some algebraic structures, as algebraic geometers do. For example, it would be easier, in general, to check the dimension of some vector space instead of proving if every unfolding of a map germ is equivalent to the trivial unfolding. In any case, this *translation* is far from being straightforward.

Let us begin introducing the structures.

Definition 1.2.16. For a map germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ we denote by

$$tf:\theta_n\to\theta(f)$$

the map $\xi\mapsto df\circ\xi$ and by

$$\omega f:\theta_p\to\theta(f)$$

the map $\eta\mapsto\eta\circ f$, where $\theta(f)$ is the module of vector fields along f.

With this notation, the $\mathcal{O}_{\mathbb{C}^p,0}$-module

$$N\mathscr{A}_e(f)\coloneqq\frac{\theta(f)}{tf(\theta_{X,S})+\omega f(\theta_{\mathbb{C}^p,0})}$$

is the $\mathscr{A}_e$-*normal space* and its dimension as vector space is the $\mathscr{A}_e$-*codimension of* f, $\mathscr{A}_e$-codim(f).

In some sense, the $\mathscr{A}_e$-codimension, sometimes referred simply as *codimension*, measures how far a germ is from being stable. This expression of $N\mathscr{A}_e f$ is an elegant way of writing

$$\frac{\left\{\left.\dfrac{df_t}{dt}\right|_{t=0}:F(x,t)=\big(f_t(x),t\big)\text{ is any unfolding of }f\right\}}{\left\{\left.\dfrac{d(\psi_t\circ f\circ\phi_t)}{dt}\right|_{t=0}:\psi_0=\mathrm{id},\phi_0=\mathrm{id}\right\}},$$

or, in other words, it compares what can be achieved *infinitesimally* using a trivial unfolding with what can be achieved using any unfolding (see [MNB20, 3.2] for all the details). With this in mind, the following definition is very natural.

Definition 1.2.17. A map germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ is *infinitesimally stable* if it has $\mathscr{A}_e$-codimension 0.

[2] The condition of corank 1 is highly useful in many problems. For example, it gives more structure to many objects, as we will see.

This definition is not completely new for us, recall Mather's work:

Theorem 1.2.18 ([MNB20, Theorem 3.1])**.** *A map germ is stable if, and only if, it is infinitesimally stable.*

This gives us a way of computing whether a germ is stable or not, as it was the case for Mather. Therefore, from now on, we use indistinctly the terms stable or infinitesimally stable.

The $\mathscr{A}_e$-codimension is a central piece in the study of $\mathscr{A}$-equivalence of germs, almost every basic object in this topic has a strong relation with it. For example, finite determinacy and finite $\mathscr{A}_e$-codimension are equivalent, what was studied by first time in [Mat68b]. Furthermore, even the degree of determinacy and the codimension are related (for a precise estimate for the determinacy degree see [MNB20, Theorem 6.2]).

Theorem 1.2.19 (see [MNB20, Theorem 6.1])**.** *A germ is finitely determined if, and only if, it has finite $\mathscr{A}_e$-codimension.*

Roughly speaking, if the $\mathscr{A}_e$-codimension measures the difference between a trivial unfolding and a versal unfolding, there has to be a relation of the codimension with a versal unfolding because the latter contains all the information of the near perturbations of the germ. However, the parameter space of a versal unfolding could carry redundant information if it is not miniversal. Here is where the relation is clear, the dimension of the parameter space in a miniversal unfolding is exactly the codimension of the germ if it is $\mathscr{A}$-finite. This is a corollary of the following theorem (proved first in [Mar76, Theorem 3.3]).

Theorem 1.2.20 (see [MNB20, Theorem 5.1])**.** *Consider a map germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ and an unfolding $F(x,u)=\big(f_u(x),u\big)$ of f, where $u=(u_1,\dots,u_d)$. Then, F is a versal unfolding if, and only if,*

$$\left\{\left.\frac{\partial f_u}{\partial u_1}\right|_{u=0},\dots,\left.\frac{\partial f_u}{\partial u_d}\right|_{u=0}\right\}$$

generates the $\mathbb{C}$-vector space

$$\frac{\theta(f)}{tf(\theta_n)+\omega f(\theta_p)}.$$

At this point we have two ways to determine if a map germ is finitely determined: we can try to find a versal unfolding or we can compute if the $\mathscr{A}_e$-codimension is finite. There are more ways, and a very useful one is using the *Mather-Gaffney criterion*. This is a geometric criterion to determine if a map germ is finitely determined, although it does not give the degree of determinacy. Furthermore, the $\mathscr{A}_e$-codimension also appears in the usual proof, which is based on a *sheafification* of the quotient that appears in the codimension (see [MNB20, Section 4.5] for the details).

Theorem 1.2.21 (Mather-Gaffney criterion, see [MNB20, Theorem 4.5])**.** *A germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ such that $S \subseteq \Sigma(f)$ has finite $\mathscr{A}_e$-codimension if, and only if, there is a small enough representative of f, $f : X \to Y$, such that*

(i) $\Sigma(f) \cap f^{-1}(0) = S$, *and*

(ii) the induced map $f| : X - f^{-1}(0) \to Y - \{0\}$ *is locally stable.*

This theorem is also expressed as *the germ f has isolated instability*, because the germs are stable near the point 0.

Remark 1.2.22. We generalize many of these notions for the case of map germs with an isolated complete intersection singularity (ICIS) in the source in Section 4.1.

1.2.2. Topology

Until now, we have not mentioned anything about the topology of a germ. In this work we focus on finitely determined germs, so we leave the rest of the cases aside from now on.

A good first question to ask would be if two $\mathscr{A}$-equivalent germs share some interesting topological property, or if there is a topological property that characterizes $\mathscr{A}$-equivalence. A first approach to answer this question would be to look at the image of the map germ, but this has some problems.

An obvious problem is that the image of a map germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is equal to the set germ $(\mathbb{C}^p, 0)$ if $n \geq p$, we shall return to this later. Moreover, the image of a germ is not that interesting, it has *locally conical structure*. Locally conical structure means that if we take a representative f and consider its image, $\operatorname{im}(f)$, then $\operatorname{im}(f) \cap \overline{B}_\varepsilon$ is homeomorphic to the cone on $\operatorname{im}(f) \cap S_\varepsilon$, where $\overline{B}_\varepsilon$ is the closure of a small enough ball and S_ε the sphere of the same radius. The details of this can be found in [MNB20, Theorem B.7], but see also the *conic structure lemma* of Burghelea and Verona ([BV72, Lemma 3.2]) stated for *Whitney stratified sets* in the smooth case and the classical results of Milnor in [Mil68] using vector fields. Therefore, if we study the image of a germ we should focus on the *link*, i.e., $\operatorname{im}(f) \cap S_\varepsilon$, which is expected to be a knotted sphere. There are classical results on this topic, such as [Bra28], and the theory was further developed in [Fuk85, Fuk82, MNnB16, NnB18]. However, this theory is neither very rich nor developed compared to the theory of *Milnor fibrations* we can reproduce with map germs. Furthermore, the link is still present in this theory as well (see Lemma 3.1.4).

We have presented in a dissimulated way an interesting object from the topological point of view, without saying anything about it. The reader may know that we are referring to Example 1.2.9 and Figures 1.1 and 1.2. We can find our topological object there.

Definition 1.2.23 (see [MNB20, Chapter 8])**.** Consider an $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with (n, p) nice dimensions or f of corank one. Then:

- The discriminant of a stable perturbation of f, with $n \geq p$, has the homotopy type of a wedge of spheres of dimension $p-1$, whose number is the *discriminant Milnor number* and is denoted by $\mu_\Delta(f)$.

- The image of a a stable perturbation of f, with $p = n+1$, has the homotopy type of a wedge of spheres of dimension n, whose number is the *image Milnor number* and is denoted by $\mu_I(f)$. Note that in this case the discriminant coincides with the image.

- In both cases, the discriminant of the stable perturbation is called the *disentanglement* of f.

Remark 1.2.24. We will cover the case $p > n+1$ later in the text, in a more general setting (see, in particular, Section 4.2). This case is more complicated because the reduced homology is not concentrated in one dimension.

Example 1.2.25. Consider the map germ

$$\begin{aligned} f : (\mathbb{C}, 0) &\to (\mathbb{C}^2, 0) \\ x &\mapsto (x^2, x^3). \end{aligned}$$

This germ has $\mathscr{A}$-codimension one and a stable perturbation is $f_t(x) = (x^2, x^3+tx)$, with $t \neq 0$. We have a real representation of this stabilisation in Figure 1.2 that suggest that $\mu_I(f) = 1$. However, as the complex spaces $\mathrm{im}(f_t)$ have dimension one, we can represent them faithfully and confirm this intuition (see Figure 1.3).

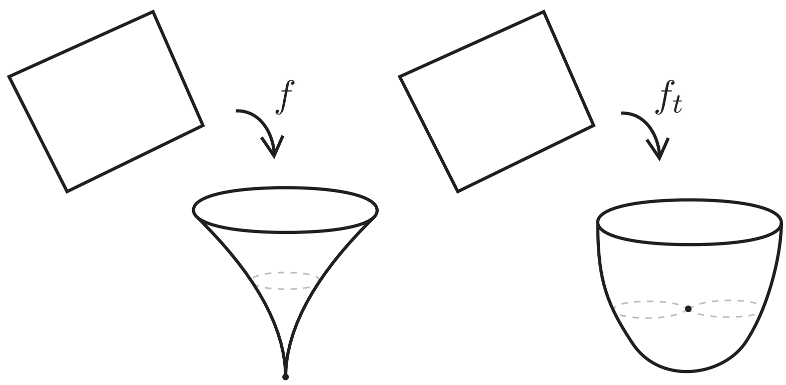


Figure 1.3: Representation of the perturbations $f_t(x) = (x^2, x^3+tx)$.

These two objects, μ_I and μ_Δ, are invariant by $\mathscr{A}$-equivalence and do not depend on the stable perturbation[3]. All the technicalities concerning the structure of a wedge

[3]Sometimes, within the area, these objects are wrongly called *topological invariants*. This is not because they are invariant by *topological left-right equivalence*, it is because they are invariant by $\mathscr{A}$-equivalence and they refer to the topology of something.

of spheres are given in [MNB20, Section 8.3], but the main point of the prove is to use Thom-Mather's isotopy lemmas and the properness, as a subset, of the bifurcation set. There are also some considerations to be made so we take stable perturbations that are near enough to f to have all the topological information, see in particular [MNB20, Propositions 8.1 and 8.2].

Remark 1.2.26. The original definition of the image Milnor number is due to Mond (see [Mon91, Theorem 1.4]) using a stabilisation, say (f_t, t), but it is equivalent to our definition using any stable perturbation. Given a stable unfolding (f_s, s) we can take *the sum of unfoldings*,

$$F(x,t,s) = \big(f_t(x) + f_s(x) - f(x), t, s\big),$$

which is also stable and, if f_t was stable for any $t \neq 0$, then $(t, 0)$ is outside the bifurcation set of F. See Lemma 3.1.4 and its following comments for more details.

The resemblance of this theory to the theory of isolated singularities of hypersurfaces is striking. Consider a hypersurface with isolated singularity H and an $\mathscr{A}$-finite germ f, then, many objects related to H have an equivalent object related to f (see Table 1.1): the conical structure of the non-perturbed object, the wedge structure of the generic fiber and the disentanglement, the Tjurina number and the $\mathscr{A}_e$-codimension, etc. Furthermore, it inspires us to study new aspects of the singularities of mappings, for example study if the Milnor-Tjurina relation for hypersurfaces has an equivalent relation for singularities of map germs. Following this question, recall that we have said that the $\mathscr{A}_e$-codimension is related to almost everything and the equivalent concept of the Tjurina number is the $\mathscr{A}_e$-codimension, so one expects a relation between μ_Δ or μ_I with the $\mathscr{A}_e$-codimension similar to the Milnor-Tjurina relation (see [MNB20, Section 8.9.4]).

Hypersurface H	**Germ f**	
Conical structure of H	Conical structure of the discriminant of f	
A miniversal unfolding of H has parameter space of dimension the Tjurina number τ	A miniversal unfolding of f has parameter space of dimension the $\mathscr{A}_e$-codimension	
The fiber H_t has the homotopy type of a wedge of spheres	The disentanglement of f has the homotopy type of a wedge of spheres	
The number of spheres is the Milnor number, μ	The number of spheres is the discriminan or image Milnor number, μ_Δ or μ_I	
Milnor-Tjurina relation	Damon-Mond results	Mond's conjecture
$\tau \leq \mu$	$\mu_\Delta \geq \mathscr{A}_e$-codim	$\mu_I \geq \mathscr{A}_e$-codim
with equality in the weighted homogeneous case	with equality in the weighted homogeneous case	

Table 1.1: Comparison between hypersurface singularities and singularities of map germs.

For the discriminant Milnor number the relation is known and it is due to Damon and Mond.

Theorem 1.2.27 (see [DM91, Theorem 1 and Corollary 3])**.** *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be $\mathscr{A}$-finite and (n,p) be nice dimensions with $n \geq p$. Then*

$$\mu_\Delta(f) \geq \mathscr{A}_e\text{-codim}(f),$$

with equality if f is weighted homogeneous.

On the other hand, for the image Milnor number we only know the relation in two pairs of dimensions.

Theorem 1.2.28 (see [dJvS91, Theorem 4.2] and [Mon95, Theorem 2.3])**.** *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite, $n = 1$ or 2. Then*

$$\mu_I(f) \geq \mathscr{A}_e\text{-codim}(f),$$

with equality if f is weighted homogeneous.

In the rest of the dimensions little to nothing is known about this proposition, and it is known as *Mond's conjecture*:

Conjecture 1.2.29 (Mond's conjecture)**.** Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite, $(n, n+1)$ nice dimensions. Then

$$\mu_I(f) \geq \mathscr{A}_e\text{-codim}(f),$$

with equality if f is weighted homogeneous.

There is also a weaker version of Mond's conjecture that we call *weak Mond's conjecture*:

Conjecture 1.2.30 (Weak Mond's conjecture)**.** Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite, $(n, n+1)$ nice dimensions. Then $\mu_I(f) = 0$ if, and only if, f is stable.

This conjecture is weaker in the sense that when a germ is stable it has $\mathscr{A}_e$-codimension equal to zero. In a joint work with Nuño-Ballesteros we solved this conjecture for the corank one case (see [GCNB21, Theorem 3.9]), and this can be seen in Theorem 3.1.22 of Section 3.1.2. We also have an unpublished proof for any corank that we will publish soon.

It is worth mentioning that there is an obvious obstacle to draw these kind of objects. For example in Figure 1.2 we cannot draw the actual object, but a *real representation* of what it is like. In these cases, we draw an object that condensates the main topological traits we want to emphasize[4]. There is a case where we can actually represent the object as it is two-dimensional: when the object is a complex curve. Figure 1.3 is a nice example of this, and similar examples will appear later.

Remark 1.2.31. We also study these notions for the case of map germs with an ICIS in the source in Section 4.1.

[4]The *good real pictures* are real representation of these kind of objects that have all the important information. See, for example, [Mon96].

1.2.3. A final observation

This text pays especial attention to problems in the dimensions $(n, n+1)$. This seems to be the hardest case to study things in general because, usually, things behave properly if $n \geq p$ (and we know it) and things behave badly if $p > n+1$ (and we know it), but we do not know what happens for $p = n+1$. Take, for example, the Milnor-Tjurina relation for map germs.

- If $n \geq p$ the relation is $\mu_\Delta \geq \mathscr{A}_e$-codimension with equality in the weighted homogeneous case, and it is known true (see [DM91, Theorem 1 and Corollary 3]).
- If $p = n+1$ the relation is $\mu_I \geq \mathscr{A}_e$-codimension with equality in the weighted homogeneous case, and few cases are known (see [dJvS91, Theorem 4.2] and [Mon95, Theorem 2.3]).
- If $p > n+1$ one can take the relation $\sum_{i\geq 1} \beta_i\big(\operatorname{im}(f_t)\big) \geq \mathscr{A}_e$-codimension with equality in the weighted homogeneous case, and it is known false (see, for example, [Hou97, Example 4.26] for an example with $\mathscr{A}_e$-codimension 5 and disentanglement homotopically equal to the wedge of two 2-spheres and two 3-spheres).

This is the reason this text deals especially with these dimensions.

1.3. Stratifications

Now, we are going to shortly introduce the basics of stratification theory, given that almost any result on this work relies on a stratified structure in some way. For more details we suggest [Tro07], [GWdPL76] and [Mat12] among others.

1.3.1. Regularity conditions

A stratification is, essentially, a partition of a set into manifolds, with some regularity conditions. For example, a variety V can be decomposed into manifolds by considering $V - \Sigma(V)$, $\Sigma(V) - \Sigma\big(\Sigma(V)\big)$, etc. (see [Whi57]). For a nice introduction to this topic see [Tro07].

Definition 1.3.1. Let X be a closed subset of a manifold M. A *(smooth) stratification* of X is a partition $\mathcal{S}$ of X into submanifolds S_i of M, called the *strata*, such that every point of X has a neighbourhood which meets finitely many strata, i.e., the partition is *locally finite*. In that case X is said to be *stratified* and the stratification is $\mathcal{S} = \{S_i\}_i$.

A usual control condition on a stratification is the *frontier condition*.

Definition 1.3.2. Given a stratification $\mathcal{S} = \{S_i\}_i$, it satisfies the *frontier condition* if $S_r \subseteq \overline{S_t} - S_t$ for any two strata S_r and S_t such that $S_r \cap \overline{S_t} \neq \varnothing$.

Finally, some conditions that guarantee certain topological structure of the stratification are the *Whitney conditions*.

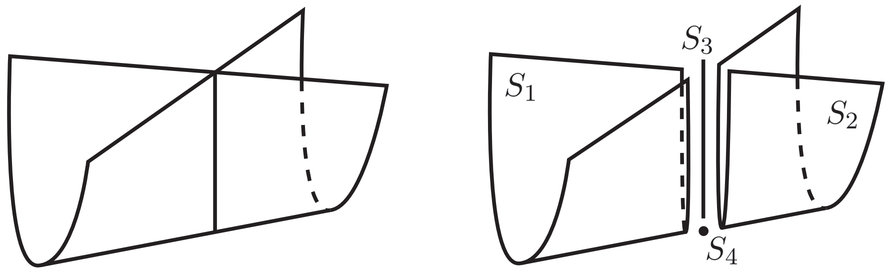


Figure 1.4: Stratification of a Whitney umbrella.

Definition 1.3.3. Consider a stratification of the subset X of M, say $\mathcal{S} = \{S_i\}_i$, and two strata S_r and S_t such that $S_r \subseteq \overline{S_t} - S_t$. Furthermore, say that $\dim M = m$ and $\dim S_t = t$. Then:

(a) The pair (S_r, S_t) satisfies *Whitney's condition* (a) at $p \in S_r$, or they are (a)-*regular* at p, if for any sequence $\{q_n\}_n \subseteq S_t$ converging to p such that $\{T_{q_n} S_t\}$ has limit τ one has $T_p S_r \subseteq \tau$, using a local chart at p and the Grassmanian G_t^m.

(b) The pair (S_r, S_t) satisfies *Whitney's condition* (b) at $p \in S_r$, or they are (b)-*regular* at p, if for any two sequences $\{p_n\}_n \subseteq S_r$ and $\{q_n\}_n \subseteq S_t$ converging to p such that $\{T_{q_n} S_t\}$ has limit τ and the lines $\overline{p_n q_n}$ tend to ν one has $\nu \in \tau$, using a local chart at p.

When any pair of strata as above satisfies the frontier condition and are Whitney (b)-regular at all points we say that the stratification is a *Whitney stratification.*

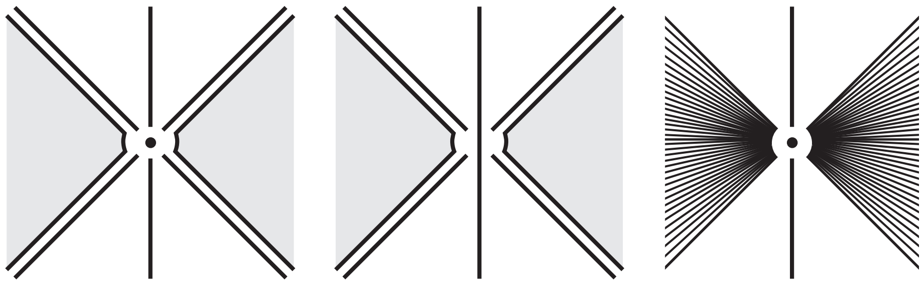


Figure 1.5: A set with: a Whitney stratification (left); a stratification without frontier condition (center); and a partition that is not locally finite, hence not a stratification (right).

Note. Whitney's (b) condition implies Whitney's (a) condition (the converse is not true) and if, in addition, the stratification is locally finite[5], as we have defined stratifications, then the frontier condition is also satisfied. See [Tro07] for more information.

[5]Some texts do not consider the condition of locally finite strata in the definition of stratifications.

Remark 1.3.4. Note that the Whitney (b)-regularity is not a topological property, it depends on how the strata is arranged along the ambient space. More precisely, it is an invariant by diffeomorphisms but not by homeomorphisms (actually, it is a C^1-invariant, see [Tro79, Corollary 3.3]). The following example illustrates this.

Example 1.3.5. If we consider the subset of $\mathbb{R}^3$

$$X = \left\{(x, y, 0) \in \mathbb{R}^3 : y \geq 0\right\},$$

it is obviously Whitney (b)-regular with the strata $S_1 := \left\{(x, 0, 0) \in \mathbb{R}^3\right\}$ and $S_2 := \left\{(x, y, 0) \in \mathbb{R}^3 : y > 0\right\}$. Although, if we consider the subset depicted in Figure 1.6, we see that it is not Whitney (b)-regular even though it is homeomorphic to X by a map that restricts to a homeomorphism on each stratum (even on the ambient). This subset can be obtained with a bump function and a homothecy, and the problem is that it is not *diffeomorphic* to X.

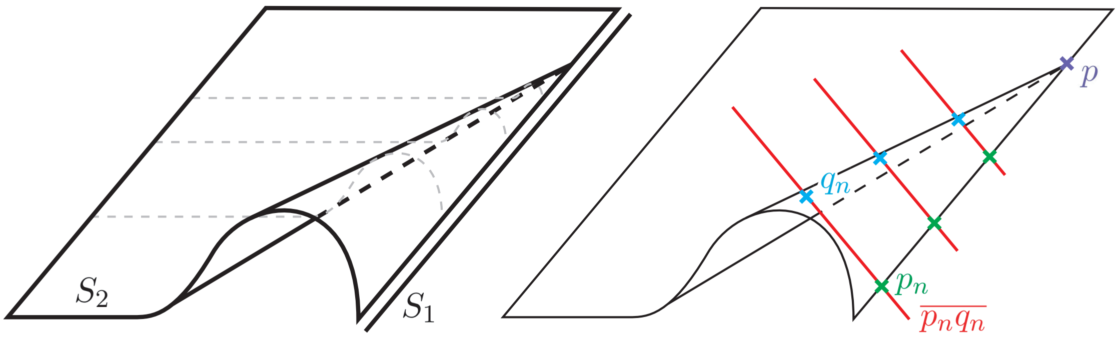


Figure 1.6: Stratified set that is not Whitney (b)-regular.

In general, in particular if one has a categorical way of thinking, once an object is given the following think to ask about are the morphisms, in this case we have the *stratified mappings*.

Definition 1.3.6. A mapping $f : X \to X'$ is *stratified* if X and X' are Whitney stratified sets such that the restriction of the mapping on each stratum of X is submersive onto a stratum of X', i.e., $f(S_\alpha) \subseteq S'_\beta$ and the induced map $f|_{S_\alpha} : S_\alpha \to S'_\beta$ is submersive where S_α is a stratum of X and S'_β is a stratum of X'. In this case, we say that the stratifications of X and X' are *a stratification of f*. A stratified mapping is also called a *stratified submersion*.

One should expect a control condition on stratified mappings to assure some structure, the *Thom condition* (see Figure 1.7).

Definition 1.3.7 (see [GWdPL76, p. 23])**.** Let f be a stratified submersion between X, X' and consider S_r and S_t two strata of the stratification of X. We say that S_t is

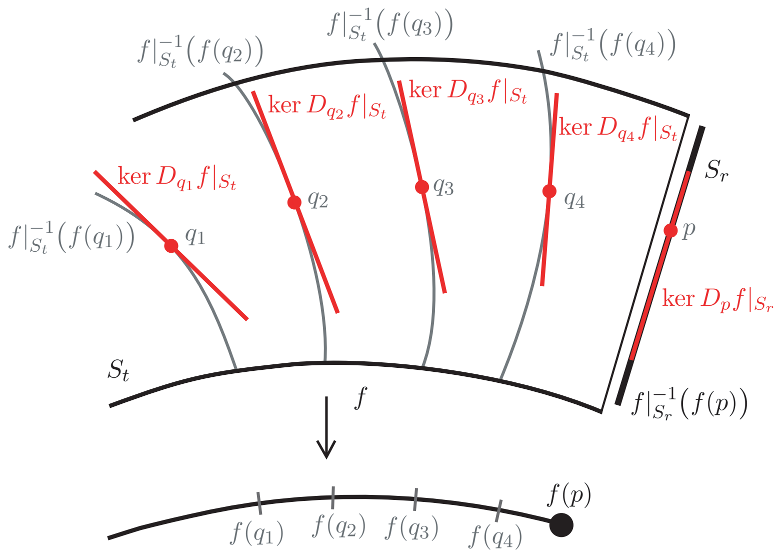


Figure 1.7: Representation of the Thom condition.

Thom regular over S_r at $p \in S_r$ relatively to f if any sequence of points $\{q_n\}_n \subseteq S_t$ converging to p is such that

$$\ker D_p\, f|_{S_r} \subseteq \lim_n \ker D_{q_n}\, f|_{S_t}\,,$$

when the limit exists.

If f is Thom regular for any pair of strata at every point we simply say that f is a *Thom map*, that it satisfies the *Thom A_f condition* or that it satisfies the *Thom condition*. In this case, the stratification of f is *a Thom stratification of f*.

Remark 1.3.8. It could be convenient to notice that

$$\ker D_q f|_S = T_q f|_S^{-1}\big(f(q)\big),$$

so the Thom condition of Definition 1.3.7 can be translated to

$$T_p f|_{S_r}^{-1}\big(f(p)\big) \subseteq \lim_n T_{q_n} f|_{S_t}^{-1}\big(f(q_n)\big).$$

Example 1.3.9. Not every mapping has a stratification such that it is a Thom map, for example the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ so that $f(x, y) = (x, xy)$ does not admit a Thom stratification (see [GWdPL76, p. 24])[6].

[6]Indeed, Thom maps were called *maps without blow-ups* by Thom.

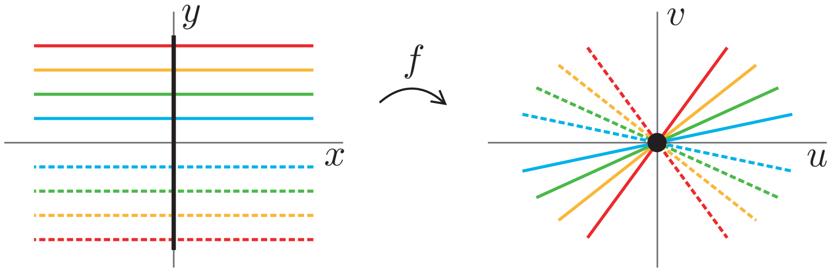


Figure 1.8: Depiction of $f : \mathbb{R}^2 \to \mathbb{R}^2$ so that $f(x, y) = (x, xy)$.

1.3.2. Relation with map germs

We are interested in map germs in the pair of dimensions (n, p) with $p > n$, as they are the dimensions where *everything behaves badly* (recall our comments in Section 1.2.3). One technical obstacle we face when we study the images of germs in those dimensions is that we do not have isolated singularity as complex spaces, for this reason we need to find a stratification with good properties. There is a natural stratification that is Whitney (b)-regular.

Definition 1.3.10. Consider a stable map germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ such that it has corank one or (n, p) are in the nice dimensions. Then, the *stratification by stable types of the discriminant of* f has two points, y_1 and y_2, in the same stratum if, and only if, the germs

$$(f)_{y_1} : \big(\mathbb{C}^n, f^{-1}(y_1) \cap \Sigma(f)\big) \to (\mathbb{C}^p, y_1) \text{ and}$$
$$(f)_{y_2} : \big(\mathbb{C}^n, f^{-1}(y_2) \cap \Sigma(f)\big) \to (\mathbb{C}^p, y_2)$$

are $\mathscr{A}$-equivalent. There is an induced stratification in the source of f, these two stratifications are the *stratification by stable types of* f.

As we were saying, this stratification is Whitney (b)-regular (see [MNB20, Corollary 7.5]). Also, the name of *stable types* is taken because, obviously, this stratification identifies stable singularities of the same $\mathscr{A}$-class. Indeed, there is an analogous definition of the *stratification by stable types* of a locally stable map (recall Definition 1.2.10).

In contrast, it could happen that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ has instabilities. However, if the instability is isolated we can extend the stratification by stable types of $f : (\mathbb{C}^n - S, S) \to (\mathbb{C}^p - 0, 0)$, that is locally stable, with the strata S and 0 in source and target, respectively, and call this stratification *stratification by stable types of* f. If the instability is not isolated we may reproduce this algorithm in some cases and we still call this stratification *stratification by stable types of* f, for example if we have an *excellent unfolding in Gaffney's sense* (see Figure 1.9).

Definition 1.3.11 (see [Gaf93, Definition 6.2])**.** We say that a one-parameter unfolding F of an $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is *excellent* if there exists a representative

$F : \mathcal{X} \to Y \times T$, where Y and T are open neighbourhoods of the origin in $\mathbb{C}^p$ and $\mathbb{C}$ respectively, such that $f_t^{-1}(0) = S$ and $f_t : X_t - S \to Y - \{0\}$ is a locally stable mapping with no 0-stable singularities, where $t \in T$ and f_t are the perturbations given by F.

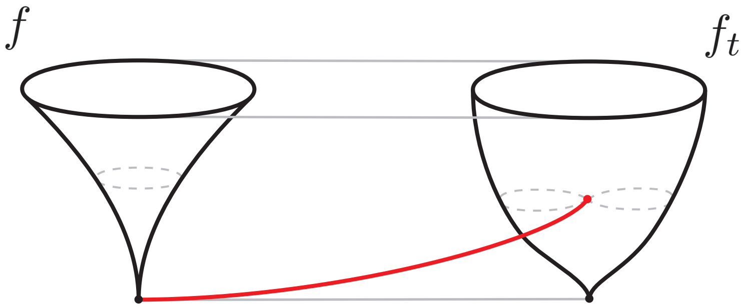


Figure 1.9: A depiction of the failure of excellency because there are new 0-stable singularities in f_t for $t \neq 0$.

Remark 1.3.12. When the unfolding is excellent, $F : \mathcal{X} - S \times T \to (Y - \{0\}) \times T$ is locally stable so we have a well defined stratification by stable types. This extends to $F : \mathcal{X} \to Y \times T$ just by adding $S \times T$ and $\{0\} \times T$ as strata in the source and target, respectively. These are, in fact, the only 1-dimensional strata.

A common situation is having an unfolding, or perturbations, of a map germ f. Say that it has only one parameter, so it induces the family $\{f_t\}_{|t|<\varepsilon}$. A first immediate question is if this family is trivial (as unfolding, recall Definition 1.2.3). If it is not trivial it could occur that all the terms of the family are *still the same in the stratified sense.*

Definition 1.3.13. We say that a one-paremeter unfolding F as in Definition 1.3.11 is *Whitney equisingular* if $F : \mathcal{X} \to Y \times T$ is a Thom stratified map with the stratification by stable types.

Remark 1.3.14. We study the relation between excellency and Whitney equisingularity in Chapter 4, in particular in Section 4.5.

If a one-parameter family is Whitney equisingular we can use Thom-Mather's second isotopy lemma to prove that the family is *topologically trivial*, using a projection to the parameter space. This means that the family (f_t, t) is *topologically left-right equivalent as unfolding* to the constant unfolding of f_0.

Definition 1.3.15. A one-parameter unfolding $(f_t, t) : (\mathbb{C}^n \times \mathbb{C}, S \times 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ is *topologically trivial* if there are two homeomorphisms, Φ and Ψ, that make the following diagram commutative,

$$\begin{array}{ccc}
(\mathbb{C}^n \times \mathbb{C}, S \times 0) & \xrightarrow{(f_t,t)} & (\mathbb{C}^p \times \mathbb{C}, 0) \\
\downarrow \Phi & & \downarrow \Psi \\
(\mathbb{C}^n \times \mathbb{C}, S \times 0) & \xrightarrow{(f_0,t)} & (\mathbb{C}^p \times \mathbb{C}, 0) \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
(\mathbb{C}^n, S) & \xrightarrow{\quad f_0 \quad} & (\mathbb{C}^p, 0)
\end{array}.$$

As we have mentioned, a Whitney equisingular unfolding is topologically trivial. The converse was known as *Ruas' conjecture* (with an equivalent statement, see [Rua94]) but was proved false in [RS19, Section 5].

Chapter 2

Multiple points and ICSS

> Much work has been done on the study of the topology of fibres of maps and this sequence allows us deep insights into the topology of images.
>
> Kevin Houston, *An introduction to the image computing spectral sequence* [Hou99]

In this chapter we introduce the multiple point spaces and the Image-Computing Spectral Sequences (ICSS in short).

Great part of this text deals with ICSS because, as we have said in Section 1.2.3, things are usually more difficult to prove or compute when $p = n + 1$. In particular, μ_Δ can be, in general, computed with algebraic tools while μ_I cannot be easily computed most of the time (see for example [MNB20, Sections 8.7 and 8.8]). We can use different techniques to compute μ_I, and one of them is using an ICSS if the multiple point spaces behave well (for example, as we will see, in corank one).

For a nice introduction to these topics, although it does not cover some recent work, the reader is referred to [Hou99], which is our main reference in this section. See also [MNB20, Sections 9 and 10] for a self-contained review with the approach of [CMM19].

2.1. Introduction

The ICSS are spectral sequences that compute the homology of the image of a mapping (see Definition B.2.3). Therefore, as we did with the *Generic theorem of spectral sequences* at the end of Appendix B.2, theorems of ICSS are of the following form:

Generic theorem of ICSS. *Let $f : X \to Y$ be good, then*

$$E^1_{p,q} \cong AH_q\big(D^{p+1}(f)\big) \Longrightarrow H_{p+q}\big(\operatorname{im}(f)\big)$$

with d_1 *induced from the projections* $\pi : D^k(f) \to D^{k-1}(f)$.

The coefficients of the homology vary from one theorem to another, as the definition of *good mapping* at the beginning of the theorem. In any case, many maps are *good enough* for our purposes.

Observe that the entries of the first page of the spectral sequence are the homology of alternating chains, AH_*, of the multiple point spaces of f, $D^*(f)$. This is foreseeing that in the chains of the spaces $D^r(f)$ we will have an action of a group of permutations Σ_k (see Appendix A.1).

2.2. Multiple points

In this section we introduce the *multiple point spaces*, whose *alternating chains* give homology groups that are the entries of the ICSS we discuss later.

When we talk about *multiple point spaces* of a map $f : X \to Y$ we do not want a partition of X or Y by counting preimages of f because, as we will see in Sections 2.4 and 4.4, these sets could present more complicated singularities than the spaces we will define.

With this in mind, there is one immediate (and naive) definition we could give. We are inspired by Houston's comments in [Hou99, p. 309] for the following definition.

Definition 2.2.1. Let $f : X \to Y$ be a finite map, then the *idiot k-th multiple point space* is

$$ID^k(f) \coloneqq \left\{(x_1, \dots, x_k) \in X^k : f(x_i) = f(x_j),\ \forall i, j\right\}.$$

Remark 2.2.2. This definition is equivalent to taking recursively fibred products, starting with $X \times_Y X$. For this reason this is called in [CMM19] the *k-fold product of X fibred over f*, and it is denoted as $W^k(f)$.

The problem with this definition is that it always includes some *noise* that provides no information, i.e., the *diagonals*

$$\Delta(X^k) \coloneqq \left\{(x_1, \dots, x_k) \in X^k : x_i = x_j,\ \forall i, j\right\}$$

are always included in $ID^k(f)$. To solve this we could take the following definition.

Definition 2.2.3. Let $f : X \to Y$ be a finite map, then the *strict k-th multiple point space* is

$$D^k_S(f) \coloneqq \left\{(x_1, \dots, x_k) \in X^k : f(x_i) = f(x_j) \text{ and } x_i \neq x_j,\ \forall i \neq j\right\}.$$

The problem with the definition of the strict multiple point spaces is that it is not algebraic or analytic in general, therefore we would like to take the closure of this space.

Definition 2.2.4. Let $f : X \to Y$ be a finite map, then the *k-th multiple point space* is

$$D^k(f) := \overline{\left\{(x_1, \dots, x_k) \in X^k : f(x_i) = f(x_j) \text{ and } x_i \neq x_j,\ \forall i \neq j\right\}}.$$

Finally, the problem with this definitions is that, although it is analytic, the equations are very hard to find in general. Despite this, we will usually work with this definition.

Remark 2.2.5. We modify the definition of multiple point spaces in Definition 2.4.2, where we deal with map germs. However, we leave this for Section 2.4 and keep the previous definitions until then.

With the three definitions above we have that

$$\overline{D^k_S(f)} = D^k(f) \subseteq ID^k(f) = W^k(f),$$

and each definition has its issues. However, observe that every definition has a high symmetric structure. Indeed, we can take Σ_k acting by permutation of the entries of $(x_1, \dots, x_k)$ and this defines an action of Σ_k in $ID^k(f), D^k_S(f)$, and $D^k(f)$.

As there are some common developments for the different definitions of multiple point spaces, let Z^k be $ID^k(f), D^k_S(f)$ or $D^k(f)$. Now, assume that the action is compatible with a cellular structure given in Z^k in the sense that the action takes cells to cells and, whenever a cell is fixed as a set, the cell is point-wise fixed. In this case, there is an induced action of Σ_k in the chain complex $C_*(Z^k)$, where we are omitting the coefficients on purpose. Then we can take the submodule

$$C^{\text{Alt}}_*(Z^k) := \left\{c \in C_*(Z^k) : \sigma c = \operatorname{sgn}(\sigma) c, \forall \sigma \in \Sigma_k\right\},$$

called the *alternating chains* of Z^k.

We have left the coefficients omitted because this is the alternating isotype of the representation of Σ_k in $C_*(Z^k, \mathbb{C})$ if we take coefficients in $\mathbb{C}$ (see Appendix A.1). Hence, we can compute it with the projection to the alternating isotype given in Theorem A.4.5:

$$\begin{aligned} P_{\text{Alt}} : C_*(Z^k, \mathbb{C}) &\longrightarrow C_*(Z^k, \mathbb{C}) \\ c &\longmapsto \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma) \sigma c, \end{aligned}$$

or, as we are in a field, we can take $\text{Alt}_{\mathbb{Z}} := k! P_{\text{Alt}}$ instead of P_{Alt}.

On the other hand, if we take coefficients in $\mathbb{Z}$, we can do something similar, as [Hou07, Theorem 2.12] shows:

$$C^{\text{Alt}}_*(Z^k, \mathbb{Z}) \cong \text{Alt}_{\mathbb{Z}} C_*(Z^k, \mathbb{Z}), \tag{2.1}$$

where $\text{Alt}_{\mathbb{Z}}$ is defined as we have done above (this isomorphism is exemplified in Examples 2.3.2 and 2.3.3). This is, in fact, the definition Houston takes in [Hou99] and the reason to define $\text{Alt}_{\mathbb{Z}}$ as $k! P_{\text{Alt}}$.

If the differential of the chain complex commutes with the action of the group, we can take the homology of $C_*^{\mathrm{Alt}}(Z^k)$, denoted as $AH_*(Z^k)$[1]. At first glance, it depends on what is Z^k between the three multiple point space definitions. However, all these homologies coincide (see [Hou99, Theorem 2.7]):

$$AH_*\big(D_S^k(f)\big) \cong AH_*\big(D^k(f)\big) \cong AH_*\big(ID^k(f)\big).$$

So, as Houston said in [Hou99], *it turns out that the idiot's definition is not so idiotic after all*. Indeed, it can be useful to use ID^k instead of D^k when the equations of D^k are hard to determine, as the equations of ID^k are easier.

Remark 2.2.6. To finish this section, observe that we take the alternating part of the chain complex and, then, its homology to obtain AH. Nevertheless, we could have taken the homology of the chain complex and then the alternating part, what is called the *alternating homology*[2]. The alternating homology, denoted as H^{Alt}, does not coincide in general with AH (see Example 2.3.2). As a matter of fact, we do not know in general when they are isomorphic or not and, in the case they are not isomorphic, what the difference is.

However, there are some situations where we do know that they coincide. The first case was proven by Goryunov (see [Gor95, Theorem 2.1.2]), and it happens when we take AH, or H^{Alt}, on the fiber of a Σ_k-invariant ICIS (isolated complete intersection singularity). Another case happens when the coefficients are taken in a field of characteristic zero (see [MNB20, Proposition 10.1]). With this last case we can prove the first one, as the integer homology and rational homology coincide in that circumstance.

2.3. ICSS and examples

Studying the multiple point spaces of stable perturbations gives a lot of information of their images by means of the ICSS. The theorems that give an ICSS have been evolving through the years, being increasingly general and with new approaches. We show below the most general version we have so far.

The theory of ICSS starts with a theorem given by Goryunov and Mond in [GM93, Proposition 2.3] for rational homology and $\mathscr{A}$-finite map germs. This being a promising technique at the moment, Goryunov extended the results on ICSS with [Gor95, Corollary 1.2.2] for finite maps and integer homology. Years later, Houston proved [Hou07, Theorem 5.4] with a bigger class of maps, an action of an additional group, any coefficients and homology of the pair. Finally, Cisneros Molina and Mond developed a new approach in [CMM19] and Mond and Nuño-Ballestero gave a self-contained introduction to this topic with the new approach in [MNB20, Section 10].

[1] Observe that the notation AH is to avoid confussion with Alt$H$ given by an operator Alt. This was done in [CMM19, MNB20] as well.

[2] This notation is not common in the area, usually the alternating homology is the homology of the alternating chains (i.e., AH), what is in deep contradiction with the notation of representation theory.

As we were saying, the most general theorem of ICSS takes into account the action of another group H. In particular, the chains we take to compute the homology have to be alternating for H as well. We will state the theorem omitting the group H, as it is not necessary for our immediate purposes. Also, the theorem considers a large class of maps, the so-called *good maps* (see [Hou07, Definition 5.1]). However, finite simplicial maps between locally finite simplicial complexes are good (see [Hou07, Proposition 5.3]) and that is general enough for us.

Theorem 2.3.1 (see [Hou07, Theorem 5.4]). *Let $f : X \to Y$ be a continuous map and $\tilde{X}$ a subspace of X. Assume that $f : X \to Y$ and $f| : \tilde{X} \to Y$ are good maps in the sense of [Hou07, Definition 5.1] and that $D^k(f|)$ is a subcomplex of $D^k(f)$ for all $k \geq 1$. Then, there exists a spectral sequence*

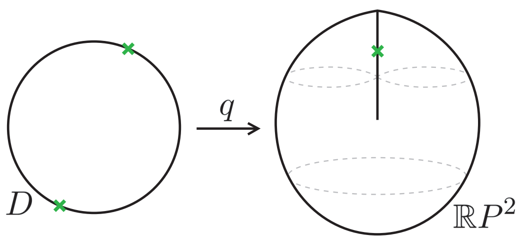
$$E^1_{p,q} = AH_q\big(D^{p+1}(f), D^{p+1}(f|); G\big) \Longrightarrow H_*\big(f(X), f(\tilde{X}); G\big),$$

where G is a coefficient group and the differential d_1 is induced by the projections $\pi : D^k(f) \to D^{k-1}(f)$ for any k.

The definition of convergence of a spectral sequence, what is happening in the theorem, is given in Definition B.2.7.

We give some examples now.

Example 2.3.2 (see [Hou99, Example 5.1] and [MNB20, Example 10.1]). The homology of the projective plane $\mathbb{R}P^2$ can be computed with Theorem 2.3.1 if we consider the quotient map



$$q : D \to \mathbb{R}P^2,$$

which identifies antipodal points on the boundary of the disc D.

First of all, observe that

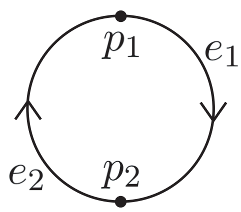
$$D^2(q) = \big\{(x, y) \in D \times D : q(x) = q(y), x \neq y\big\} \subseteq \mathbb{S}^1 \times \mathbb{S}^1.$$

Therefore, using angular coordinates,

$$D^2(q) = \big\{(z, z + \pi) : z \in \mathbb{S}^1\big\} \cong \mathbb{S}^1$$

and the action of Σ_2 is simply the antipodal action.

With the cellular decomposition given here, it is easy to check that



$$C_0^{\mathrm{Alt}}\big(D^2(q), \mathbb{Z}\big) = \langle p_1 - p_2 \rangle$$
$$C_1^{\mathrm{Alt}}\big(D^2(q), \mathbb{Z}\big) = \langle e_1 - e_2 \rangle$$

and that $\partial(e_1 - e_2) = (p_2 - p_1) - (p_1 - p_2) = 2(p_2 - p_1)$.

Therefore, taking quotients, $AH_0\big(D^2(q),\mathbb{Z}\big) = \mathbb{Z}/2\mathbb{Z}$ and $AH_1\big(D^2(q),\mathbb{Z}\big) = 0$. The remaining multiple points are empty, so the spectral sequence given by Theorem 2.3.1 has the shown first page.

$E^1_{*,*} =$

AH_2	0	0	0
AH_1	0	0	0
AH_0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
	D^1	D^2	D^3

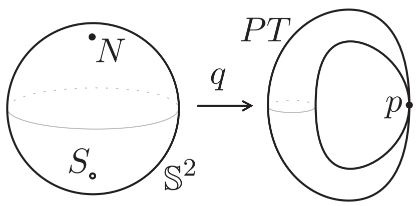
Finally, this spectral sequence collapses at the first page, because the differential between $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ has to be zero, and there are no extension problems (see Definition B.2.5 and the end of Appendix B.2). Hence, as $H_i(\mathbb{R}P^2) = \bigoplus_{r+s=i} AH_s\big(D^{r+1}(q)\big)$, one has that

$$H_i(\mathbb{R}P^2,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}/2\mathbb{Z}, & i=1 \\ 0, & i>1 \end{cases}.$$

This example also illustrates that AH does not coincide in general with H^{Alt}. To be more precise, H^{Alt} is always a subgroup of H and, in this example, $AH_0\big(D^2(q),\mathbb{Z}\big) = \mathbb{Z}/2\mathbb{Z}$ is not a subgroup of $H_0\big(D^2(q),\mathbb{Z}\big) = \mathbb{Z}$.

Example 2.3.3 (see [Hou99, Example 5.2]).

We can compute the homology of the pinched torus PT with Theorem 2.3.1 as well, but the map that collapses a generatrix of the torus to a point is not finite. On the other hand, we can use the identification of the north pole and the south pole of $\mathbb{S}^2$,



$$\begin{aligned} q : \mathbb{S}^2 &\longrightarrow PT \\ N,S &\longmapsto p \end{aligned}$$

Obviously, $D^2(q) = \big\{(N,S),(S,N)\big\}$, Σ_2 acts by permutation of the two points and $C_0^{\mathrm{Alt}}\big(D^2(q),\mathbb{Z}\big)$ is generated by $(N,S)-(S,N)$. Hence, as $D^k(q) = \varnothing$ for $k \geq 3$, the first page of the spectral sequence is as shown.

$E^1_{*,*} =$

AH_2	$\mathbb{Z}$	0	0
AH_1	0	0	0
AH_0	$\mathbb{Z}$	$\mathbb{Z}$	0
	D^1	D^2	D^3

The problem is that the spectral sequence could not collapse at the first page if some differential is not zero, and between the entries $(1,0)$ and $(0,0)$ we could have something non-trivial. However, recall that the first differential is induced by the projection $\pi : D^k(q) \to D^{k-1}(q)$, so this differential is, indeed, zero because $AH_0\big(D^2(q)\big)$ is generated by $[(N,S)-(S,N)]$. Again, as $H_i(PT) = \bigoplus_{r+s=i} AH_s\big(D^{r+1}(q)\big)$, one has that

$$H_i(PT,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i=0,1,2 \\ 0, & i>3 \end{cases}.$$

Why the alternating part?

After seeing Theorem 2.3.1 and these examples, the reader could ask what the alternating part has that do not have other parts to be present in these considerations. Actually, we will answer this question partially in Chapter 6, where *only* the alternating isotype has the properties we need (see, in particular, Lemmas 6.3.2 and 6.3.3). Nevertheless, the fundamental reasons can be seen in the approach of the different theorems of ICSS:

- The approach of [Gor95, Hou07] is constructing a *semi-simplicial resolution* of $f(X)$, i.e., a space with the same homology than $f(X)$ with better properties (see [Hou07, Definition 4.4]). By construction, this *realization* is related in a natural way to the multiple point spaces by means of the alternated chains, after some consideration regarding their dimensions (see [Gor95, Section 1.2] or [Hou07, Proposition 4.8]).
- The approach of [GM93] is similar to the previous one but in algebraic terms (as noted by Goryunov in [Gor95, p. 45]). It is an *alternating subcomplex* that makes certain complex of sheaves exact, allowing us to compute the homology of $f(X)$ (see [GM93, pp. 49–51]).
- Finally, the approach of [CMM19] is constructing a double complex whose associated spectral sequence is the ICSS. In this case, only the alternating chains give the double complex structure (see [CMM19, Lemmas 2.2 and 2.3]).

It is possible, however, that these reasons are expressions of some fundamental fact in a deep sense.

2.4. Multiple points of map germs

Going back to our setting of singularities of map germs, to compute the image Milnor number (and, in general, images of germs where $p > n$) we can use Theorem 2.3.1 in two ways: with a stable perturbation and $\tilde{X}$ being empty or with a stable unfolding and $f|$ in the theorem being a stable perturbation (hence $\tilde{X}$ will be the domain of a stable perturbation).

However, if one is given an $\mathscr{A}$-finite germ f, it could be very difficult to find a stable perturbation f_s of f, in general, so we would like to relate the multiple points of f and those of f_s in some way. This can be done in some cases, but the definition of multiple point spaces for map germs has to be adapted to fit with our ambitions. The relation does not exist without this modification, as the following well-known example shows.

Example 2.4.1. Consider the map germ $f:(\mathbb{C},0)\to(\mathbb{C}^2,0)$ given by $f(x)=(x^2,x^3)$. Then, with the notation of Definitions 2.2.3 and 2.2.4, any small enough representative of the germ is such that $D^2(f)=D^2_S(f)=\varnothing$. In contrast, a stable perturbation f_s has $D^2(f_s)=D^2_S(f_s)\neq\varnothing$.

Indeed, we can take $f_s(x)=(x^2,x^3-sx)$ as a stable perturbation and it maps the points $P=s^{1/2}$ and $Q=-s^{1/2}$ to $(s,0)$. More precisely, $D^2(f_s)=D^2_S(f_s)=\big\{(P,Q),(Q,P)\big\}$ (see Figure 1.3 for a proper illustration of f_s).

We follow [NBPS17, Proposition-definition 2.5] to state the modified definition.

Definition 2.4.2. The *kth-multiple point space* of a mapping or a germ f, denoted as $D^k(f)$, is defined as follows:

- Let $f\colon X \to Y$ be a locally stable mapping between complex manifolds. Then, $D^k(f)$ is equal to the closure[3] of the set of points $\left(x^{(1)}, \dots, x^{(k)}\right)$ in X^k such that $f\left(x^{(i)}\right) = f\left(x^{(j)}\right)$ but $x^{(i)} \neq x^{(j)}$, for all $i \neq j$.
- When $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a stable germ, then $D^k(f)$ is defined analogously but in this case it is a set germ in $\left((\mathbb{C}^n)^k, S^k\right)$.
- Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be finite[4] and let $F(x,u) = \left(f_u(x), u\right)$ be a stable unfolding of f. Then, $D^k(f)$ is the complex space germ in $\left((\mathbb{C}^n)^k, S^k\right)$ given by

$$D^k(f) = D^k(F) \cap \{u = 0\}\,.$$

The fact that $D^k(f)$ is independent of the choice of the stable unfolding F can be found in [NBPS17, Lemma 2.3 and Proposition-definition 2.5].

There is an interesting simplification that can always be done. Suppose $F\colon X \times U \to Y \times U$ is a mapping of the form $F(x,u) = \left(f_u(x), u\right)$. Then, $D^k(F)$ contains only k-tuples $\left(x^{(1)}, u, \dots, x^{(k)}, u\right)$ with the same parameter u. So, it is more convenient to embed $D^k(F)$ in $X^k \times U$ by identifying such a k-tuple with the point $\left(x^{(1)}, \dots, x^{(k)}, u\right)$.

In the particular case of a corank 1 mono-germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, we have explicit equations for the multiple point spaces $D^k(f)$. These are given by the so-called *divided differences of f*, which were introduced by Mond in [Mon87, Section 3] (see also [MNB20, Section 9.5]). The multi-germ version is similar (also in corank 1), it can be found in [MM89, p. 555].

Example 2.4.3. With this definition, the germ given in Example 2.4.1 has $D^2(f) = \left\{(0,0)\right\}$.

We can confirm this with a stabilisation. For example, the one given by $f_s(x) = (x^2, x^3 + sx)$ is well represented in Figure 1.2 and, together with Figure 1.3, validates our claim. On the other hand, we can also use the divided differences we were talking about above, because they give the equations of $D^2(f)$. In this case, they are

$$\left(\frac{f_1(x_1) - f_1(x_2)}{x_1 - x_2}, \frac{f_2(x_1) - f_2(x_2)}{x_1 - x_2}\right) = \left(x_1 + x_2, x_1^2 + x_1 x_2 + x_2^2\right) = 0,$$

which has only the point $\left\{(0,0)\right\}$ as solution.

Recall that we said in Section 2.2 that the different multiple point spaces we could consider ($ID^k(f), D^k_S(f)$ and $D^k(f)$, as defined in Definitions 2.2.1, 2.2.3 and 2.2.4) had a natural action of Σ_k, and the homology of the alternating chains allowed us to compute

[3] As these sets are constructible, this coincides with the Zariski closure.
[4] It is here where there are stable unfoldings, see [MNB20, Theorem 7.2].

the ICSS. With the new definition for germs nothing changes, because we will only use the multiple point spaces of a stable perturbation to compute the ICSS, and in this case the definitions coincide.

Observe also that, when we are studying disentanglements, we can take the alternating homology, H^{Alt}, instead of the homology of the alternating chains, AH, because the disentanglement has the homotopy type of a wedge of spheres and we can take rational homology to compute the number of spheres (see Remark 2.2.6).

The relation that emerges with this new definition can be seen in the *Marar-Mond criterion* (see [MM89, Theorem 2.14]), which was later generalized for multi-germs by Houston (see [Hou10, Theorem 2.4 and Corollary 2.6]):

Theorem 2.4.4. *For $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ of corank 1 and finite, with $n<p$:*

(i) f is stable if, and only if, $D^k(f)$ is smooth of dimension $p-k(p-n)$, or empty, for any $k\geq 1$.

(ii) If $\mathscr{A}_e$-codim(f) is finite, then, for each k with $p-k(p-n)\geq 0$, $D^k(f)$ is empty or an ICIS *of dimension $p-k(p-n)$. Furthermore, for those k such that $p-k(p-n)<0$, $D^k(f)$ is a a subset of S^k, possibly empty.*

There is also an important piece of information hidden in this result. Given an ICIS $D^k(f)$ from Item *(ii)*, as $D^k(f_s)$ is smooth if f_s is a stable perturbation of f by Item *(i)*, it is not difficult to see that the Milnor fiber of $D^k(f)$ is $D^k(f_s)$. This will be a central idea along the following chapters.

Remark 2.4.5. See also Section 4.2 for a generalized version of this theorem for germs with an ICIS in the source, specifically Definition 4.2.1 and Lemma 4.2.3.

In contrast with the corank one case, where we have the divided differences and the Marar-Mond criterion, multiple point spaces of germs with corank greater than one are horrendously behaved, *as far as we know*. Even the stable germs have considerably bad algebraic properties. Notwithstanding this, the double point space of a map germ f still has a known algebraic structure. The $n\times n$ minors of $(\alpha_{ij})_{ij}$; for α_{ij} given by

$$f_i(x_1)-f_i(x_2)=\sum_{j=1}^{n}\alpha_{ij}(x_1,x_2)\big((x_1)_j-(x_2)_j\big),\ \text{for } i=1,\dots,p;$$

together with $f_i(x_1)-f_i(x_2)$ give the equations of $D^2(f)$ (see [Mon87, Section 3] or [MNB20, Section 9.4]).

On the other hand, although the homology of $D^k(f_t)$ is present in different dimensions (see [Mon16]), the homology of the alternating chains is only present in middle dimension and it is free, similarly to the corank one case. This is proven in [Hou97, Theorem 4.6].

Remark 2.4.6. We will exploit further the symmetric structure of the multiple point spaces in Sections 4.2 and 4.4 and Chapter 6.

Chapter 3

Excellent unfoldings

There is an $\mathscr{A}$-orbit open in the $\mathscr{K}$-orbit.

Raúl Oset Sinha

This chapter contains the results of a joint work with Nuño-Ballesteros, [GCNB21]. This paper was completed with [GCNB20], as we had in mind a final objective: to prove that we have Whitney equisingularity under some (few) conditions.

In particular, this chapter gives conditions under we have an excellent unfolding and Chapter 4, which contains the results of [GCNB20], uses this to solve our equisingularity problem in corank 1. However, some results of the first paper were extended on the second one after we encountered some technical problems, so we will cover these extended results in Chapter 4 as well.

3.1. Basic properties of the image Milnor number

Some properties of the image Milnor number were part of the folklore for some time, but formal proofs of these facts were never published. During the first stage of our research the need of these properties became evident and instead of simply accepting them we began to work on formal and detailed proofs. Surprisingly, they turned out to need deep arguments to be proved.

3.1.1. Conservation of the image Milnor number

The first basic property we will need is the *conservation of the image Milnor number*. Loosely speaking, this *conservation* means that when you consider an $\mathscr{A}$-finite germ, for example f, and you perturb it to something that keeps having instabilities, say f_t, the homology that you can see in the image of f_t plus the homology that has to appear if we stabilize its remaining instabilities is equal to the original image Milnor number of f. In short, *the homology of the image of a perturbation plus its image Milnor numbers is* *always constant* (see Theorem 3.1.7 for a formal statement).

To prove this property, we recall the definition of the Milnor fibration (the basic references are [Mil68, Theorems 4.8 and 5.11], but see also our more detailed introduction of this topic in Section 5.2). Let $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic non-zero function which defines a hypersurface $X = g^{-1}(0)$ in $(\mathbb{C}^{n+1}, 0)$ with arbitrary singularities (either isolated or non-isolated). We fix a Whitney stratification on X, as defined in Definition 1.3.3. We denote by B_ε the closed ball of radius ε centred at 0 in $\mathbb{C}^{n+1}$, with boundary $S_\varepsilon = \partial B_\varepsilon$ and interior $\mathring{B}_\varepsilon = B_\varepsilon - S_\varepsilon$.

A *Milnor radius* is a number $\varepsilon > 0$ such that $S_{\varepsilon'}$ is transverse to X, for all ε' such that $0 < \varepsilon' \leq \varepsilon$. This implies that $X \cap B_\varepsilon$ is homeomorphic to the cone on $X \cap S_\varepsilon$ (recall the comments of the beginning of Section 1.2.2).

Once we have fixed $\varepsilon > 0$, there exists $\eta > 0$ such that

$$g : g^{-1}\big(\mathring{D}_\eta\big) \cap B_\varepsilon \to \mathring{D}_\eta$$

is a locally trivial fiber bundle over $\mathring{D}_\eta - \{0\}$. Here, $\mathring{D}_\eta$ is the open disk of radius η centred at 0 in $\mathbb{C}$. The choice of η has to be made in such a way that t is a regular value of g and S_ε is transverse to $g^{-1}(t)$ for all t such that $0 < |t| < \eta$. This is called the *Milnor fibration* and the fibres are called *Milnor fibres*.

As we need to study a *family* of images, now we consider an r-parameter deformation of g, that is, a holomorphic germ $G\colon (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \to (\mathbb{C}, 0)$ written as $G(y, u) = g_u(y)$ and such that $g_0 = g$. Then, G defines a hypersurface $\mathcal{X} = G^{-1}(0)$ in $(\mathbb{C}^{n+1} \times \mathbb{C}^r, 0)$, which is a deformation of X. We assume that $\mathcal{X}$ also has a Whitney stratification whose restriction to $\{u = 0\}$ coincides with that of X.

Definition 3.1.1 (see [Sie91, p. 2])**.** We say that the deformation G is *topologically trivial over the Milnor sphere* S_ε if, for η and ρ small enough,

$$\big(S_\varepsilon \times \mathring{B}_\rho\big) \cap G^{-1}\big(\mathring{D}_\eta\big) \xrightarrow{(G,\mathrm{id})} \mathring{D}_\eta \times \mathring{B}_\rho \tag{3.1}$$

is a stratified submersion with strata $\{0\} \times \mathring{B}_\rho$ and $\big(\mathring{D}_\eta - \{0\}\big) \times \mathring{B}_\rho$ on $\mathring{D}_\eta \times \mathring{B}_\rho$ and the induced stratification on $\big(S_\varepsilon \times \mathring{B}_\rho\big) \cap G^{-1}\big(\mathring{D}_\eta\big)$.

Since we have a Whitney stratification on $\mathcal{X}$, the restriction of Equation (3.1) to each stratum in the target is a locally trivial C^0-fibration, by Thom-Mather's first isotopy lemma (see [GWdPL76, Theorem 5.2]), hence the words *topologically trivial* in Definition 3.1.1.

Now, we introduce the primary tool we will use (see also the previous work of Lê Dũng Tráng in [Trá87, Trá92]):

Theorem 3.1.2 (see [Sie91, Theorem 2.3])**.** *With the notation of Definition 3.1.1, let G be a deformation of g which is topologically trivial over a Milnor sphere. Let $u \in \mathring{B}_\rho$ and suppose that all the fibres of g_u are smooth or have isolated singularities except for one* *special fibre* $X_u := g_u^{-1}(0) \cap B_\varepsilon$. *Then* X_u *is homotopy equivalent to a wedge of spheres of dimension* n *and its number is the sum of the Milnor numbers over all the fibres different from* X_u.

The condition that G is topologically trivial over a Milnor sphere is necessary in Theorem 3.1.2, as the following example shows.

Example 3.1.3. Consider $G : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ given by $G(x, y, z, u) = xy - u$. For $u \neq 0$, $X_u = g_u^{-1}(0) \cap B_\varepsilon$ has not the homotopy type of a wedge of 2-spheres (in fact, it has the homotopy type of $\mathbb{S}^1$). See its representation in Figure 3.1.

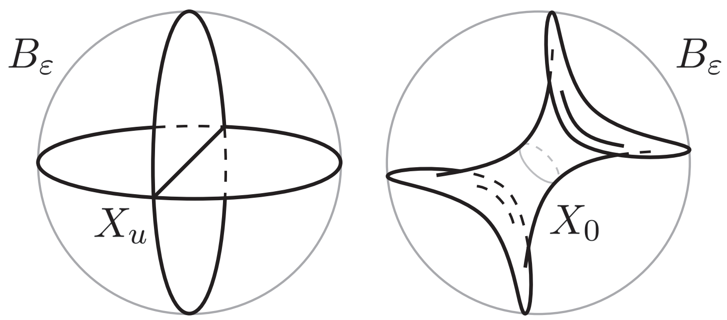


Figure 3.1: Representation of the fibers of Example 3.1.3. Notice the failure of the topological triviality over a Milnor sphere.

Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathscr{A}$-finite germ, that is, with finite $\mathscr{A}_e$-codimension. By the Mather-Gaffney criterion (see Theorem 1.2.21), this is equivalent to that f has isolated instability. In particular, f is finite and, hence, its image is an analytic hypersurface $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$. We take a holomorphic function $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ such that $g = 0$ is a reduced equation for X. We will assume that either $(n, n+1)$ are nice dimensions in Mather's sense or f has corank 1. In both cases, X has a natural stratification given by the stable types (see Section 1.3.2). This stratification is analytically trivial, so it is a Whitney stratification (see [MNB20, Corollary 7.5]).

Consider now an unfolding $F\colon (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\}) \to (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0)$ of f. Write $F(x, u) = \big(f_u(x), u\big)$, as usual, with $f_0 = f$. We denote by $(\mathcal{X}, 0)$ the image of F in $(\mathbb{C}^{n+1} \times \mathbb{C}^r, 0)$ and choose a holomorphic function $G\colon (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \to (\mathbb{C}, 0)$ such that $G = 0$ is a reduced equation of $\mathcal{X}$ and $g_0 = g$, where $g_u(y) = G(y, u)$. We also consider in $\mathcal{X}$ the natural Whitney stratification by stable types outside the instability locus and some stratification in the instability locus. This stratification of $\mathcal{X}$ has the property that its restriction to $u = 0$ coincides with the stratification of X by stable types. We say that G is a deformation of g *induced* by the unfolding F.

Lemma 3.1.4. *Let* f *be* $\mathscr{A}$*-finite such that either* $(n, n+1)$ *are nice dimensions or* f *has corank 1. Any deformation* G *induced by an unfolding* F *is topologically trivial over a Milnor sphere.*

Proof. The proof of this lemma is basically the same that appears in [Mon91, proof of Theorem 1.4] in the particular case that F is a stabilisation of f. On one hand, f is $\mathscr{A}$-finite, hence it has isolated instability, so f is locally stable on S_ε (recall Definition 1.2.10). On the other hand, g is regular on S_ε by definition of Milnor radius. Since S_ε is compact, we can assume, after shrinking ρ if necessary, that f_u is locally stable on S_ε and g_u has no critical points on S_ε, for all $u \in \mathring{B}_\rho$. Now we prove that

$$\left(S_\varepsilon \times \mathring{B}_\rho\right) \cap G^{-1}\left(\mathring{D}_\eta\right) \xrightarrow{(G,\mathrm{id})} \mathring{D}_\eta \times \mathring{B}_\rho$$

is a stratified submersion.

In fact, let $(y,u) \in \left(S_\varepsilon \times \mathring{B}_\rho\right) \cap G^{-1}\left(\mathring{D}_\eta\right)$. If $y \in X_u$, then f_u is stable at y and, hence, F is (analytically) trivial in a neighbourhood of (y,u). This implies that the induced stratification in $\left(S_\varepsilon \times \mathring{B}_\rho\right) \cap \mathcal{X}$ is also (analytically) trivial in a neighbourhood of (y,u). In particular, the map

$$\left(S_\varepsilon \times \mathring{B}_\rho\right) \cap \mathcal{X} \xrightarrow{0\times\mathrm{id}} \{0\} \times \mathring{B}_\rho$$

is a stratified submersion at (y,u). Otherwise, if $y \notin X_u$, then y is a regular point of g_u, therefore (y,u) is a regular point of (G,id). It follows that

$$\left(S_\varepsilon \times \mathring{B}_\rho\right) \cap G^{-1}\left(\mathring{D}_\eta - \{0\}\right) \xrightarrow{(G,\mathrm{id})} \left(\mathring{D}_\eta - \{0\}\right) \times \mathring{B}_\rho$$

is a submersion at (y,u). QED

It follows from Theorem 3.1.2 that, for all u small enough, X_u is homotopy equivalent to a wedge of spheres of dimension n and its number is the Betti number

$$\beta_n(X_u) = \sum_{y \in B_\varepsilon - X_u} \mu(g_u; y).$$

Remark 3.1.5. Note that, since f is $\mathscr{A}$-finite in this case, we can consider a stabilisation of f and find the image Milnor number, $\mu_I(f)$ (see Definition 1.2.23). This is in fact the definition of $\mu_I(f)$ given originally by Mond in [Mon91, Theorem 1.4] in terms of a stabilisation instead of a stable unfolding.

Remark 3.1.6. When $(n, n+1)$ are not nice dimensions and f has corank > 1, the definition of $\mu_I(f)$ can be done analogously by taking Mather's canonical stratification of the image instead of the stratification by stable types and taking a parameter u such that f_u is *topologically stable* instead of stable. However, we will not consider these cases.

The following property is the so-called *conservation of the image Milnor number.*

Theorem 3.1.7. *Let f be $\mathscr{A}$-finite such that either $(n, n+1)$ are nice dimensions or f has corank 1. Let F be any unfolding of f and take $u \in \mathring{B}_\rho$, with $\rho > 0$ small enough. Then,*

$$\mu_I(f) = \beta_n(X_u) + \sum_{y \in X_u} \mu_I(f_u; y),$$

where $\mu_I(f_u; y)$ is the image Milnor number of f_u at $y \in X_u$ and $X_u := \mathrm{im}(f_u)$.

Proof. By taking the sum of F with a stable unfolding, we can assume that F is itself stable. Since f is $\mathscr{A}$-finite, f has isolated instability at the origin by the Mather-Gaffney criterion. This implies that f_u has only finitely many unstable singularities, which we denote by $y_1, \dots, y_k \in X_u$, hence,

$$\sum_{y \in X_u} \mu_I(f_u; y) = \sum_{i=1}^{k} \mu_I(f_u; y_i).$$

Also, by Theorem 3.1.2, the equation of X_u, g_u, has only finitely many (isolated) critical points on $B_\varepsilon - X_u$, which we denote by $z_1, \dots, z_m$, so that

$$\beta_n(X_u) = \sum_{j=1}^{m} \mu(g_u; z_j).$$

For each $i = 1, \dots, k$, we choose a Milnor ball B_{ε_i} for g_u at y_i contained in B_ε. Analogously, for each $j = 1, \dots, m$, we choose also a Milnor ball B_{δ_j} for g_u at z_j contained in $B_\varepsilon - X_u$. We will assume that the balls $B_{\varepsilon_1}, \dots, B_{\varepsilon_k}, B_{\delta_1}, \dots, B_{\delta_m}$ are pairwise disjoint (see Figure 3.2).

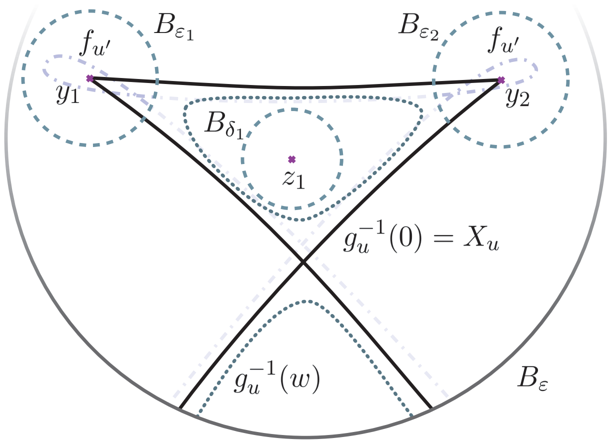


Figure 3.2: Balls in the target.

Again by Theorem 3.1.2, for each $i = 1, \dots, k$, there exists an open ball $\mathring{B}_{\rho_i}$ centered at u and contained in $\mathring{B}_\rho$ such that

$$\mu_I(f_u; y_i) = \beta_n(X_{u'} \cap B_{\varepsilon_i}) = \sum_{w \in B_{\varepsilon_i} - X_{u'}} \mu(g_{u'}; w),$$

for all $u' \in \mathring{B}_{\rho_i} - B(F)$. We set $U_1 = \mathring{B}_{\rho_1} \cap \dots \cap \mathring{B}_{\rho_k}$ (see Figure 3.3).

For each $j = 1, \dots, m$, z_j is an isolated critical point of g_u and $X_u \cap B_{\delta_j} = \varnothing$. By the conservation of the Milnor number of a function, there exists another open ball $\mathring{B}_{\tau_j}$

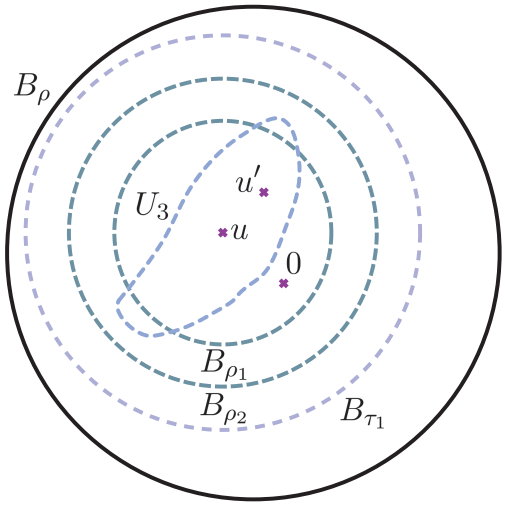


Figure 3.3: Balls in the parameter space.

centered at u and contained in $\mathring{B}_\rho$ such that

$$\mu(g_u; z_j) = \sum_{w \in B_{\delta_j}} \mu(g_{u'}; w),$$

and also $X_{u'} \cap B_{\delta_j} = \varnothing$, for all $u' \in \mathring{B}_{\tau_j}$. As above, we set $U_2 = \mathring{B}_{\tau_1} \cap \cdots \cap \mathring{B}_{\tau_m}$.

Consider the compact set

$$K = B_\varepsilon - \left(\bigcup_{i=1}^{k} \mathring{B}_{\varepsilon_i} \cup \bigcup_{j=1}^{m} \mathring{B}_{\delta_j} \right).$$

Since g_u has no critical points on $K - X_u$, there exists another open neighbourhood U_3 of u in $\mathring{B}_\rho$ such that $g_{u'}$ has no critical points on $K - X_{u'}$, for all $u' \in U_3 - B(F)$.

Finally, again by Theorem 3.1.2,

$$\begin{aligned} \mu_I(f) = \beta_n(X_{u'}) &= \sum_{w \in B_\varepsilon - X_{u'}} \mu(g_{u'}; w) \\ &= \sum_{i=1}^{k} \sum_{w \in B_{\varepsilon_i} - X_{u'}} \mu(g_{u'}; w) + \sum_{j=1}^{m} \sum_{w \in B_{\delta_j}} \mu(g_{u'}; w) \\ &= \sum_{i=1}^{k} \mu_I(f_u; y_i) + \sum_{j=1}^{m} \mu(g_u; z_j), \end{aligned}$$

for all $u' \in U_1 \cap U_2 \cap U_3 - B(F)$. QED

Remark 3.1.8. There is another proof of this theorem using the additivity of the Euler-Poincaré characteristic, due to Nuño-Ballesteros and Peñafort Sanchis. The main idea is to split X_u into two parts, the union of balls around the instabilities and the complementary, and compare them with the image of a stable perturbation. Outside the little balls there is no change because the mapping is locally stable, and inside the little balls we have the wedge of spheres that gives the image Milnor number of each instability. Comparing both characteristics gives the desired result.

A straightforward consequence of Theorem 3.1.7 is that the image Milnor number is upper semi-continuous.

Corollary 3.1.9. *With the conditions and notation of Theorem 3.1.7,*

$$\mu_I(f) \geq \mu_I(f_u; y),$$

for all $y \in X_u$.

The upper semi-continuity of $\mu_I(f)$ has been also obtained by Houston in [Hou10, Theorem 4.3] but in the particular case that f has corank 1 and either $s(f_u) \leq d(f_u)$ or $s(f_u)$ and $d(f_u)$ are constant (see Definition 3.1.14 for the definitions of $s(f_u)$ and $d(f_u)$).

Another consequence of the conservation is the topological invariance of the image Milnor number for unfoldings. Recall Definition 1.3.15: we say that an unfolding F is topologically trivial if it is *topologically $\mathscr{A}$-equivalent as an unfolding* to the constant unfolding.

Corollary 3.1.10. *With the conditions and notation of Theorem 3.1.7, if F is topologically trivial, then*

$$\mu_I(f) = \sum_{y \in X_u} \mu_I(f_u; y).$$

Proof. Write $F(x,u) = \big(f_u(x), u\big)$, $\Phi(x,u) = \big(\phi_u(x), u\big)$ and $\Psi(y,u) = \big(\phi_u(y), u\big)$. Then $\psi_u \circ f_u \circ \phi_u^{-1}$, for all u. Hence, X_u is homeomorphic to X, which is contractible. QED

Remark 3.1.11. Recently, Fernández de Bobadilla, Peñafort Sanchis and Sampaio have published a note proving that the image Milnor number is a topological invariant, not only for families, when the dimensions are $(2,3)$ (see [FdBPnSS19, Theorem 3.3]).

We can say more when F is *good in Gaffney's sense*, see [Gaf93, Definition 2.1]. Roughly speaking it means that F has isolated instability uniformly. We will assume that F is a one-parameter unfolding which is origin-preserving, that is, $f_t(S) = \{0\}$ for all t.

Definition 3.1.12. We say that an origin-preserving one-parameter unfolding $F(x,t) = \big(f_t(x), t\big)$ is *good* if there exists a representative $F : U \to W \times T$, where U is an open neighbourhood of $S \times \{0\}$ in $\mathbb{C}^n \times \mathbb{C}$ and W, T are open neighbourhoods of the origin in $\mathbb{C}^{n+1}$, $\mathbb{C}$ respectively, such that *(i)* F is finite,

(ii) $f_t^{-1}(0) = S$, for all $t \in T$,

(iii) f_t is locally stable on $W - \{0\}$, for all $t \in T$.

Corollary 3.1.13. *If F is a topologically trivial and good unfolding of an $\mathscr{A}$-finite germ f, then $\mu_I(f_t)$ is constant for the family of germs $f_t\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$.*

3.1.2. Weak form of Mond's conjecture

In this subsection we prove the weak version of Mond's conjecture in corank one (see Conjecture 1.2.30). In order to do this, we need the multiple point spaces and the ICSS we presented in Chapter 2, in particular the multiple point spaces for map germs that we defined in Definition 2.4.2.

With that in mind, we follow Houston for the following definition.

Definition 3.1.14 (see [Hou10, Definition 3.9])**.** Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, be $\mathscr{A}$-finite of corank 1 and let $F(x,t) = \big(f_t(x), t\big)$ be a stabilisation of f. We set the following notation:

- $s(f) = |S|$, the number of branches of the multi-germ;
- $d(f) = \sup\big\{k:\ D^k(f_t) \neq \varnothing\big\}$, where f_t is a stable perturbation of f.

The *k-th alternating Milnor number of f*, denoted by $\mu_k^{\mathrm{Alt}}(f)$, is defined as

$$\mu_k^{\mathrm{Alt}}(f) \coloneqq \begin{cases} \dim_{\mathbb{Q}} H^{\mathrm{Alt}}_{n+1-k+1}\left(D^k(F), D^k(f_t); \mathbb{Q}\right), & \text{if } k \leq d(f),\\ \left|\displaystyle\sum_{\ell=d(f)+1}^{s(f)} (-1)^\ell \binom{s(f)}{\ell}\right|, & \text{if } k = d(f)+1 \text{ and } s(f) > d(f),\\ 0, & \text{otherwise.}\end{cases}$$

The value of $\mu_k^{\mathrm{Alt}}(f)$ when $k = d(f)+1$ and $s(f) > d(f)$ can be simplified using

$$\left|\sum_{\ell=d+1}^{s} (-1)^\ell \binom{s}{\ell}\right| = \binom{s-1}{d}.$$

This equality can be proven easily by using elementary properties of binomial numbers.

Another useful property is the following lemma, which gives a relation between $s(f)$ and $d(f)$.

Lemma 3.1.15. *In terms of Definition 3.1.14, the inequality $s(f) > d(f)$ can only happen when $d(f)$ has the maximal possible value.*

Proof. Suppose that the maximal possible value for $d(f)$, for map germs of the type $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$, is m. We will assume, for the sake of the contradiction, that for a map germ f in this pair of dimensions $d(f)<m$ but $d(f)<s(f)$.

Let k be $\min\{s(f),m\}$, hence $d(f)<k$. If we prove that $D^k(f_t)$ is not empty, for f_t a stable perturbation of f, we arrive to a contradiction. To simplify the argument, assume that we have a stabilization $F(x,t)=\big(f_t(x),t\big)$ of f, so that f_t is a stable perturbation for every $t\neq 0$, and this unfolding is inside another unfolding $\mathcal{F}$ of f that is stable as a map germ, i.e., $\mathcal{F}(x,t,u)=\big(f_{t,u}(x),t,u\big)$ such that $f_{t,0}=f_t$.

Given that $k\leq s(f)$, necessarily $D^k(f)$ has at least a point. In fact, since f has $s(f)$ branches passing through the origin of $\mathbb{C}^p$, any subset of S with k distinct points determines a point in $D^k(f)$. However, notice that

$$\begin{aligned} D^k(f) =& D^k(\mathcal{F})\cap\{t=0,u=0\} \text{ and}\\ D^k(f_{t_0}) =& D^k(\mathcal{F})\cap\{t=t_0,u=0\}=D^k(F)\cap\{t=t_0\}\end{aligned}$$

as well, by definition. Hence, if $D^k(F)$ has bigger dimension than $D^k(f)$ we finish because, then, the intersection with $\{t=t_0\}$ will contain at least a point, otherwise the dimensions would be equal.

In fact, since $k\leq m$ and f is $\mathscr{A}$-finite, it follows by [MM89, Theorem 2.14] that $\dim D^k(f)=nk-p(k-1)$. The stabilisation F of f is also $\mathscr{A}$-finite (see [MNB20, Exercise 5.4.2]), so that $\dim D^k(F)=(n+1)k-(p+1)(k-1)$. Since both sets are non-empty, $\dim D^k(F)>\dim D^k(f)$, and this finishes the proof. QED

For instance, for a germ $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$, we have $s(f)>d(f)$ only when $d(f)=n+1$. We discuss this further in Remark 4.2.2.

The motivation for the definition of $\mu_k^{\mathrm{Alt}}(f)$ is the following result by Houston which shows that, for a corank 1 germ $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$, the image Milnor number $\mu_I(f)$ is equal to the sum of all the alternating Milnor numbers.

Proposition 3.1.16 (see [Hou10, Definition 3.11])**.** *Let $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ be $\mathscr{A}$-finite of corank 1. Then,*

$$\mu_I(f)=\sum_k \mu_k^{\mathrm{Alt}}(f).$$

As the reader could imagine, the proof of this result is based on an ICSS such as the one in Theorem 2.3.1. Moreover, the above equality leads to a general definition when we consider the situation of a germ $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$, with $p\geq n+1$. In that case, $\mu_I(f)$ can be also interpreted in terms of the homology of the disentanglement of f (see [Hou10, Remark 3.12] for the details).

Another thing we need to prove the weak version of Mond's conjecture is the following result, due to Wall.

Suppose $g\colon(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ has isolated singularity at 0. Let $U=\mathcal{O}_{n+1}/J_g$ be the *Milnor algebra*, where J_g is the Jacobian ideal, generated by the partial derivatives $\partial g/\partial y_i$, $1\leq i\leq n+1$. Denote by $X_t=g^{-1}(t)\cap B_\varepsilon$ the Milnor fiber, where $0<\delta\ll\varepsilon\ll 1$

and $0<|t|<\delta$. We assume G is a finite group of automorphisms of $(\mathbb{C}^{n+1},0)$ that leaves g invariant. This implies that we have induced actions of G on X_t and on U.

Theorem 3.1.17 (see [Wal80, Theorem of p. 170]). *With the above notation, we have an isomorphism of* $\mathbb{C}G$*-modules*

$$H^n(X_t;\mathbb{C})\cong U\otimes_{\mathbb{C}}\Lambda^{n+1}(\mathbb{C}^{n+1})^*,$$

where $\Lambda^{n+1}(\mathbb{C}^{n+1})^*$ *is the* $(n+1)$*th exterior power of the dual* $(\mathbb{C}^{n+1})^*$*.*

Obviously, the same is true if we replace $\mathbb{C}$ by $\mathbb{Q}$ and consider homology instead of cohomology.

We are now able to state and prove the following essential lemma about the structure of the alternating homology of the multiple point spaces:

Lemma 3.1.18. *Let* $f:(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ *be unstable of corank 1 and* $\mathscr{A}$*-finite, which admits a* 1*-parameter stable unfolding* $F(x,t)=\big(f_t(x),t\big)$*. Take* f_t *a stable perturbation of* f *and* $k=2,\dots,d(f)$*. Then,* $H^{\mathrm{Alt}}_{n-k+1}\big(D^k(f_t);\mathbb{Q}\big)\neq 0$ *if, and only if,* $D^k(f)$ *is singular. Furthermore, if* $H^{\mathrm{Alt}}_{n-k+1}\big(D^k(f_t);\mathbb{Q}\big)\neq 0$*, then* $H^{\mathrm{Alt}}_{n-k'+1}\big(D^{k'}(f_t);\mathbb{Q}\big)\neq 0$ *for all* $k'=k,\dots,d(f)$*.*

Proof. To prove the first part we begin with the case $S=\{0\}$. We use the Marar-Mond criterion, Theorem 2.4.4. Since F is stable, $D^k(F)$ is smooth and $D^k(f)$ is a hypersurface in $D^k(F)$ with isolated singularity and with Milnor fibre $D^k(f_t)$. Moreover, the symmetric group Σ_k leaves invariant the defining equation of $D^k(f)$ in $D^k(F)$. By Theorem 3.1.17, we have an isomorphism of $\mathbb{C}\Sigma_k$-modules

$$H^{n-k+1}(D^k(f_t);\mathbb{C})\cong U\otimes_{\mathbb{C}}\Lambda^{n-k+2}V^*,$$

where U is the Milnor algebra of $D^k(f)$ in $D^k(F)$ and $V=T_0D^k(F)$ is the tangent space of $D^k(F)$ at the origin. If $D^k(f)$ is singular, then $U\neq 0$ and contains the constants. Now, we will see that these constants, after tensoring with $\Lambda^{n-k+2}V^*$, are contained in the alternating part.

From [MM89, Proposition 2.3], we can take Σ_k-invariant equations for $D^k(F)$ in $\mathbb{C}^n\times\mathbb{C}^k$. Since F has corank 1, we assume that $D^k(F)$ is embedded in $\mathbb{C}^n\times\mathbb{C}^k$ with coordinates $x_1,\dots,x_n,y_1,\dots,y_k$ and that Σ_k acts by permuting $y_1,\dots,y_k$. It follows that the tangent space V has Σ_k-invariant linear equations of the form

$$a_i(y_1+\dots+y_k)+\sum_{j=1}^n b_{i,j}x_j=0,\ \text{for } i=1,\dots,n.$$

Hence, we can split V as $V=V_1\oplus V_2$, where

$$\begin{aligned}V_1&=\{x_1=0,\dots,x_n=0,y_1+\dots+y_k=0\},\\ V_2&=V\cap\{y_i=y_j,1\le i<j\le k\}.\end{aligned}$$

If $\omega_1, \dots, \omega_\ell$ is any basis of V_2^*, then

$$\lambda = (dy_1 - dy_2) \wedge \cdots \wedge (dy_{k-1} - dy_k) \wedge \omega_1 \wedge \cdots \wedge \omega_\ell$$

generates $\Lambda^{n-k+2}V^*$ and is Σ_k-alternating. This shows that $H^{n-k+1}\big(D^k(f_t);\mathbb{C}\big)$ has non-zero alternating part in the mono-germ case.

Suppose now that S is any finite set. Let $D_1^k(F), \dots, D_m^k(F)$ be the connected components of $D^k(F)$. Each $D_i^k(F)$ is a mono-germ at a point $\big(z^{(1)}, \dots, z^{(k)}, 0\big) \in S^k \times \{0\}$. We also denote by $D_1^k(f), \dots, D_m^k(f)$ the connected components of $D^k(f)$ such that $D_i^k(f) \subset D_i^k(F)$ for any i.

As $D^k(f)$ is singular, without loss of generality, we can suppose that $D_1^k(f)$ is singular. Assume that $D_1^k(f)$ is a mono-germ at $\big(z^{(1)}, \dots, z^{(k)}\big) \in S^k$ and let $G \leq \Sigma_k$ be the stabilizer of this point. By following the same argument as in the mono-germ case but with $D_1^k(F)$, $D_1^k(f)$ and G instead of $D^k(F)$, $D^k(f)$ and Σ_k, respectively; we find a non-zero element v in the homology of $D_1^k(f_t)$ which is G-alternating.

Now, for each $i = 1, \dots, m$, we choose a permutation $\sigma_i \in \Sigma_k$ that takes $D_1^k(f_t)$ into $D_i^k(f_t)$. We claim that $\omega = \sum_i \operatorname{sgn}(\sigma_i)\sigma_i v$ is a non-zero element in the homology of $D_k(f_t)$ which is alternating.

Let τ be an element of Σ_k. For each $i = 1, \dots, m$, τ takes $\sigma_i\big(z^{(1)}, \dots, z^{(k)}\big)$ into some other $\sigma_{j(i)}\big(z^{(1)}, \dots, z^{(k)}\big)$, where $j(i) = 1, \dots, m$. We can write $\tau\sigma_i$ as $\tau\sigma_i = \sigma_{j(i)}\Big(\sigma_{j(i)}^{-1}\tau\sigma_i\Big)$, and $\Big(\sigma_{j(i)}^{-1}\tau\sigma_i\Big) \in G$. Hence,

$$\begin{aligned}
\tau\omega &= \tau\sum_i \operatorname{sgn}(\sigma_i)\sigma_i v \\
&= \sum_i \operatorname{sgn}(\sigma_i)^2 \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma_{j(i)})\sigma_{j(i)} v \\
&= \operatorname{sgn}(\tau)\sum_i \operatorname{sgn}(\sigma_{j(i)})\sigma_{j(i)} v.
\end{aligned}$$

But if $j(i_1) = j(i_2)$, for some $i_1 \neq i_2$, then

$$g = (\tau\sigma_{i_1})^{-1}(\tau\sigma_{i_2}) = \sigma_{i_1}^{-1}\sigma_{i_2}$$

is in G, as it fixes $\big(z^{(1)}, \dots, z^{(k)}\big)$. We have $\sigma_{i_2} = \sigma_{i_1} g$ and both σ_{i_1} and σ_{i_2} take $D_1^k(f_t)$ to the same component, which is absurd. Hence, $\tau\omega = \operatorname{sgn}(\tau)\omega$.

This concludes the proof that if $D^k(f)$ is singular, then $H^{n-k+1}\big(D^k(f_t);\mathbb{C}\big)$ has non-zero alternating part. The converse is obvious, for if $D^k(f)$ is smooth then the homology is trivial and it cannot have alternating part.

For the second part, take k such that $D^k(f)$ is singular. Then, $D^k(f)$ is a subspace of $\big((\mathbb{C}^n)^k, S^k\big)$, with coordinates $x_i^{(j)}$, with $i = 1, \dots, n$ and $j = 1, \dots, k$, and whose equations are the divided differences, which we represent by $\phi_1, \dots, \phi_r$ with $r = (n+1)(k-1)$. Moreover, $D^k(f)$ has codimension r and, by the Jacobian criterion, the Jacobian matrix A of the functions $\phi_1, \dots, \phi_r$ has rank less than r at some point in S^k.

With this setting, $D^{k+1}(f)$ is defined in $\big((\mathbb{C}^n)^{k+1}, S^{k+1}\big)$ by adding n new coordinates $x_1^{(k+1)}, \dots, x_n^{(k+1)}$ and $n+1$ new equations $\phi_{r+1}, \dots, \phi_{r+n+1}$. Since the old equations do not depend on the new variables, the Jacobian matrix of $\phi_1, \dots, \phi_{r+n+1}$ is

$$\left(\begin{array}{c|c} A & 0 \\ \hline * & B \end{array}\right),$$

where B is the Jacobian matrix of the new equations with respect to the new variables. Obviously, this matrix has rank $< r+n+1$ at some point in S^{k+1} and, thus, $D^{k+1}(f)$ is also singular, since it has codimension $r+n+1$. We can proceed recursively for $D^{k'}(f)$, with $k \le k' \le d(f)$. QED

Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite and assume that $F = (f_u, u)$ is another $\mathscr{A}$-finite germ that unfolds f with p parameters. Consider a simultaneous stable unfolding $\mathcal{F} = (f_{u,v}, u, v) = (F_v, v)$ of f and F (with $p+q$ and q parameters, respectively). Now, choose $\mathcal{G}\colon (\mathbb{C}^{n+1} \times \mathbb{C}^p \times \mathbb{C}^q, 0) \to (\mathbb{C}, 0)$ such that $\mathcal{G}(\bullet, \bullet, v) = 0$ are reduced equations of the image of F_v and $\mathcal{G}(\bullet, u, v) = 0$ are reduced equations of the image of $f_{u,v}$. In that case, y is a critical point of $\mathcal{G}(\bullet, u, v)$ whenever (y, u) is a critical point of $\mathcal{G}(\bullet, \bullet, v)$. In particular, by Theorem 3.1.2, this proves the following lemma:

Lemma 3.1.19. *If $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{A}$-finite and F is an $\mathscr{A}$-finite unfolding of f, then*

$$\mu_I(f) \ge \mu_I(F).$$

However, when we wrote [GCNB21], we only cared about the positivity of these image Milnor numbers, so we gave an overcomplicated proof with the following two lemmas[1].

Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite. Take F a stable unfolding and choose $G\colon (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \to (\mathbb{C}, 0)$ such that $G(y, u) = 0$ is a reduced equation of the image of F. The relative Jacobian ideal is the ideal $J_y(G)$ generated by the partial derivatives of G with respect to the variables $y_1, \dots, y_{n+1}$.

Lemma 3.1.20. *We have:*

$$\mu_I(f) = 0 \iff G \in \sqrt{J_y(G)}.$$

Proof. We follow the notation of Section 3.1.1. If $G \in \sqrt{J_y(G)}$, then $V\big(J_y(G)\big) \subseteq V(G)$. Hence, for any (y, u) such that y is a singular point of g_u, we have $g_u(y) = 0$. In particular, for $u \notin B(F)$,

$$\mu_I(f) = \beta_n(X_u) = \sum_{y \in B_\varepsilon - X_u} \mu(g_u; y) = 0.$$

Conversely, if $G \notin \sqrt{J_y(G)}$, then $V(J_y(G)) \nsubseteq V(G)$. Hence, there exists (y, u) such that y is a singular point of g_u and $g_u(y) \ne 0$. This gives

$$\mu_I(f) \ge \beta_n(X_u) = \sum_{y \in B_\varepsilon - X_u} \mu(g_u; y) \ge 1.$$

QED

[1] This was kindly noted by Mond.

Lemma 3.1.21. *Let $h\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite and let f be any unfolding of h which is also $\mathscr{A}$-finite. If $\mu_I(f) > 0$, then $\mu_I(h) > 0$.*

Proof. Assume that $f(x, v) = \big(h_v(x), v\big)$ and denote by (y, v) the coordinates of f in the target. Let F be a stable unfolding of f. If $\mu_I(h) = 0$, then $G \in \sqrt{J_y(G)} \subseteq \sqrt{J_{y,v}(G)}$, so $\mu_I(f) = 0$. QED

Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite and assume that either $(n, n + 1)$ are nice dimensions or f has corank 1. Here we prove the following weak version of Mond's Conjecture in the corank 1 case (see Conjecture 1.2.29 for the original version of the conjecture).

Theorem 3.1.22 (Weak Mond's conjecture)**.** *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite of corank 1. Then, $\mu_I(f) = 0$ if, and only if, f is stable.*

Proof. Obviously, $\mu_I(f) = 0$ when f is stable. Assume that f is not stable.

If $s(f) > d(f)$, we know that $d(f) = n+1$ by Lemma 3.1.15, and also that $\mu^{\text{Alt}}_{n+2}(f) > 0$. Hence, we can suppose that $s(f) \leq d(f)$.

By the Marar-Mond criterion, either $D^k(f)$ is singular for some $k = 2, \dots, d(f)$ or $D^k(f) \subseteq S^k$ for some $k \geq n + 2$. We suppose first that $D^k(f)$ is singular, for some $k < n + 1$.

If f admits a 1-parameter stable unfolding $F(x, t) = \big(f_t(x), t\big)$, then $H_{n-k+1}\big(D^k(f_t)\big)$ has non-zero alternating part for $t \neq 0$, by Lemma 3.1.18. Since $D^k(F)$ is contractible and $k < n + 1$, it follows from the exact sequence of the pair $\big(D^k(F), D^k(f_t)\big)$ that

$$H^{\text{Alt}}_{n-k+2}\big(D^k(F), D^k(f_t); \mathbb{Q}\big) \cong H^{\text{Alt}}_{n-k+1}\big(D^k(f_t); \mathbb{Q}\big),$$

so $\mu^{\text{Alt}}_k(f) > 0$.

If f does not admit a 1-parameter stable unfolding, we consider a minimal stable unfolding F. By taking a generic section on the parameter space, we get a finitely determined germ F_0 which is an unfolding of f and which admits the 1-parameter stable unfolding F. Now $\mu_I(F_0) > 0$ by the above argument and, hence, also $\mu_I(f) > 0$ by Lemma 3.1.19 (or Lemma 3.1.21).

The next case to consider is when $D^{n+1}(f)$ is singular. Again, we use the exact sequence of the pair $\big(D^k(F), D^k(f_t)\big)$, but, in this case,

$$H^{\text{Alt}}_1\big(D^{n+1}(F), D^{n+1}(f_t); \mathbb{Q}\big)$$

is isomorphic to the kernel of the mapping

$$H^{\text{Alt}}_0\left(D^{n+1}(f_t); \mathbb{Q}\right) \longrightarrow H^{\text{Alt}}_0\left(D^{n+1}(F); \mathbb{Q}\right) \tag{3.2}$$

induced by the inclusion. Take a singular 0-dimensional component of $D^{n+1}(f)$, with multiplicity $m > 1$. Such component will split into m distinct points in $D^{n+1}(f_t)$, which correspond to m distinct generators of $H^{\text{Alt}}_0\left(D^{n+1}(f_t); \mathbb{Q}\right)$. But these m points are in the same connected component of $D^{k+1}(F)$, for $F(x, t) = \big(f_t(x), t\big)$. Hence, we get a non-trivial element of the kernel of Equation (3.2), thus $\mu^{\text{Alt}}_{n+1}(f) > 0$.

Finally, it only remains to consider the case where $D^{n+1}(f)$ is smooth but $D^k(f) \subseteq S^k$ for some $k \geq n+2$. Since $s(f) \leq d(f)$, $D^{n+1}(f)$ must contain a point $\left(x^{(1)}, \dots, x^{(n+1)}\right)$ such that $x^{(i)} = x^{(j)}$ for some $i \neq j$, as the projections from the previous $D^k(f)$ to this $D^{n+1}(f)$ cover all the possible points in the last space and we have less than $n+2$ points in S. This point will also split into several distinct points in $D^{n+1}(f_t)$, which is not possible if $D^{k+1}(f)$ is smooth. We deduce that this case cannot occur when $s(f) \leq d(f)$. QED

Note. The proof of Theorem 3.1.22 is inspired in the proof of [CMWA02, Proposition 4.4]. Here, it is proved that a corank 1 mono-germ of $\mathscr{A}_e$-codimension 1 has image Milnor number equal to 1, based on the same result of Wall (i.e., Theorem 3.1.17).

The following corollary can be deduced easily from Lemma 3.1.18, Theorem 3.1.22 and their proofs and it gives a sharper estimate of $\mu_I(f)$ when f is unstable.

Corollary 3.1.23. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite of corank 1 and unstable. Assume $H_{n-k+1}\left(D^k(f_t); \mathbb{Q}\right)$ has non-zero alternating part for some k:*

(i) If $s(f) \leq d(f)$, then $\mu_I(f) \geq d(f) - k + 1$.

(ii) If $s(f) > d(f)$, then $\mu_I(f) \geq d(f) - k + 1 + \binom{s(f)-1}{d(f)}$.

Furthermore, in Item (i), there always exists such a k and, in Item (ii), $d(f)$ has to be equal to $n+1$ and such a k could not exist.

A straightforward consequence of the weak Mond's conjecture is about the dimension of the relative Jacobian module of f considered in [FdBNnBPnS19]. It is defined as

$$M_y(G) = \frac{J(G) + (G)}{J_y(G)},$$

where $G\colon (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \to (\mathbb{C}, 0)$ is a function such that $G(y, u) = 0$ is a reduced equation of the image of a stable unfolding of f. It is not difficult to see that the dimension of $M_y(G)$ is always $\leq r$ when f is $\mathscr{A}$-finite. Moreover, it is shown in [FdBNnBPnS19, Theorem 6.1] that Mond's conjecture holds for f when $M_y(G)$ is Cohen-Macaulay of dimension r.

Corollary 3.1.24. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite of corank 1 and unstable. Then, $M_y(G)$ has dimension r.*

Proof. It follows from [FdBNnBPnS19, Theorem 6.1] that

$$\mu_I(f) = e_{\mathcal{O}_r}\left((u_1, \dots, u_r); M_y(G)\right),$$

the Samuel multiplicity of the $\mathcal{O}_r$-module $M_y(G)$ with respect to the parameter ideal $(u_1, \dots, u_r)$. But it is well known that an R-module has multiplicity > 0 if, and only if, it has dimension equal to $\dim R$. QED

3.2. Houston's conjecture on excellent unfoldings

It is not difficult to see that, if we add a new branch to an unstable multi-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, then its $\mathscr{A}_e$-codimension increases strictly (see for instance [MNB20, Exercise 3.4.1]). We show the same property for the image Milnor number, instead of the $\mathscr{A}_e$-codimension. The idea of the proof is easy to visualize, as we can see in Figure 3.4.

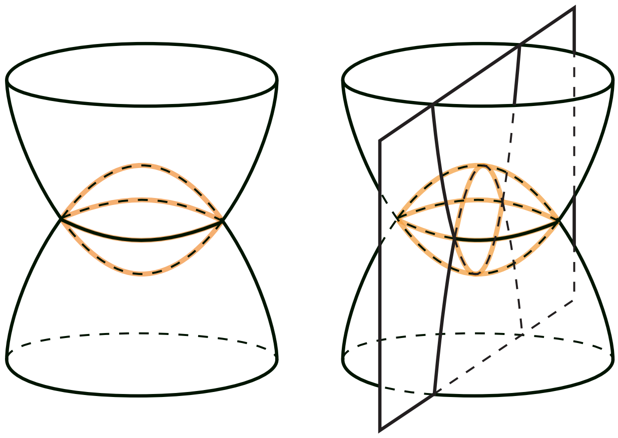


Figure 3.4: Real representation of the creation of more homology via the addition of more branches. Note that in the complex case this happens in middle dimension.

Given two germs $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ and $g\colon (\mathbb{C}^n, z) \to (\mathbb{C}^{n+1}, 0)$, we denote by $\{f, g\}\colon \big(\mathbb{C}^n, S \sqcup \{z\}\big) \to (\mathbb{C}^{n+1}, 0)$ the new multi-germ obtained as the disjoint union of f and g. If f and g are both of corank 1 and $\mathscr{A}$-finite, then

$$\mu_k^{\mathrm{Alt}}(f) \leq \mu_k^{\mathrm{Alt}}\big(\{f, g\}\big),$$

for all k, since adding a new branch does not kill the corresponding alternating homology of the k-multiple point space because the new branch just adds more connected components disjoint from the ones we had before. By Proposition 3.1.16, this implies that

$$\mu_I(f) \leq \mu_I\big(\{f, g\}\big).$$

We may have $\mu_I(f) = \mu_I\big(\{f, g\}\big)$ when f is stable and g is transverse to f, so that $\{f, g\}$ is also stable. In the next lemma, we show that, if f is unstable, then the inequality is strict.

Lemma 3.2.1. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ and $g\colon (\mathbb{C}^n, z) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite. If f has corank 1 and $\mu_I(f) > 0$, then*

$$\mu_I(f) < \mu_I\big(\{f, g\}\big).$$

Proof. By the upper semi-continuity of the image Milnor number (see Corollary 3.1.9), we can assume that the image of g is a generic hyperplane H in $\mathbb{C}^{n+1}$ through the origin.

Let f_t be a stable perturbation of f with image X_t. Since H is a generic hyperplane, the disjoint union $\{f_t, g\}$ gives a stable perturbation of $\{f, g\}$, with image $X_t \cup H$.

Furthermore, $X_t \cap H$ is also the image of a stable perturbation of the restriction $\tilde{f} : \big(f^{-1}(H), S\big) \to (H, 0)$. Since H is generic and f is $\mathscr{A}$-finite of corank 1, $\big(f^{-1}(H), S\big)$ is smooth and $\tilde{f}$ is also $\mathscr{A}$-finite of corank 1. Moreover, $\tilde{f}$ cannot be stable because f is a 1-parameter unfolding of $\tilde{f}$. Hence $\mu_I(\tilde{f}) > 0$, by the weak Mond's conjecture (Theorem 3.1.22).

Now, just apply the Mayer-Vietoris sequence:

$$0 \longrightarrow H_n(X_t) \longrightarrow H_n(X_t \cup H) \longrightarrow H_{n-1}(X_t \cap H) \longrightarrow 0$$

so

$$\mu_I\big(\{f, g\}\big) = \mu_I(f) + \mu_I(\tilde{f}) > \mu_I(f).$$ QED

We recall, now, the notion of excellent unfolding in Gaffney's sense we gave in Definition 1.3.11, as the reader will see new connections with previous concepts. Indeed, we give a reformulation of the first definition.

Definition 3.2.2 (see [Gaf93, Definition 6.2])**.** A one-parameter origin-preserving unfolding F is called *excellent* if it is good and it has a representative as in Definition 3.1.12 such that, in addition, f_t has no 0-stable singularities on $W - \{0\}$ (i.e., stable singularities whose isosingular locus is 0-dimensional).

Excellent unfoldings play an important role in the theory of equisingularity of families of germs. In fact, recall that in the stratification by stable types of an excellent unfolding outside the singularity locus extends to the whole space if we add the axes in source and target (see Remark 1.3.12). See Figure 3.5 for a representation of an excellent unfolding.

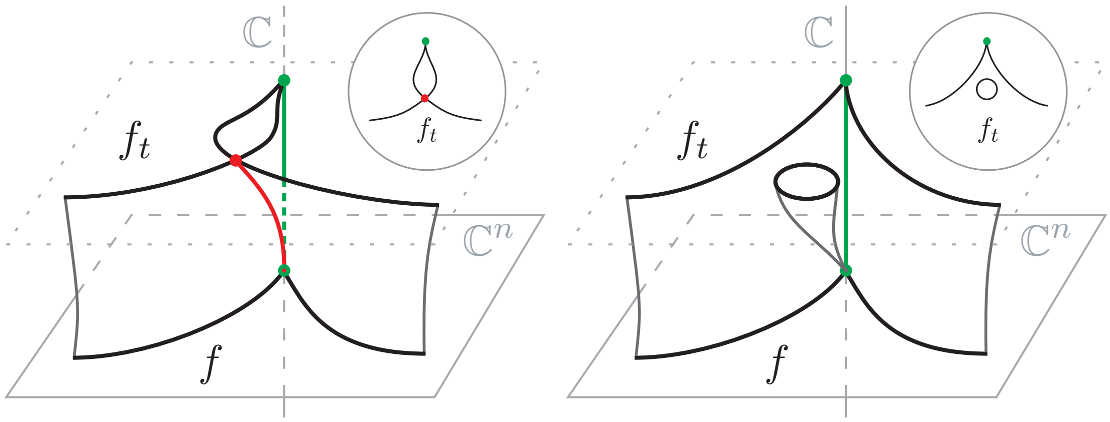


Figure 3.5: A non-excellent unfolding (left) due to the presence of a 1-dimensional stratum, of stable or unstable points, distinct from the parameter axis (red and green, respectively), and an excellent unfolding (right) with only one stratum of dimension one (green).

The above lemma together with the conservation of the image Milnor number and the weak Mond's conjecture allow us to prove *Houston's conjecture on excellent unfoldings* for the pair of dimensions $(n, n+1)$ (see [Hou10, Conjecture 6.2]), which we state now.

Theorem 3.2.3. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite of corank 1 and let $F(x,t) = \big(f_t(x), t\big)$ be an origin-preserving one-parameter unfolding. Consider the family of germs $f_t\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. Then $\mu_I(f_t)$ constant implies F excellent.*

Proof. We will use [Hou10, Corollary 5.9], so we only need to show that F is good and that either $s\big((f_t)_0\big) \leq d\big((f_t)_0\big)$ for all t or $s\big((f_t)_0\big)$ and $d\big((f_t)_0\big)$ are both constant, where $(f_t)_0$ is the germ $(f_t)_0 : \big(\mathbb{C}^n, f_t^{-1}(0) \cap \Sigma(f_t)\big) \to (\mathbb{C}^{n+1}, 0)$ (we keep the notation f_t for the germ at S).

We can suppose that f is not stable, otherwise the result is trivial. We first prove that $s\big((f_t)_0\big)$ is constant, that is, $f_t^{-1}(0) = S$ and, hence, $(f_t)_0 = f_t$. We have $S \subseteq f_t^{-1}(0)$ and, if the inclusion was strict, then $\mu_I(f_t) < \mu_I\big((f_t)_0\big)$ by Lemma 3.2.1. But the upper semi-continuity of Corollary 3.1.9 implies that $\mu\big((f_t)_0\big) \leq \mu_I(f)$, in contradiction with the constancy of $\mu_I(f_t)$.

The inequality $s(f_{t_0}) > d(f_{t_0})$ for some t_0 can only happen when $d(f_{t_0}) = n+1$ (recall Lemma 3.1.15). But $s(f_t)$ is constant, so $s(f_t) > n+1 \geq d(f_t)$ and, again, we have $d(f_t) = n+1$. This shows that either $s(f_t) \leq d(f_t)$ for all t or $s(f_t)$ and $d(f_t)$ are both constant.

Finally, we use the conservation of the image Milnor number, Theorem 3.1.7, to show that F is good. In fact, we get

$$\mu_I(f_t; 0) = \mu_I(f) \geq \sum_{y \in X_t} \mu_I(f_t; y),$$

so $\mu_I(f_t; y) = 0$ for all $y \in X_t - \{0\}$. By the weak Mond's conjecture Theorem 3.1.22, f_t is locally stable on $X_t - \{0\}$. QED

One can ask if the converse is true, that is, if excellency implies constant image Milnor number. We have the following partial result:

Proposition 3.2.4. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite with $n = 1, 2$ and let $F(x,t) = \big(f_t(x), t\big)$ be an origin-preserving one-parameter unfolding. Then, F excellent implies $\mu_I(f_t)$ constant.*

Proof. Let $n = 1$. We have $\mu_I(f_t) = \delta(f_t) - s(f_t) + 1$, where $\delta(f_t)$ is the delta invariant (see, for example, [Mon95, Lemma 2.2]). Obviously, $s(f_t) = |S|$ is constant and we also have conservation of the delta invariant, which means that

$$\delta(f) = \sum_{y \in \Sigma(X_t)} \delta(f_t; y),$$

where $\Sigma(X_t)$ is the singular locus of the image of f_t and $\delta(f_t; y)$ is the delta invariant of the germ of f_t at $f_t^{-1}(y)$. Since F is excellent, we have $\Sigma(X_t) = \{0\}$ and $f_t^{-1}(0) = S$, so $\delta(f_t) = \delta(f_t; 0)$ is also constant.

Let $n = 2$. We consider the double point curve in the source $D(f_t)$, defined as $p_1(D^2(f_t))$, where $p_1 : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ is the projection onto the first component. Then, $D(f_t)$ is a family of germs of plane curves in $(\mathbb{C}^2, S)$. Since F is excellent, we can choose representatives of $D(f_t)$ on some open neighbourhood U of S in $\mathbb{C}^2$ such that $\Sigma\big(D(f_t)\big)$ is equal to S for all t. This implies that the (usual) Milnor number $\mu\big(D(f_t); x\big)$ at each point $x \in S$ must be constant. By a theorem of Fernández de Bobadilla and Pe-Pereira, see [FdBP08, Theorem C], the unfolding F is topologically trivial. So, $\mu_I(f_t)$ is constant by Corollary 3.1.13. QED

We also have the following partial counterexample.

Example 3.2.5. The family $f_t(x, y) = \big(x, y^2, yp_t(x, y^2)\big)$ with

$$p_t(x, y^2) = \left(x - \frac{t}{2}\right)^2 + \left(y^2 - \frac{t}{2}\right)^2 - \frac{t^2}{8}$$

yields an excellent unfolding over $\mathbb{R}$, but not over $\mathbb{C}$ because $y = 0$ and $x = \frac{1}{4}\left(2t \pm i\sqrt{2}t\right)$ are curves of non-immersive points (of f_t). Furthermore, its image Milnor number is not constant, $\mu_I(f_0) > \mu_I(f_t)$ for $t \neq 0$ (see Figure 3.6).

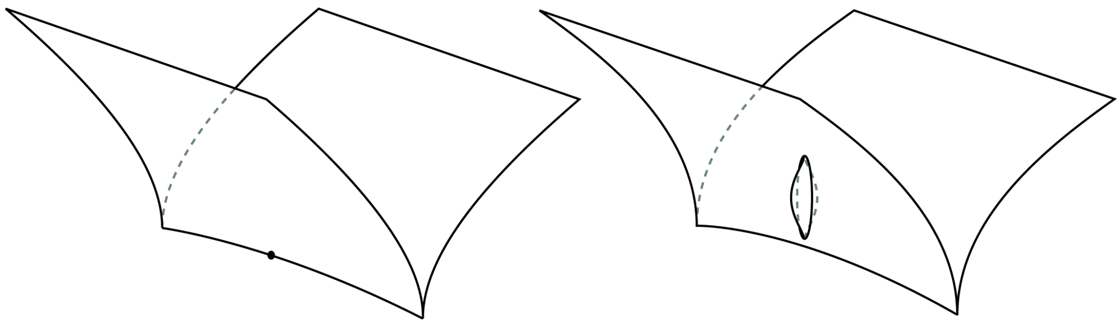


Figure 3.6: f_0 (left) and f_t with $t \neq 0$ (right) as real maps.

Theorem 3.2.3 and Proposition 3.2.4 motivate the following more general conjecture, where we consider not only the converse of Theorem 3.2.3 in higher dimensions, but also drop the corank 1 condition.

Conjecture 3.2.6. For every $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ and every origin-preserving one-parameter unfolding $F(x, t) = \big(f_t(x), t\big)$, F is excellent if, and only if, $\mu_I(f_t)$ is constant.

Chapter 4

Whitney equisingularity

> The basic question is the following: *what shall we mean by saying that the two singularities P, P' are equivalent?*
>
> Oscar Zariski, *Some open questions in the theory of singularities* [Zar71]

This chapter is the natural continuation of Chapter 3, and contains the results of [GCNB20]. Here, we use the results on excellent families to prove that a family is Whitney equisingular provided a few conditions.

In order to solve this equisingularity problem, we need to extend the theory of map germs with an ICIS in the source, in particular, we extend some results we have given in Chapter 3 for this setting. Moreover, we study a new $\mathscr{A}$-invariant that appears naturally when we study this setting.

4.1. Map germs with an ICIS in the source

In [MM94], Mond and Montaldi developed the Thom-Mather theory of singularities of mappings defined on an isolated complete intersection singularity (ICIS). They also extended Damon's results in [Dam91], which related the $\mathscr{A}_e$-versal unfolding of a map germ f with the $\mathscr{K}_{D(G)}$-versal unfoldings of an associated map germ which induces f from a stable map G. In particular, when the target has greater dimension than the source or both dimensions coincide, they proved that the discriminant Milnor number $\mu_\Delta(X, f)$ is greater than or equal to the $\mathscr{A}_e$-codimension, with equality in the weighted homogeneous case. This is a generalisation of Damon and Mond's result Theorem 1.2.27. Here, we study what happens when the dimension of the source is one less than the dimension of the target and we consider the image Milnor number $\mu_I(f)$ instead of $\mu_\Delta(f)$.

First of all, we fix a bit of notation to get rid of some details. Along this chapter, (X, S) will be a multi-germ of an ICIS and $f\colon (X, S) \to (\mathbb{C}^p, 0)$ will be a holomorphic map germ, written also as (X, f). This kind of germs will be called *germs on* ICIS as well, and we may omit the base set of the germ if it does not provide relevant information or it is clear from the context.

Definition 4.1.1 (see [MM94, p. 4])**.** We will say that $x \in X$ is a *critical point* of (X, f) if either X is smooth at x and f is not submersive at x or if x is a singular point of X. Besides, we will denote the set of critical points by $\Sigma(X, f)$, in a similar fashion as in the case of a smooth source. Furthermore, we will say that (X, f) has *finite singularity type* if the restriction of f to $\Sigma(X, f)$ is finite-to-one.

We want to develop the theory we introduced in Sections 1.2.1 and 1.2.2 for germs on ICIS. Many definitions are similar to the smooth case, but the reader should be cautious because the fact that we have to carry the ICIS structure forces us to give slightly different definitions. However, the definition of $\mathscr{A}$-equivalence in this setting is straightforward.

Definition 4.1.2. Two map germs $f, g : (X, S) \to (\mathbb{C}^p, 0)$ are *$\mathscr{A}$-equivalent* if there are germs of biholomorphisms ϕ of (X, S) and ψ of $(\mathbb{C}^p, 0)$ such that the following diagram is commutative:

$$\begin{array}{ccc} (X, S) & \xrightarrow{f} & (\mathbb{C}^p, 0) \\ \phi \big\downarrow \wr & & \wr \big\downarrow \psi \\ (X, S) & \xrightarrow{g} & (\mathbb{C}^p, 0) \end{array} .$$

As we already know from Chapter 1, the Thom-Mather theory is hugely developed and well established (see, for example, [MNB20] or [Wal81]). In particular, we know that the next important notion one should look for regarding $\mathscr{A}$-equivalence is the concept of *unfolding*, as we have seen in Definition 1.2.3.

Compared with unfoldings in the smooth case, unfoldings of germs on ICIS need to be compatible with the extra structure we are carrying with us. We cover this in the next definition, which generalizes Definitions 1.2.3 and 1.2.8.

Definition 4.1.3 (see [MM94, Definition 1])**.** Let $f\colon (X, S) \to (\mathbb{C}^p, 0)$.

(i) An *unfolding of the pair* (X, f) *over a smooth space germ* $(W, 0)$ is a map germ $F\colon (\mathcal{X}, S') \to (\mathbb{C}^p \times W, 0)$ together with a flat projection $\pi\colon (\mathcal{X}, S') \to (W, 0)$ and an isomorphism $j\colon (X, S) \to \big(\pi^{-1}(0), S'\big)$ such that the following diagram commu-

tes

$$\begin{array}{ccccc} & & (X,S) & & \\ & \overset{j}{\swarrow} & & \overset{f\times\{0\}}{\searrow} & \\ \big(\pi^{-1}(0), j(S)\big) & & & & \big(\mathbb{C}^p \times \{0\}, 0\big) \\ \big\downarrow & & & & \big\downarrow \\ \big(\mathcal{X}, j(S)\big) & & \xrightarrow{\quad F \quad} & & (\mathbb{C}^p \times W, 0) \\ & \underset{\pi}{\searrow} & & \underset{\pi_2}{\swarrow} & \\ & & (W,0) & & \end{array},$$

where $\pi_2 : \mathbb{C}^p \times W \to W$ is the Cartesian projection. In this case, W is the *parameter space of the unfolding*, and in general we use $(\mathbb{C}^d, 0)$ instead of $(W, 0)$. In short, we will use also $(\mathcal{X}, \pi, F, j)$ to denote the unfolding.

(ii) Given an unfolding $(\mathcal{X}, \pi, F, j)$ of (X, f), the map $f_t : X_t \to \mathbb{C}^p$ induced from F on $X_t := \pi^{-1}(t)$ is called the *perturbation of* (X, f) *induced by the unfolding*, and is abbreviated to the pair (X_t, f_t).

(iii) In this context, an *unfolding of* f is an unfolding of (X, f) with $\mathcal{X} = X \times \mathbb{C}^d$ and with $\pi : \mathcal{X} \to \mathbb{C}^d$ the Cartesian projection. This coincides with the usual definition for smooth spaces, Definition 1.2.3.

(iv) Two unfoldings $(\mathcal{X}, \pi, F, j)$ and $(\mathcal{X}', \pi', F', j')$ over W are *isomorphic* if there are isomorphisms $\Phi : \mathcal{X} \to \mathcal{X}'$ and $\Psi : \mathbb{C}^p \times \mathbb{C}^d \to \mathbb{C}^p \times \mathbb{C}^d$ such that Ψ is an unfolding of the identity over $\mathbb{C}^d$ and the following diagram commutes:

$$\begin{array}{ccccccc} & & \big(\mathcal{X}, j(S)\big) & & \xrightarrow{\quad F \quad} & & (\mathbb{C}^p \times \mathbb{C}^d, 0) \\ & \overset{j}{\nearrow} & & \overset{\pi}{\searrow} & & \overset{\pi_2}{\swarrow} & \\ (X,S) & & \Phi \big\downarrow \wr & & (\mathbb{C}^d, 0) & & \wr \big\downarrow \Psi \\ & \underset{j'}{\searrow} & & \underset{\pi'}{\nearrow} & & \underset{\pi_2}{\nwarrow} & \\ & & \big(\mathcal{X}', j'(S)\big) & & \xrightarrow{\quad F' \quad} & & (\mathbb{C}^p \times \mathbb{C}^d, 0) \end{array}.$$

(v) If $(\mathcal{X}, \pi, F, j)$ is an unfolding of (X, f) over $(\mathbb{C}^d, 0)$, a germ $\rho : (\mathbb{C}^r, 0) \to (\mathbb{C}^d, 0)$ induces and unfolding $(\mathcal{X}_\rho, \pi_\rho, F_\rho, j_\rho)$ of (X, f) by a *base change* or, in other words, by the fibre product of F and $\mathrm{id}_{\mathbb{C}^p} \times \rho$:

$$\begin{array}{ccc} \mathcal{X}_\rho := \mathcal{X} \times_{\mathbb{C}^p \times \mathbb{C}^d} (\mathbb{C}^p \times \mathbb{C}^s) & \xrightarrow{F_\rho} & \mathbb{C}^p \times \mathbb{C}^s \\ \big\downarrow & & \big\downarrow \mathrm{id}_{\mathbb{C}^p} \times \rho \\ \mathcal{X} & \xrightarrow{\quad F \quad} & \mathbb{C}^p \times \mathbb{C}^d \end{array},$$

where we omit the points of the germs for simplicity.

(vi) The unfolding $(\mathcal{X}, \pi, F, j)$ is *versal* if every other unfolding, e.g. $(\mathcal{X}', \pi', F', j')$, is isomorphic to an unfolding induced from the former by a base change, $(\mathcal{X}_\rho, \pi_\rho, F_\rho, j_\rho)$. A versal unfolding is called *miniversal* if it has a parameter space with minimal dimension.

As we have seen in the smooth case, an unfolding shows information about the deformations of the germ it unfolds. Hence, again mimicking the smooth case, there is a type of unfolding that does not contain relevant information: a *trivial unfolding*. We arrive to the following definition taking into account the ICIS structure and the previous definition of unfolding in Definition 4.1.3.

Definition 4.1.4. A *trivial unfolding* of a map germ f is an unfolding that is isomorphic to the constant unfolding $(X \times \mathbb{C}^d, \pi_2, f \times \mathrm{id}_{\mathbb{C}^d}, i)$,

$$\begin{array}{ccccc} & & (X, S) & & \\ & \swarrow^{i} & & \searrow^{f\times\{0\}} & \\ \big(X \times \mathbb{C}^d, S \times \{0\}\big) & & \xrightarrow{f\times \mathrm{id}_{\mathbb{C}^d}} & & (\mathbb{C}^p \times \mathbb{C}^d, 0) \\ & \searrow_{\pi_2} & & \swarrow_{\pi_2} & \\ & & (\mathbb{C}^d, 0) & & \end{array} ,$$

where π_2 is the projection on the second factor and i is the inclusion $(X, S) \hookrightarrow \big(X \times \mathbb{C}^d, S \times \{0\}\big)$.

On this regard, a map germ is *stable* if every unfolding is trivial. If the map germ is not stable, we say that it has an *instability* or that it is *unstable*, as in the smooth case.

Following the idea that an unfolding shows information about the perturbations of a germ, it is evident from the definition of stability of a map germ that a germ is stable if, and only if, it is its own miniversal unfolding. Another way of seeing this is that every deformation is $\mathscr{A}$-equivalent to the original map germ if, and only if, it is stable. As a consequence of this, we see that, if a map germ is stable, then (X, S) is smooth and f is stable in the usual sense. This is related to Theorem 4.1.7 below, where the Tjurina number of $(X, 0)$ appears.

We already know that the concept of *stabilisation* is of high interest in the study of singularities of map germs:

Definition 4.1.5 (see [MM94, Definition 2])**.** A *stabilisation of a map germ* $f : (X, S) \to (\mathbb{C}^p, 0)$ is an unfolding $(\mathcal{X}, \pi, F, j)$ such that the parameter space has dimension one and $f_s : X_s \to \mathbb{C}^p$ has only stable singularities for $s \neq 0$, where f_s is the induced map by F.

How far (X, f) is of being stable is measured by means of its $\mathscr{A}$-codimension, as in the smooth case (see Definition 1.2.16). However, there are some changes compared with the smooth case, for example the *$\mathscr{A}_e$-codimension of f* is not equal to the dimension of the parameter space of a miniversal unfolding when f is a germ on a (non-smooth) ICIS (see Theorem 1.2.20).

Definition 4.1.6. Let $(X,S) \subset (\mathbb{C}^N,S)$ be a germ of an ICIS and consider a map germ $f:(X,S) \to (\mathbb{C}^p,0)$. We define the following objects:

(i) $\theta_{\mathbb{C}^p,0}$ is the module of vector fields on $(\mathbb{C}^p,0)$,

(ii) the module of tangent vector fields defined on (X,S) is

$$\theta_{X,S} := \frac{\operatorname{Der}\big(\text{-log}(X,S)\big)}{I(X)\operatorname{Der}\big(\text{-log}(X,S)\big)},$$

where $\operatorname{Der}\big(\text{-log}(X,S)\big)$ are the vector fields on $(\mathbb{C}^N,S)$ tangent to (X,S),

(iii) $\theta(f)$ is the module of vector fields along f,

(iv) $\omega f:\theta_{\mathbb{C}^p,0} \to \theta(f)$ is the composition with f, and

(v) $tf:\theta_{X,S} \to \theta(f)$ is the composition with the differential of a smooth extension of f.

Then, the $\mathcal{O}_{\mathbb{C}^p,0}$-module

$$N\mathscr{A}_e(f) := \frac{\theta(f)}{tf(\theta_{X,S}) + \omega f(\theta_{\mathbb{C}^p,0})}$$

is the $\mathscr{A}_e$-*normal space* and its dimension as vector space is the $\mathscr{A}_e$-*codimension of* f, $\mathscr{A}_e$-codim(f). As usual, we will say that f is $\mathscr{A}$-finite if this dimension is finite.

In contrast, the $\mathscr{A}_e$-*codimension of the pair* (X,f), $\mathscr{A}_e$-codim(X,f), is the dimension of the parameter space of a miniversal unfolding of the pair (X,f), if it exists, and it is infinite otherwise. If the $\mathscr{A}$-codimension of (X,f) is finite, we say that (X,f) is $\mathscr{A}$-finite.

It is reasonable to ask for the relation between the $\mathscr{A}_e$-codimension of (X,f), the $\mathscr{A}_e$-codimension of f and the Tjurina number of X. This is addressed in [MM94, Theorem 1.4] for the case of mono-germs, i.e., when S is a point:

Theorem 4.1.7. *Let $(X,0)$ be an ICIS and $f:(X,0) \to (\mathbb{C}^p,0)$ of finite singularity type, then f is $\mathscr{A}$-finite if, and only if, (X,f) is $\mathscr{A}$-finite. Furthermore, in this case,*

$$\mathscr{A}_e\text{-codim}(X,f) = \mathscr{A}_e\text{-codim}(f) + \tau(X,0).$$

Note. With this result, we see clearly that if a map germ has smooth source and it is stable in the usual sense then it is stable (and vice versa).

This theorem allows us to prove a very useful result, the *Mather-Gaffney criterion* for germs with an ICIS in the source (recall the smooth version in Theorem 1.2.21).

Proposition 4.1.8. *A map germ $f:(X,S) \to (\mathbb{C}^p,0)$ is $\mathscr{A}$-finite if, and only if, it has isolated instability.*

Proof. The instabilities could come from points where X is not smooth, which are isolated. On the other hand, on the smooth points we have the usual Mather-Gaffney criterion (see Theorem 1.2.21), therefore, these points are isolated as well. Using Theorem 4.1.7 finishes the proof. QED

As in the smooth case, we would like to relate those algebraic $\mathscr{A}$-invariants with some invariants with a topological flavour. We study what happens when the dimension of the target is greater than the dimension of the source, especially when the difference is 1.

Of course, one expects that, in the case where the source is an ICIS, the situation is similar to the smooth case, where we have the image Milnor number (see Definition 1.2.23). However, let us begin with a hypothesis that simplifies many arguments because it gives extra structure to the objects we study, as we have seen in Theorems 2.4.4, 3.1.22 and 3.2.3 and Definition 3.1.14 among others: the corank one hypothesis.

Definition 4.1.9. We say that $f : (X, S) \to (\mathbb{C}^p, 0)$ has *corank* r if it has a smooth extension of corank r.

Indeed, Goryunov proved that the image of a stable perturbation of $f : (X, 0) \to (\mathbb{C}^p, 0)$ has non-trivial homology only in certain degrees if $n < p$ and f has corank one, see [Gor95, Theorem 3.3.1]. Furthermore, in [MM94, p. 13], Mond and Montaldi proved that, for a map germ $f : (X, 0) \to (\mathbb{C}^p, 0)$, the discriminant locus of a stable perturbation has the homotopy type of a wedge of spheres if $\dim X = n \geq p$, but the same proof works when $p = n + 1$.

For the sake of completeness, we outline a proof when $p = n + 1$ based on the same arguments we used in Section 3.1.1 for the smooth case (recall Remark 3.1.5).

Proposition 4.1.10. *Let $f\colon (X, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite, where X is an* ICIS *with $\dim(X) = n$. Suppose also that f has corank one or $(n, n+1)$ are nice dimensions in the sense of Mather. In this case, if (X_s, f_s) is a perturbation from a stabilisation of (X, f) with $s \neq 0$, the image of f_s intersected with a Milnor ball has the homotopy type of a wedge of spheres of dimension n.*

As we were saying, the techniques to prove this result are Theorem 3.1.2 and the ones that appear in Lemma 3.1.4. There are routine technical details of the proof that can be found there. In any case, to prove this, note that the case of $\mathscr{A}_e$-codimension equal to 0 is trivial. Otherwise, X_s is smooth and f_s is stable outside the origin by Theorem 4.1.7, so we can take a Milnor sphere S_ε such that the image of the stabilisation of the pair (X, f) is topologically trivial, seen as the zero-set of its defining equation G (see Lemma 3.1.4). One can conclude applying Theorem 3.1.2 to prove that the stable perturbation has the homotopy type of a wedge of spheres.

The definition of *image Milnor number* in this setting follows naturally from here.

Definition 4.1.11. For $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$ as in Proposition 4.1.10, if (X_s, f_s) is a perturbation given by a stabilisation of f, we will say that the *image Milnor number* of (X, f) is the number of spheres, in the homotopy type, of the image of (X_s, f_s) on a Milnor ball, for $s \neq 0$ (see Figure 4.1). This number will be denoted by $\mu_I(X, f)$.

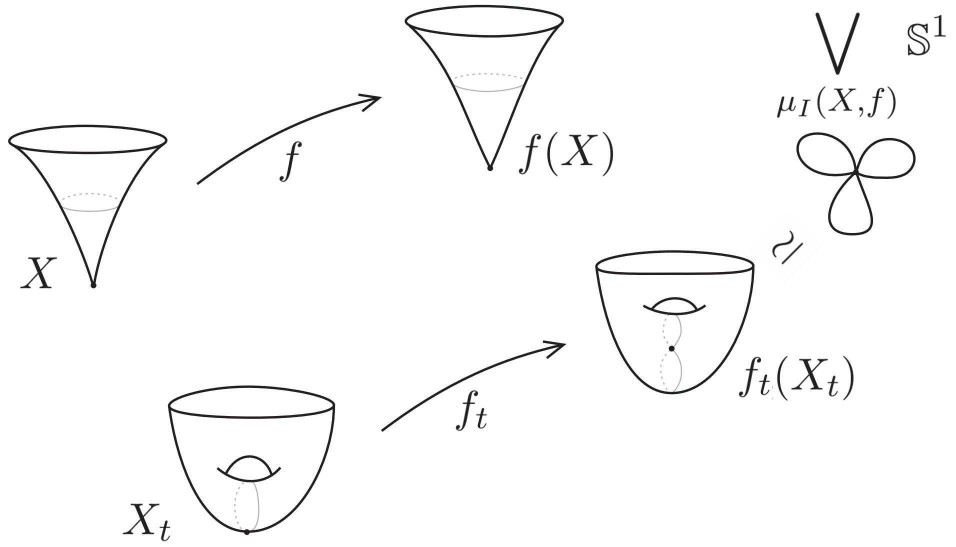


Figure 4.1: Illustration of how $\mu_I(X, f)$ works, i.e., of the homology of the image of a stable perturbation (X_t, f_t).

Actually, an equivalent definition can be given replacing (X_s, f_s) from the stabilisation for any stable (X_u, f_u) from a versal unfolding (or, in general, a stable unfolding) because any stabilisation can be found inside a versal unfolding by means of a base change, as it happened in the smooth case (see Remark 1.2.26). This may simplify some arguments or intuitions and we will use both indistinctly.

For the same reason, the definition does not depend on the stabilisation (see [MM94, p. 12]). Finally, a stabilisation always exists when f has corank one or $(n, n+1)$ are nice dimensions[1]. In fact, the *bifurcation set* $\mathcal{B}$ of a versal unfolding $(\mathcal{X}, \pi, F, j)$ over $(\mathbb{C}^d, 0)$ is the set germ in $(\mathbb{C}^d, 0)$ of parameters u such that (X_u, f_u) has some instability (as in the smooth case). It is enough to show that $\mathcal{B}$ is analytic and proper in $(\mathbb{C}^d, 0)$.

On one hand, we consider the set germ $\mathcal{C}$ in $(\mathbb{C}^p \times \mathbb{C}^d, 0)$ of pairs (y, u) such that (X_u, f_u) is unstable at y. We fix a small enough representative $F : \mathcal{X} \to Y \times U$, where Y and U are open neighbourhoods of the origin in $\mathbb{C}^p$ and $\mathbb{C}^d$, respectively. Then, $\mathcal{C}$ is the support of the relative normal module on $Y \times U$, defined as

$$N\mathscr{A}_e(F|U) := \frac{\theta(F|U)}{t_{rel}F(\theta_{\mathcal{X}|U}) + \omega_{rel}F(\theta_{Y\times U|U})},$$

where $\theta(F|U)$, $\theta_{\mathcal{X}|U}$ and $\theta_{Y\times U|U}$ are, respectively, the submodules of $\theta(F)$, θ_X and $\theta_{Y\times U}$ of relative vector fields (see [MNB20, Definition 3.9]) and $t_{rel}(F)$ and $\omega_{rel}(F)$ are the respective restrictions of tF and ωF. The fact that (X, f) has finite singularity type implies that $N\mathscr{A}_e(F|U)$ is coherent (see the proof of [MNB20, Lemma 5.3]) and, hence, $\mathcal{C}$

[1] This (routine) detail is missing in [MM94].

is analytic in $Y \times U$. Moreover, the projection $\pi_2 : \mathcal{C} \to U$ given by $\pi_2(y,u) = u$ is a finite mapping, because (X,f) has isolated instability. Therefore, $\mathcal{B} = \pi(\mathcal{C})$ is also analytic in U, by Remmert's finite mapping theorem (see, for example, [BBH+98, p. 5]).

On the other hand, we prove that $\mathcal{B}$ cannot be equal to U. Since $(\mathcal{X}, \pi, F, j)$ is a versal unfolding of (X,f), $(\mathcal{X},\pi)$ is a versal unfolding of X. Hence, there exists $u_0 \in U$ such that X_{u_0} is smooth. Now, we can apply the classical Thom-Mather theory to the mapping $f_{u_0} \colon X_{u_0} \to \mathbb{C}^p$. If either (n,p) are nice dimensions or f_{u_0} has only corank one singularities, then, for almost any u in a neighbourhood of u_0, the mapping $f_u \colon X_u \to \mathbb{C}^p$ has only stable singularities (see, for example, [MNB20, Propositions 5.5 and 5.6]).

A desirable property of this topological $\mathscr{A}$-invariant is that it is conservative, as it was for the usual image Milnor number (see Theorem 3.1.7). The reasoning that proves the conservation of the usual image Milnor number can be applied verbatim for the general version, and is based as well on Theorem 3.1.2. Here, we give a sketch of the proof.

Theorem 4.1.12. *Let $f : (X,S) \to (\mathbb{C}^{n+1}, 0)$ be as in Proposition 4.1.10, and (X_{u_0}, f_{u_0}) a perturbation in a one-dimensional unfolding of (X,f). Take a representative of the unfolding such that its codomain is a Milnor ball B_ε. Then,*

$$\mu_I(X,f) = \beta_n\big(f_{u_0}(X_{u_0})\big) + \sum_{y \in B_\varepsilon} \mu_I(X_{u_0}, f_{u_0}; y),$$

where β_n is the nth Betti number, if u_0 is small enough.

Sketch of the proof. If (X_{u_0}, f_{u_0}) is stable the result is trivial.

Assume that (X_{u_0}, f_{u_0}) is not stable. Then, take a versal unfolding of (X,f) such that it unfolds the original one-dimensional unfolding and $(X_{u_0,v}, f_{u_0,v})$ is stable for $v \neq 0$ small enough. Consider the defining equations, G, of $f_{u,v}(X_{u,v})$. Now, as f is stable outside the origin and X is smooth outside the points of S, we can take a Milnor radius ε such that the family of equations G is topologically trivial over S_ε. Now, we are in the conditions of applying Theorem 3.1.2 and follow the reasoning of Theorem 3.1.7, but working on $X_{u_0,v}$ and the corresponding instabilities of (X_{u_0}, f_{u_0}). QED

In particular, this implies the upper semi-continuity of the image Milnor number (see also Figure 4.2).

Corollary 4.1.13. *Using the notation and hypotheses of Theorem 4.1.12, $\mu_I(X,f)$ is upper semi-continuous, i.e.,*

$$\mu_I(X,f) \geq \mu_I(X_{u_0}, f_{u_0}; y).$$

4.2. Multiple points and the ICSS

One may ask what happens when we have a map germ $f : (X,S) \to (\mathbb{C}^p, 0)$ with X ICIS but $\dim X = n < p$, in general. Houston studied this for the case of smooth source in [Hou10] using the multiple point spaces of the map germ and an Image-Computing

Spectral Sequence (ICSS). We take a similar path, therefore, we need the machinery of the multiple point spaces we have seen in Chapter 2 for the smooth case.

To simplify notation, (X_t, f_t) will be a stable perturbation of (X, f). We also recall the notion of multiple point space of a locally stable mapping (see Definition 2.4.2).

Definition 4.2.1 (see Definitions 2.4.2 and 3.1.14)**.** The *kth-multiple point space*, $D^k(f)$, of a mapping or a map germ f is defined as follows:

(i) Let $f\colon X \to Y$ be a locally stable mapping between complex manifolds. Then, $D^k(f)$ is equal to the closure of the set of points $(x_1, \dots, x_k)$ in X^k such that $f(x_i) = f(x_j)$ but $x_i \neq x_j$, for all $i \neq j$.

(ii) Given $f : (X, S) \to (\mathbb{C}^p, 0)$ with finite singularity type, if $(\mathcal{X}, \pi, F, j)$ is a stable unfolding, then
$$D^k(f) = J_k^{-1}\big(D^k(F)\big),$$
where $J_k = \overbrace{j \times \cdots \times j}^{k}$.

(iii) For a map germ $f\colon (X, S) \to (\mathbb{C}^p, 0)$, we will denote as $d(f)$ the maximal multiplicity of the stable perturbation of f (i.e., $d(f) := \max\{k : D^k(f_t) \neq \varnothing\}$) and $s(f)$ the number of points of the set S.

These definitions behave properly under isomorphisms and base change of unfoldings, i.e., Items *(ii)* and *(iii)* do not depend on the stable unfolding (this is proved in [NBPS17, Lemma 2.3], which is stated for the smooth case but the proof works for our case).

Remark 4.2.2. As we know from Section 2.4, the multiple point spaces have some useful properties. In the smooth case, they were exceptionally good because they gave the Marar-Mond criterion (see Theorem 2.4.4). We prove that the criterion is still true for germs on ICIS of corank one in Lemma 4.2.3.

In contrast, in any corank, we know a few facts about them. For example, the proof of [Hou97, Theorem 4.3 and Corollary 4.4] still works with an $\mathscr{A}$-finite map germ f : $(X, S) \to (\mathbb{C}^p, 0)$, $\dim X = n$, and we can deduce that the dimension of $D^k(f)$ is $p - k(p-n)$ (if this number is not negative nor $D^k(f)$ is empty). Furthermore, in the pair of dimensions (n, p), we deduce from [Hou97, Theorem 4.3] that $d(f)$ is at most the integer part of $\frac{p}{p-n}$. Finally, taking into account these previous remarks, this maximum is attained when $s(f) \geq d(f)$ because the proof of Lemma 3.1.15 only relies on the dimension of the multiple point spaces.

This lemma generalizes Marar-Mond criterion for the context of multi-germs and an ICIS in the source (see Theorem 2.4.4).

Lemma 4.2.3. *For* $f : (X, S) \to (\mathbb{C}^p, 0)$ *of corank* 1 *and finite singularity type,* $\dim X = n < p$*:*

(i) *(X, f) is stable if, and only if, $D^k(f)$ is smooth of dimension $p - k(p-n)$, or empty, for $k \geq 1$.*

(ii) If $\mathscr{A}_e$-codim(X, f) is finite, for each k with $p - k(p - n) \geq 0$, $D^k(f)$ is empty or an ICIS *of dimension $p - k(p - n)$. Furthermore, for those k such that $p - k(p - n) < 0$, $D^k(f)$ is a a subset of S^k, possibly empty.*

Proof. For the first statement, if (X, f) is stable, X is smooth and it follows from Theorem 2.4.4. For the converse, if every $D^k(f)$ is smooth, then, in particular, so is $D^1(f) = X$ and, again, the result follows from Theorem 2.4.4.

For the second statement, note that for the case of X being smooth the statement is contained in Theorem 2.4.4 as well. Fortunately, if we take a versal unfolding $(\mathcal{X}, \pi, F, j)$ of (X, f) we can apply that result. Also, for k with $p - k(p - n) \geq 0$, observe that the codimension of $D^k(F)$ coincides with the one of $D^k(f)$ because outside the isolated singularities they are smooth and of the same codimension. Furthermore, the only singularities that can appear in $D^k(f)$ are at the problematic points (the ones that come from S) by the Mather-Gaffney criterion (see Proposition 4.1.8). It only remains to check that $D^k(f)$ is a complete intersection, and this is the case as it can be constructed as a pull back of a complete intersection, $D^k(F)$, and both have the same codimension.

The other case is trivial. QED

Note that this is the best we can aim for: we need to study $D^1(f) = X$. The following example illustrates this.

Example 4.2.4. Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of an ICIS of dimension n. Then, the inclusion $i : (X, 0) \to (\mathbb{C}^N, 0)$ is stable in the sense that the $\mathscr{A}_e$-codimension of i is zero, since ωi is surjective. However, the $\mathscr{A}_e$-codimension of (X, i) is equal to the Tjurina number of $(X, 0)$. Therefore, after taking a stable perturbation of (X, i), say (X_t, i_t), every $D^k(i_t)$ is empty for $k \geq 2$ because we can set $i_t = i$ and X_t is smooth, and this implies that $D^k(i)$ is empty for every $k \geq 2$ as well.

Remark 4.2.5. The multiple point spaces in corank one are specially friendly, as we have seen. Indeed, if we consider $(X, S) \subseteq (\mathbb{C}^N, S)$ and a map germ $f : (X, S) \to (\mathbb{C}^p, 0)$ of corank 1 and finite singularity type, $\dim X = n < p$, we can simplify even more the structure of the different $D^k(f)$. This is because $D^k(f)$ is a subset of X^k, therefore a subset of $\mathbb{C}^{Nk}$ and, if f is of corank one, we can assume that f has the form

$$\begin{aligned} f : (X, S) &\to (\mathbb{C}^p, 0) \\ (x_1, \dots, x_N) &\mapsto \big(x_1, \dots, x_{N-1}, h_1(x), h_2(x)\big). \end{aligned}$$

Therefore, $D^k(f)$ has many duplicated entries at each point, omitting these duplicates we can see $D^k(f)$ as a subset of $X \times \mathbb{C}^{k-1}$. Finally, observe that this identification preserves the ICIS structure.

Furthermore, there is a natural action of Σ_k in $D^k(f)$ by permutation of entries of the k-tuples of points, as in the smooth case.

Also, recall that all the elements in Σ_k can be decomposed into disjoint cycles in a unique way, called the *cycle shape*, and this inspires a refinement of the kth-multiple point space based on the relations an element $\sigma \in \Sigma_k$ gives. To be precise, if we take a partition of k, $\gamma(k) = (r_1, \dots, r_m)$, and $\alpha_i = \#\{j : r_j = i\}$, where $\#A$ denotes the number of points in A, one can find an element $\sigma \in \Sigma_k$ such that it can be decomposed into α_i pair-wise disjoint cycles of length i.

Notation 4.2.6. In this case, the partition $\gamma(k)$ is called the *cycle type* of σ (see [Sag01, pp. 2–3]).

Example 4.2.7. The cycle type of $\sigma \in \Sigma_{10}$, such that

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 6 & 1 & 5 & 4 & 3 & 7 & 9 & 8 & 10 \end{pmatrix},$$

is the partition $(4, 2, 2, 1, 1)$, because $\sigma = (1\ 2\ 6\ 3)(4\ 5)(8\ 9)(7)(10)$ and $\alpha_4 = 1, \alpha_3 = 0, \alpha_2 = 2$ and $\alpha_1 = 1$.

Definition 4.2.8. We define $D^k\big(f, \gamma(k)\big)$ as the subspace of $D^k(f)$ given by the equations of the fixed points of σ with the usual action, for σ of cycle type $\gamma(k)$. We may also use $D^k(f)^\sigma$ instead of $D^k\big(f, \gamma(k)\big)$ to specify the element.

Remark 4.2.9. By symmetry, $D^k(f)^\sigma$ is isomorphic to $D^k(f)^{\sigma'}$ if σ and σ' have the same cycle type (recall that the cycle types determine the conjugacy classes). Hence, the definition of $D^k\big(f, \gamma(k)\big)$ works modulo isomorphism. We will omit this detail in general (see also Lemma 4.2.10).

If we take into account the group action and the subspaces $D^k(f)^\sigma$, we have a refinement of Lemma 4.2.3. It is a generalization of [MM89, Corollary 2.15] for the context of multi-germs and an ICIS in the source and [Hou10, Corollary 2.8] for the context of ICIS in the source.

Lemma 4.2.10. *With the hypotheses of Lemma 4.2.3 and $\gamma(k)$ a partition of k, we have the following.*

(i) If f is stable, $D^k\big(f, \gamma(k)\big)$ is smooth of dimension $p - k(p-n) - k + \sum_i \alpha_i$, or empty.

(ii) If $\mathscr{A}_e - codim(X, f)$ is finite, then:

- *(a) for each k with $p - k(p-n) - k + \sum_i \alpha_i \geq 0$, $D^k\big(f, \gamma(k)\big)$ is empty or an ICIS of dimension $p - k(p-n) - k + \sum_i \alpha_i$,*
- *(b) for each k with $p - k(p-n) - k + \sum_i \alpha_i < 0$, $D^k\big(f, \gamma(k)\big)$ is subset of S^k, possibly empty.*

A proof of this lemma can be seen in [Hou10, Corollary 2.8], because the same proof applies once we know Lemma 4.2.3. In any case, observe that the Item *(i)* is a consequence of Lemma 4.2.3 and the fact that we are adding $k - \sum_i \alpha_i$ equations to the ones of $D^k(f)$ to form $D^k\big(f, \gamma(k)\big)$, as each r_i of $\gamma(k)$ gives $r_i - 1$ more equations. Item *(ii)* follows the same idea of Item *(i)*.

From the homology of the Milnor fiber of the multiple point spaces, a special part will serve our purposes: its alternating part. We have seen in Chapter 2 that the alternating part of the homology of the spaces $D^k(f)$ gives a lot of information of $\operatorname{im}(f)$ but, as we will take things further, we make a general definition.

Definition 4.2.11. Given a *sing* homomorphism, $\operatorname{sgn} : G \to \{\pm 1\}$ where $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$, and a linear action of a finite group G on some $\mathbb{C}$-vectorial space H, we say that the *G-alternating part* of H is the set

$$\{h \in H : gh = \operatorname{sgn}(g)h, \text{ for all } g \in G\},$$

and we denote it by H^{Alt_G}. If the group is Σ_k, then the sign homomorphism is the usual sign of a permutation and we simply write H^{Alt_k} or H^{Alt} if the group is clear from the context.

Remark 4.2.12. Recall that, in terms of representation theory, H^{Alt_G} is the isotype of the sign representation of the representation H (see Appendix A, in particular Definition A.1.9 and Example A.1.4). Furthermore, the *sign* homomorphism can be defined as the usual signature, or sign, for permutations for every finite group, seen as a subgroup of a Σ_N by Cayley's theorem (see [Rob96, Proposition 1.6.8]).

With the study of the multiple point spaces, we are able to obtain a lot of information of images of stable perturbations in any pair of dimensions, as long as $n < p$. This is done by means of an ICSS, as the one we have shown in Theorem 2.3.1. For example, in [Hou10], Houston uses an ICSS as we are going to use it now. It will appear in later techniques as well.

The first application of Theorem 2.3.1 we will show here is a result that follows the idea of [Hou10, Theorem 3.1] and generalizes it when the source is an ICIS and we consider integer homology. Moreover, it is a generalization of [Gor95, Theorem 3.3.1], for multi-germs.

Theorem 4.2.13. *Consider a map germ $f : (X, S) \to (\mathbb{C}^p, 0)$ of finite $\mathscr{A}_e$-codimension and of corank 1, with X ICIS of dimension $\dim X = n < p$. Then, the reduced integer homology of the image of a stable perturbation of (X, f) is zero except possibly in dimensions*

(i) $p - k(p - n) + k - 1$ *for all* $2 \leq k \leq d(f)$,

(ii) $d(f) - 1$ *if* $s(f) > d(f)$, *and*

(iii) n *if* X *is non-smooth.*

Proof. Apply Theorem 2.3.1 to a versal unfolding of (X, f), say $(\mathcal{X}, \pi, F, j)$, and its restriction to $\pi^{-1}(t) = X_t$ that gives a stable perturbation, (X_t, f_t). Hence, we have the spectral sequence

$$E_1^{r,q}(F, f) \coloneqq H_q^{\mathrm{Alt}_{r+1}}\big(D^{r+1}(F), D^{r+1}(f_t); \mathbb{Z}\big) \Longrightarrow H_*\big(F(\mathcal{X}), f_t(X_t); \mathbb{Z}\big).$$

This spectral sequence, of homology type, collapses at the second page instead of the first one (see Table 4.1), because all the differentials are trivial except the ones at the bottom row which may be non-zero (e.g., when X is smooth there are some cases where these entries are non-zero, as the quadruple point of a map germ from $\mathbb{C}^2$ to $\mathbb{C}^3$). From this, we can recover the limit of the spectral sequence and deduce the result.

First of all, take into account that $F(\mathcal{X})$ is contractible. Furthermore, by Lemma 4.2.3, the reduced homology of $D^k(f_t)$ could be non-trivial only in middle dimension, therefore, the groups

$$H_i\big(F(\mathcal{X}), f_t(X_t); \mathbb{Z}\big)$$

are possibly non-trivial when

$$\begin{aligned} i &= r + \dim D^{r+1} + 1 \\ &= r + p - (r+1)(p-n) + 1 \\ &= p - (p-n)(r+1) + (r+1), \end{aligned}$$

for $2 \leq r+1 \leq d(f)$. This comes from the convergence of the spectral sequence (see Definition B.2.5, the end of Appendix B.2 and Examples 2.3.2 and 2.3.3).

The last possibly non-trivial entry after collapsing the sequence is $E_2^{d(f)+1,0}$. This comes from the fact that the bottom row of the first page is an exact sequence. This, in turn, comes from applying Theorem 2.3.1 for F and its restriction to $\varnothing$, deducing that the non-trivial part of the bottom row has to be exact because $F(\mathcal{X})$ is contractible (one can also apply the proof and statement of [Hou10, Lemma 3.3] verbatim for this case).

Finally, when $r = 0$, we have some homology apart from the 0-dimensional, because $D^1(f_t) = X_t$ and it is the stable perturbation of the ICIS X. The homology in this case appears when $i = 0 + \dim(X) + 1$ and it is equal to $\mu(X)$.

Using the exact sequence of the homology of the pair, the result follows. QED

The argument of Theorem 4.2.13 would work for any corank if we were able to prove that the alternating homology of the pairs $\big(D^k(F), D^k(f)\big)$ of any corank appear in the same dimensions as in corank one. Unfortunately, the techniques used for the smooth case do not give what we expect.

Lemma 4.2.14 (see [Hou97, Theorem 4.6]). *Let $f : (X, S) \to (\mathbb{C}^p, 0)$ be an $\mathscr{A}$-finite map germ, where (X, S) is a germ of an* ICIS *of dimension $n < p$ and codimension r. Consider a non-empty $D^k(f_t)$ and write d for $\dim_{\mathbb{C}} D^k(f_t)$. Then,*

(i) for $k \geq 2$, $H_q^{\mathrm{Alt}}\big(D^k(f_t)\big) = 0$ if $q \neq 0$ or $q \notin [d + (1-r)k, d]$, and

(ii) $H_q^{\mathrm{Alt}}\big(D^1(f_t)\big)$ is zero for $q \neq 0, d$.

$r \backslash q$	0	1	2	3	4	5	6	7	8	$\cdots$
7										
6	•									
5		•								
4			•							
3				•						
2					•					
1						•				
0							•	•	•	$\cdots$

	D^1	D^2	D^3	D^4	D^5	D^6	D^7	D^8	D^9	$\cdots$
H_7^{Alt}										
H_6^{Alt}										
H_5^{Alt}	X_t									
H_4^{Alt}		f_t								
H_3^{Alt}			f_t							
H_2^{Alt}				f_t						
H_1^{Alt}					f_t	(F, f_t)				
H_0^{Alt}							F	F	F	$\cdots$

Table 4.1: First page of the spectral sequence $E_1^{r,q} = H_q^{\text{Alt}_{r+1}}\left(D^{r+1}\left(F\right), D^{r+1}\left(f_t\right)\right)$ for a map germ $f : (X, S) \to (\mathbb{C}^6, 0)$ in the pair of dimensions $(5, 6)$ (left) and the schematic information required *after some identifications* (right).

Proof. Observe that $D^1(f_t) = X_t$ is the Milnor fibre of the ICIS $D^1(f) = X$, so we can assume that $k \geq 2$.

For $q \geq d$, the space $D^k(f_t)$ has the homotopy type of a CW-complex of dimension d, therefore, there are no alternating chains above $\dim_{\mathbb{C}} D^k(f_t)$. This proves the result for those q. It only remains to check when $q < \dim_{\mathbb{C}} D^k(f_t)$ for $k \geq 2$.

To prove the remaining part of the lemma, we take the proof of [Hou97, Theorem 4.6] as reference. This argument consists of two steps: controlling the alternating homology of the pair $\big(D^k(F), D^k(f_t)\big)$ by virtue of [Hou97, Theorem 3.30], where F is a one-parameter unfolding of f, and specify exactly when this pair can have alternating homology using [Hou97, Theorem 3.13].

The hypothesis of [Hou97, Theorem 3.13 and Theorem 3.30] are not too restrictive, so we can take a one-parameter unfolding F of f and combine these theorems to prove that

$$H_q^{\text{Alt}}\big(D^k(F), D^k(f_t)\big) = 0$$

for

$$\begin{aligned} q &\leq \min\big\{(n+1-r+1)k - (p+1)(k-1) - 1, nk - p(k-1)\big\} \\ &= nk - p(k-1) + (1-r)k = d + (1-r)k. \end{aligned}$$

Therefore, using the exact sequence of the pair and the fact that $D^k(F)$ contracts to isolated points in an equivariant way, we have that $H_0^{\text{Alt}}\big(D^k(f_t)\big) \cong H_0^{\text{Alt}}\big(D^k(F)\big)$ and $H_{q-1}^{\text{Alt}}\big(D^k(f_t)\big) \cong H_q^{\text{Alt}}\big(D^k(F), D^k(f_t)\big)$, for $q > 1$. QED

Remark 4.2.15. Actually, this lemma should be stated for the homology of alternating chains, AH, instead of alternating homology (see Section 2.2). However, if we use rational homology they coincide (see the end of Section 2.2). If one wants to use integer homology the same proof works changing H^{Alt} for AH and adding in Item *(ii)* that the homology is free if $q = 0, d$. Nevertheless, Corollary 4.2.16 is well stated as it is, precisely using AH instead of H^{Alt}.

Corollary 4.2.16. *With the hypotheses of Lemma 4.2.14, the reduced integer homology of the image of a stable perturbation of (X, f) is zero except possibly in dimensions*

(i) $p - k(p-n) + k - 1 + s$ *for all* $0 \leq s \leq (1-r)k$ *and* $2 \leq k \leq d(f)$*,*

(ii) $d(f) - 1$ *if* $s(f) > d(f)$*, and*

(iii) n *if* X *is non-smooth.*

Proof. The proof follows from Lemma 4.2.14, Theorem 2.3.1 and a careful inspection of the ICSS as in Theorem 4.2.13. QED

Observe that, if X is a hypersurface, this theorem proves that the homology of the image appears in the same dimensions than the smooth case. Also, note that this theorem may not be sharp, because [Hou97, Theorem 3.13] only gives a bound to control the homology of the alternating chains of the pair $\left(D^k(F), D^k(f_t)\right)$ and it could be a bad bound in general.

It is surprising that, in the case of corank one, the same proof does not prove Theorem 4.2.13. This makes us think that there is an argument that avoids the detail of the codimension of the ICIS:

Conjecture 4.2.17. Consider a map germ $f : (X, S) \to (\mathbb{C}^p, 0)$ of finite $\mathscr{A}_e$-codimension, with X ICIS of dimension $\dim X = n < p$. Then, the reduced integer homology of the image of a stable perturbation of (X, f) is zero except possibly in dimensions

(i) $p - k(p-n)k + k - 1$ for all $2 \leq k \leq d(f)$,

(ii) $d(f) - 1$ if $s(f) > d(f)$, and

(iii) n if X is non-smooth.

Remark 4.2.18. This conjecture and Theorems 4.2.13 and 4.2.16 are related to [LPSZ21, Theorems 2.3 and 2.8], when the source is smooth and the map germ is not necessarily $\mathscr{A}$-finite but the dimensions of the multiple point spaces are controlled. They, and Section 4.4, are also closely related with [LPSZ21, Theorem 2.4] in the particular case that X is the double point space $D^2(f)$.

Houston also uses Theorem 2.3.1 in [Hou10] with a versal unfolding F, of a multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, and a section that gives the stable perturbation, f_t. Also, taking into account that the Euler-Poincaré characteristic of every page remains invariant, see Appendix B.3 and Proposition B.3.3, and that the image of the versal unfolding is contractible, it remains to compute $\chi\left(E_1^{*,*}\right)$ to get $\mu_I(f)$, and the terms of the sum are arranged to define Houston's alternating Milnor numbers, $\mu_k^{\mathrm{Alt}}(f)$ (actually, he computes them through the limit of the spectral sequence, both ways give the same result).

These ideas and the previous proofs gave Definition 3.1.14, and it inspires us to give a definition for germs on ICIS (see also the simplification and developments of it in Section 3.1.2, particularly Lemma 3.1.15).

Definition 4.2.19. Given an $\mathscr{A}$-finite map germ $f:(X,S)\to(\mathbb{C}^{n+1},0)$ of corank one, with X ICIS of dimension n, the *k-th alternating Milnor number of* (X,f), denoted as $\mu_k^{\mathrm{Alt}}(X,f)$, is defined as

$$\mu_k^{\mathrm{Alt}}(X,f):=\begin{cases}\operatorname{rank} H_{n-k+2}^{\mathrm{Alt}_k}\big(D^k(F),D^k(f_t);\mathbb{Z}\big), & \text{if } 1\leq k\leq d(f)\\ \binom{s(f)-1}{d(f)}, & \text{if } k=d(f)+1 \text{ and } s(f)>d(f)\\ 0, & \text{otherwise,}\end{cases}$$

being F its versal unfolding and f_t a stable perturbation.

These numbers are very useful because they decompose the image Milnor number, exactly as in Proposition 3.1.16.

Proposition 4.2.20. *For $f:(X,S)\to(\mathbb{C}^{n+1},0)$ $\mathscr{A}$-finite of corank one and X an ICIS of dimension n,*

$$\sum_k \mu_k^{\mathrm{Alt}}(X,f)=\mu_I(X,f)\,.$$

Proof. From the proof of Theorem 4.2.13, we only have to check that $\mu_{d(f)+1}^{\mathrm{Alt}}(X,f)$ coincides with the (rank of the) remaining non-zero entries of the spectral sequence after collapsing, i.e., we have to check that

$$\operatorname{rank} E_2^{d(f)+1,0}=\binom{s(f)-1}{d(f)}.$$

From [Hou10, Lemma 3.3], which can be stated for general stable map germs with a verbatim proof, or the constancy of the Euler-Poincaré characteristic of the spectral sequence (see Proposition B.3.3), we have

$$\operatorname{rank} E_2^{d(f)+1,0}=\left|\sum_{\ell=d(f)+1}^{s(f)}(-1)^\ell\binom{s(f)}{\ell}\right|=\binom{s(f)-1}{d(f)}.\qquad \text{QED}$$

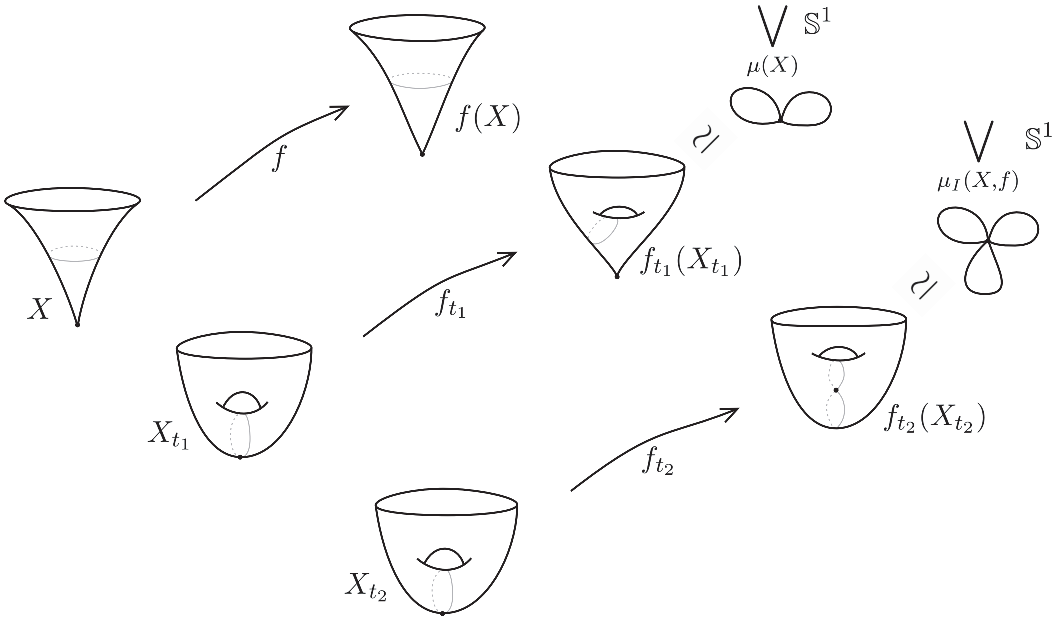


Figure 4.2: The first alternating Milnor number and its relationship with the deformations of (X, f) and the $\mu_I(X, f)$. Here, one can also appreciate the conclusions of Theorem 3.1.7 and Corollary 4.1.13.

One term deserves a bit of attention: $\mu_1^{\text{Alt}}(X, f)$. This term does not appear in the smooth case because it is zero but, in general, it is equal to $\mu(X)$ (see Figure 4.2). Proposition 4.2.20 allows us to reduce the weak Mond's conjecture for ICIS to the smooth case given in Theorem 3.1.22.

Corollary 4.2.21. *For $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$ $\mathscr{A}$-finite of corank one and X an* ICIS *of dimension n, $\mu_I(X, f) = 0$ if, and only if, (X, f) is stable.*

Proof. One direction is trivial.

If $\mu_I(X, f) = 0$ then $\mu_1^{\text{Alt}}(X, f) = \mu(X) = 0$ and we are in the case of smooth domain, Theorem 3.1.22. QED

Remark 4.2.22. Note that the weak form of Mond's conjecture for the smooth case in any corank implies, with the same proof of Corollary 4.2.21, the same conjecture for ICIS in any corank by means of Proposition 4.2.20, which can be stated for any corank using Lemma 4.2.14 and always carries a term equal to $\mu(X)$.

4.3. A Lê-Greuel type formula

Now that we have a basic building of the image Milnor number with ICIS in the source, our last preparatory step is to prove a *Lê-Greuel type formula* for $\mu_I(X, f)$. In [NBPT19], Pallarés-Torres and Nuño-Ballesteros proved a *Lê-Greuel type formula* in the setting of the image Milnor number in the smooth case and finitely determined map germs. Recall the original *Lê-Greuel formula*, see [Gre75, Trá74],

$$\mu(X,0)+\mu(X\cap H,0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{(g)+J(g,p)}.$$

Here, $(X,0)$ is an ICIS with defining equation g, p a generic linear projection with $H := p^{-1}(0)$ such that $(X\cap H,0)$ is an ICIS as well, and $J(g,p)$ is the ideal generated by the minors of maximum order of the Jacobian matrix of (g,p). Taking into account that the right hand side of the equation could be seen as the number of critical points of p restricted to the Milnor fiber of X, it is obvious that the main theorem of [NBPT19] is a similar result for the context of map germs (see Figure 4.3):

Theorem 4.3.1 (see [NBPT19, Theorem 3.2])**.** *Let $f : (\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ be a corank 1 and $\mathscr{A}$-finite map germ with $n>1$. Let $p:\mathbb{C}^{n+1}\to\mathbb{C}$ be a generic linear projection which defines a transverse slice $g:(\mathbb{C}^{n-1},0)\to(\mathbb{C}^n,0)$. Then,*

$$\#\Sigma\left(p|_{Z_s}\right)=\mu_I(f)+\mu_I(g),$$

where $\#\Sigma\left(p|_{Z_s}\right)$ is the number of critical points on all the strata of $Z_s := \operatorname{im}(f_s)$, being f_s a stable perturbation of f.

See [MNnB14, pp. 1380–1381] for the definition of *transverse slice*.

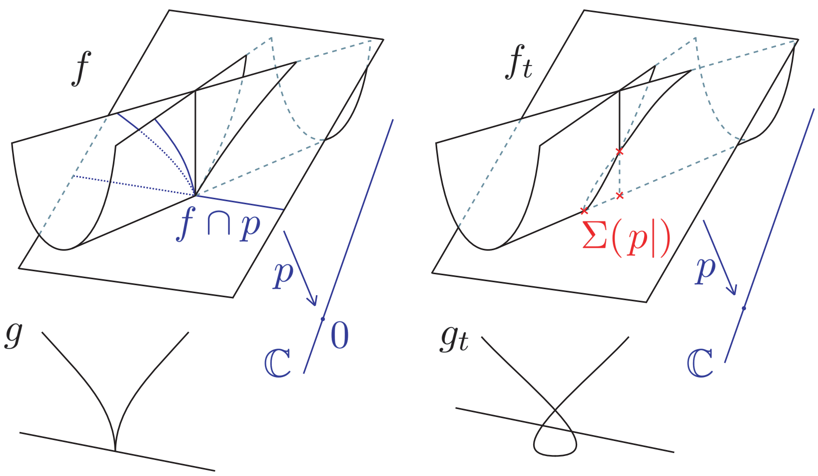


Figure 4.3: Depiction of the Lê-Greuel type formula for map germs.

The stratification considered in the image of the stable perturbation f_s in the theorem above is the stratification by stable types (see Definition 1.3.10).

We prove a similar result for multi-germs on ICIS. The first step in this direction is finding a version of *Marar's formula* for this setting. Fortunately, his proof is essentially combinatorial and one can prove the version we need with almost no modifications.

Given a partition $\gamma(k) = (r_1, \dots, r_m)$ of k and $\alpha_i := \#\{j : r_j = i\}$, *Marar's formula* is the following.

Theorem 4.3.2 (see [Mar91, Theorem 3.1]). *Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of corank 1 and $\mathscr{A}$-finite, $2 \le n < p$, and consider its stable perturbation $f_t : U_t \to \mathbb{C}^p$. Then,*

$$\chi\big(f_t(U_t)\big) = a_0 \chi(U_t) + \sum_{k\ge 2}\sum_{\gamma(k)} a_{\gamma(k)} \chi\left(D^k\big(f_t, \gamma(k)\big)\right),$$

where $a_0 = 1$ and

$$a_{\gamma(k)} = \frac{(-1)^{\sum \alpha_i + 1}}{\prod_{i\ge 1} i^{\alpha_i} \alpha_i!}$$

if $D^k\big(f_t, \gamma(k)\big)$ is non-empty, and zero otherwise.

Now, by Theorem 4.3.2 but stated for a stable perturbation of an $\mathscr{A}$-finite corank 1 map germ $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$, with $\dim X = n$, we have

$$\begin{aligned} 1 + (-1)^n \mu_I(X, f) &= \#S + (-1)^n \mu(X) \\ &\quad + \sum_{k\ge 2}\sum_{\gamma(k)} a_{\gamma(k)} \left(\beta_0^{\gamma(k)} + (-1)^{\dim D^k(f,\gamma(k))} \mu\left(D^k\big(f, \gamma(k)\big)\right)\right), \end{aligned} \tag{4.1}$$

where $\#S$ is the number of points in S and $\beta_0^{\gamma(k)}$ the zero Betti number of $D^k\big(f, \gamma(k)\big)$.

With Theorem 4.3.1 in mind, we take a generic linear projection $p : \mathbb{C}^{n+1} \to \mathbb{C}$, with kernel H, which defines a transverse slice $g := f| : \big(X \cap f^{-1}(H), S\big) \to (\mathbb{C}^n, 0)$.

Observe that $X \cap f^{-1}(H)$, which we will call $\tilde{X}$ to simplify notation, is still an ICIS. Following the steps of [NBPT19] from here, behold that, if $\dim D^k\big(f, \gamma(k)\big) > 0$, then $\dim D^k\big(f, \gamma(k)\big) - 1 = \dim D^k\big(g, \gamma(k)\big)$ and, if $\dim D^k\big(f, \gamma(k)\big) = 0$, then $D^k\big(g, \gamma(k)\big) = \varnothing$. Therefore, if we apply the previous formula to $(\tilde{X}, g)$, we get

$$\begin{aligned} 1 + (-1)^{n-1} \mu_I(\tilde{X}, g) &= \#S + (-1)^{n-1} \mu(\tilde{X}) \\ &\quad + \sum_{\substack{k\ge 2,\, \gamma(k): \\ \dim D^k(f,\gamma(k))>0}} a_{\gamma(k)} \left(\beta_0^{\gamma(k)} + (-1)^{\dim D^k(f,\gamma(k))-1} \mu\left(D^k\big(g, \gamma(k)\big)\right)\right), \end{aligned} \tag{4.2}$$

If we subtract Equation (4.2) from Equation (4.1), and use that $\dim D^k\big(f, \gamma(k)\big) = n + 1 - k - k + \sum_i \alpha_i$ (see Lemma 4.2.10), we have

$$\begin{aligned} \mu_I(X, f) + \mu_I(g, \tilde{X}) &= \mu(X) + \mu(\tilde{X}) \\ &\quad + \sum_{\substack{k\ge 2,\, \gamma(k): \\ \dim D^k(f,\gamma(k))=0}} \frac{(-1)^{\sum \alpha_i + 1 + n}}{\prod_{i\ge 1} i^{\alpha_i} \alpha_i!} \left(\beta_0^{\gamma(k)} + \mu\left(D^k\big(f, \gamma(k)\big)\right)\right) \\ &\quad + \sum_{\substack{k\ge 2,\, \gamma(k): \\ \dim D^k(f,\gamma(k))>0}} \frac{1}{\prod_{i\ge 1} i^{\alpha_i} \alpha_i!} \left(\mu\left(D^k\big(f, \gamma(k)\big)\right) + \mu\left(D^k\big(g, \gamma(k)\big)\right)\right), \end{aligned}$$

where we have simplified the signs expanding $a_{\gamma(k)}$, and $\beta_0^{\gamma(k)}$ denotes the same as before.

Once we arrive here, we can keep simplifying signs: if $\dim D^k(f,\gamma(k)) = \sum \alpha_i + 1 + n - 2k = 0$, then the first sign is positive.

On the other hand, we can choose a generic projection and coordinates on source and target so that $p(y_1,\ldots,y_{n+1}) = y_1$. Moreover,

$$D^k\big(g,\gamma(k)\big) = D^k\big(f,\gamma(k)\big) \cap \tilde{p}^{-1}(0),$$

where $\tilde{p} : X \times \mathbb{C}^{k-1} \to \mathbb{C}$ is the projection on the first coordinate for every k, seeing $D^k(f)$ as a subset of $X \times \mathbb{C}^{k-1}$ (recall Remark 4.2.5), and it is generic as well (in general it would be a mapping induced by $p \circ f$).

By the comments above, the structure of ICIS given in Lemma 4.2.10 and the Lê-Greuel-type formula for ICIS; we have

$$\mu\left(D^k\big(f,\gamma(k)\big)\right) + \mu\left(D^k\big(g,\gamma(k)\big)\right) = \#\Sigma\left(\tilde{p}|_{D^k(f_s,\gamma(k))}\right)$$

and

$$\mu(X) + \mu(\tilde{X}) = \#\Sigma\left(\tilde{p}|_{X_s}\right),$$

where f_s and X_s are the stable perturbations of f and X.

Moreover, note that, if $\dim D^k(f,\gamma(k)) = 0$, then

$$\mu\left(D^k\big(f,\gamma(k)\big)\right) = m_0\left(D^k\big(f,\gamma(k)\big)\right) - \beta_0^{\gamma(k)},$$

where $m_0\left(D^k\big(f,\gamma(k)\big)\right)$ is the multiplicity of $D^k\big(f,\gamma(k)\big)$. This can also be seen as the number of critical points of $\tilde{p}|_{D^k(f_s,\gamma(k))}$.

In conclusion,

$$\mu_I(X,f) + \mu_I(g,\tilde{X}) = \sum_{k\geq 1}\sum_{\gamma(k)} \frac{\#\Sigma\left(\tilde{p}|_{D^k(f_s,\gamma(k))}\right)}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!}.$$

This is exactly the same point Pallarés-Torres and Nuño-Ballesteros reach in [NBPT19, Theorem 3.2]. The theorem below follows from there (see Figure 4.3).

Theorem 4.3.3 (see [NBPT19, Theorem 3.2])**.** *For an $\mathscr{A}$-finite map germ $f : (X,S) \to \big(\mathbb{C}^{n+1},0\big)$ of corank 1 from an ICIS X of dimension $\dim X = n \geq 2$, let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear projection which defines a transverse slice $g : \big(X \cap (p\circ f)^{-1}(0), S\big) \to (\mathbb{C}^n,0)$. Then,*

$$\mu_I(f,X) + \mu_I\left(g, X \cap (p\circ f)^{-1}(0)\right) = \#\Sigma\left(p|_{Z_s}\right),$$

where $\#\Sigma\left(p|_{Z_s}\right)$ is the number of critical points on all the strata of $Z_s := \operatorname{im}(f_s)$, being f_s a stable perturbation of f.

We complete Theorem 4.3.3 with the case of a one-dimensional ICIS X.

Proposition 4.3.4 (see [NBPT19, Theorem 3.1]). *Let $f:(X,S)\to(\mathbb{C}^2,0)$ be an injective map germ on an* ICIS *X of dimension one. Consider a generic linear projection $p:\mathbb{C}^2\to\mathbb{C}$, then*

$$\mu_I(X,f)+m_0(f)-1=\#\Sigma\left(p|_{Z_s}\right),$$

where $\#\Sigma\left(p|_{Z_s}\right)$ is the number of critical points on all the strata of $Z_s:=\operatorname{im}(f_s)$, being f_s a stable perturbation of f and $m_0(f)=\dim_{\mathbb{C}}\mathcal{O}_{X,S}/f^*\mathfrak{m}_2$ the multiplicity of f.

Proof. We have two strata in the image Z_s of the stable perturbation (X_s,f_s): the 0-dimensional stratum Z_s^0 given by the transverse double points and the 1-dimensional stratum $Z_s^1=f_s(X_s)-Z_s^0$.

Obviously, $\#\Sigma\big(\,p|_{Z_s^0}\,\big)$ is equal to $\#Z_s^0$, which is equal to $\mu_2^{\text{Alt}}(X,f)$. Since f_s is a local diffeomorphism on Z_s^1, $\#\Sigma\big(\,p|_{Z_s^1}\,\big)$ is equal to the number of critical points of $p\circ f_s$ on X_s (here the points of Z_s^0 can be excluded by genericity of p). By the usual Lê-Greuel formula for X and $X\cap(p\circ f)^{-1}(0)$, we have

$$\#\Sigma\big(p\circ f_s\big)=\mu(X)+\deg(p\circ f)-1.$$

But, again, the genericity of p implies that $\deg(p\circ f)=m_0(f)$. Hence,

$$\begin{aligned}\#\Sigma\big(\,p|_{Z_s^0}\,\big)+\#\Sigma\big(\,p|_{Z_s^1}\,\big)&=\mu_2^{\text{Alt}}(X,f)+\mu(X)+m_0(f)-1\\&=\mu_I(X,f)+m_0(f)-1,\end{aligned}$$

by Proposition 4.2.20. QED

4.4. The double point Milnor number

4.4.1. General aspects

Finding conditions for a 1-parameter family to be Whitney equisingular (see Definition 1.3.13) requires working on the source and in the target separately. In the case of the source, we need an object to assure some structure and, as the reader could guess, the double point set is the best candidate. Furthermore, if the map is *nice enough*, this set is the projection of an ICIS, the double point space (see Figure 4.4). We have some invariants in this sense.

Definition 4.4.1. *The double point set of $f:(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$*, of finite singularity type, is the projection on the first coordinate of $D^2(f)$, and we denote it by $D(f)$. Furthermore, if f is $\mathscr{A}$-finite, we will define the *double point Milnor number* as

$$\mu_D(f):=\beta_{n-1}\big(D(f_t)\big),$$

where f_t is a stable perturbation of f.

Remark 4.4.2. The double point Milnor number was denoted as μ_{Σ_2} in [Hou01] by Houston.

Note that we have to define $\mu_D(f)$ through a stable perturbation of f because $D(f)$ is a hypersurface with (not necessarily) non-isolated singularities, hence there is no way to define the Milnor number as hypersurface. However, we can still use Siersma's result, Theorem 3.1.2, to prove that $D(f_t)$ in Definition 4.4.1 has the homotopy type of a wedge of spheres of middle dimension, as a small deformation of $D(f)$ is topologically trivial in a Milnor sphere (see Definition 3.1.1).

The main reason to use this invariant is that $\mu_D(f)$ is the image Milnor number of certain germ on an ICIS (X, f) if f has corank one and it is $\mathscr{A}$-finite (recall Marar-Mond criterion, Theorem 2.4.4). In that case, $\mu_D(f)$ coincides with $\mu_I\big(D^2(f), \pi\big)$.

Hence, we can use all the machinery we developed above if f has corank one. Firstly, $\mu_D(f)$ is well defined by Proposition 4.1.10. Secondly, there are triple points of f_t that also correspond to double points of π and give rise to more homology (as Figure 4.4 represents, there we have depicted a vague idea of the generators of the homology because some higher-dimensional properties cannot be made visible). Finally, the invariant $\mu_D(f)$ is also conservative by Theorem 4.1.12:

Corollary 4.4.3. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be finitely $\mathscr{A}$-determined, of corank 1, and f_u a one-parameter unfolding of $f = f_0$. Take a representative of the unfolding such that its codomain, B_ε, is a Milnor ball. Then,*

$$\mu_D(f) = \beta_{n-1}\big(D(f_u)\big) + \sum_{y \in B_\varepsilon} \mu_D(f_u, y).$$

Remark 4.4.4. Observe that Definition 4.4.1 can be generalized for every pair of dimensions (n, p) as long as $n < p$ and f has corank one. Despite the fact that $D(f)$ could not be a hypersurface, $D^2(f)$ is an ICIS in this case (again, Marar-Mond criterion, Theorem 2.4.4) and we can define $\mu_D(f)$ by means of the homotopy of the image of a stable perturbation of $\big(D^2(f), \pi\big)$, which is controlled by Theorem 4.2.13. One can also use the Euler-Poincaré characteristic of the image of a stable perturbation, sometimes referred as *vanishing Euler characteristic*, see [NBOOT18] for example.

Once more, we focus on the multiple point spaces but, in this case, we deal with $\big(D^2(f), \pi\big)$ and $D^k(\pi)$, where f has corank one and it is $\mathscr{A}$-finite. Using the *principle of iteration* (see [Kle81, Section 4.1]), the multiple point spaces of a perturbation of f are isomorphic to the multiple point spaces of a perturbation of π with a shift in the multiplicity, and the same is true for unfoldings F and Π (of f and π, respectively). More precisely, $D^k(\pi_t) \cong D^{k+1}(f_t)$ and $D^k(\Pi) \cong D^{k+1}(F)$, where the first isomorphism is given by

$$\begin{aligned} \phi\colon D^k(\pi_t) &\longrightarrow D^{k+1}(f_t) \\ \big((x, x_1), \dots, (x, x_k)\big) &\longmapsto (x, x_1, \dots, x_k), \end{aligned} \tag{4.3}$$

and the second one is analogous. This inspires us to compare $\mu_k^{\mathrm{Alt}}\big(D^2(f), \pi\big)$ and $\mu_{k+1}^{\mathrm{Alt}}(f)$, which determine $\mu_D(f)$ and $\mu_I(f)$, respectively. The relation is straightforward conside-

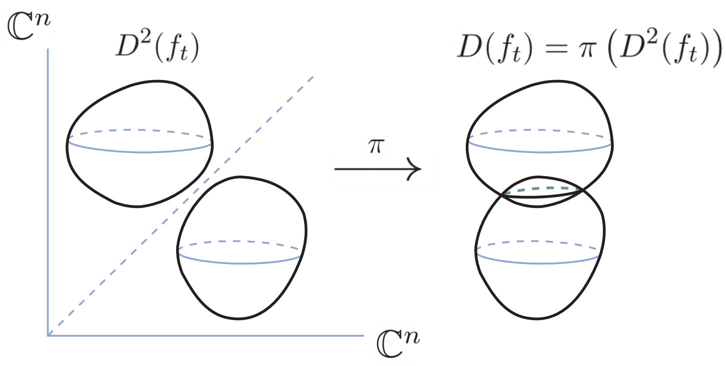


Figure 4.4: Representation of how the homology of the double point set of a stable perturbation works.

ring that

$$\begin{aligned}\mu_k^{\mathrm{Alt}}\big(D^2(f),\pi\big) &= \operatorname{rank}\, H_{n-1-k+2}^{\mathrm{Alt}_k}\big(D^k(\Pi), D^k(\pi_t)\big)\\ &= \operatorname{rank}\, H_{n-1-k+2}^{\mathrm{Alt}_k}\big(D^{k+1}(F), D^{k+1}(f_t)\big)\end{aligned}$$

and

$$\mu_{k+1}^{\mathrm{Alt}}(f) = \operatorname{rank}\, H_{n-1-k+2}^{\mathrm{Alt}_{k+1}}\big(D^{k+1}(F), D^{k+1}(f_t)\big).$$

More precisely, the difference between $\mu_k^{\mathrm{Alt}}\big(D^2(f),\pi\big)$ and $\mu_{k+1}^{\mathrm{Alt}}(f)$ is the group of permutations that acts.

To ease the notation, we will simply write H instead of $H_{n-1-k+2}\big(D^{k+1}(F), D^{k+1}(f_t)\big)$ and k will be clear from the context. Consequently, we want to compare the alternating actions of Σ_{k+1} and $\Sigma_k < \Sigma_{k+1}$ on H, where Σ_k acts as a subgroup fixing the first entry (by construction of the isomorphism of Equation (4.3)). We will use representation theory, which we introduced in Appendix A[2], to do this. For this reason, we will see H as a $\mathbb{C}$-vector space.

For each partition of N, say $\gamma(N)$, there is associated an irreducible representation of Σ_N (see Definition A.3.11 and Theorem A.3.12), which is called the $\gamma(N)$-representation of Σ_N (see Notation A.3.13). Furthermore, the representation that acts by its sign is associated to the partition $(1,\dots,1)$, which is called the alternating representation. Moreover, from the branching rules (see Appendix A.4 and Theorem A.4.4), we know that the alternating representation of Σ_N appears as a restriction of Σ_{N+1} from both the alternating representation and the $(2,1,\dots,1)$-representation. Therefore, knowing the character of the last one will be useful. Unfortunately, we could not find it in the literature, so we compute it here.

Lemma 4.4.5. *The character of the irreducible representation associated to the partition* $(2,1,\dots,1)$ *is*

$$\operatorname{sgn}(\sigma)\left(\operatorname{fix}(\sigma)-1\right),$$

[2] We recommend the reader to have a look at this appendix before continuing with this part.

where $\operatorname{fix}(\sigma)$ *is the number of entries fixed by the permutation* σ*.*

Proof. If $N = 3$, the result is trivial. Assume that $N > 3$; then, we know that the representation associated to this partition is either the standard or the tensor product of the standard and the alternating representations (see, for example, [FH91, Exercise 4.14][3]). By a careful inspection (but see also [FH91, Exercise 4.6]), we know that it is not the standard representation, therefore its character is the product of the standard and the alternating representations. QED

Given any representation V of a finite group G, Maschke's theorem allows us to decompose V into isotypes (see Theorem A.1.8). For example, any partition $\gamma(N)$ of N has its $\gamma(N)$-isotype in V, $V^{\gamma(N)}$. Recall that this is nothing more than the sum of copies of the $\gamma(N)$-representation that appear in V (see Definition A.1.9). In the particular case of the $(1,\dots,1)$-isotype, we use the name alternating isotype as well.

As part of the alternating isotype of $\Sigma_k < \Sigma_{k+1}$ comes from the $(2,1,\dots,1)$-isotype of Σ_{k+1}, we want to make explicit the projection formula onto this last isotype. Considering that we know the character and the dimension of the $(2,1,\dots,1)$-representation, finding the projection is very easy (see Theorem A.4.5):

$$P_{k+1} \coloneqq \frac{k}{(k+1)!} \sum_{\sigma \in \Sigma_{k+1}} \operatorname{sgn}(\sigma) \left(\operatorname{fix}(\sigma) - 1\right) \sigma.$$

Remark 4.4.6. Observe that one can define the projection P_{k+1} with domain any set where Σ_{k+1} acts, here we will define it on H if nothing is said.

Theorem 4.4.7. *Let* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ *be* $\mathscr{A}$*-finite of corank* 1*. Then,*

$$\mu^{\mathrm{Alt}}_{k+1}(f) \le \mu^{\mathrm{Alt}}_{k}\big(D^2(f), \pi\big)$$

for $k = 1, \dots, n$*. Furthermore,*

(i) for $k = 2, \dots, n$*,* $\mu^{\mathrm{Alt}}_{k+1}(f) = \mu^{\mathrm{Alt}}_{k}\big(D^2(f), \pi\big)$ *if, and only if,* $P_{k+1} \equiv 0$ *(or, equivalently, the* $(2,1,\dots,1)$*-isotype is zero), and*

(ii) for $k = 1$*,* $\mu^{\mathrm{Alt}}_{2}(f) = \mu\left(D^2(f)\right)$ *if, and only if, the space* $H_{n-1}\big(D^2(f_t)\big)$ *coincides with its alternating isotype, for* f_t *a stable perturbation of* f*.*

Proof. For $k = 2, \dots, n$, from the branching rules (see Appendix A.4 and Theorem A.4.4), we know that the alternating isotype of the Σ_k representation on H comes exactly from the alternating isotype and the $(2,1,\dots,1)$-isotype of the representation of Σ_{k+1}.

Moreover, the former isotype contributes with the same dimension it has but the latter makes a contribution of one dimension for each k-dimensional copy it has. This comes from the fact that, in this isotype, every copy of the $(2,1,\dots,1)$-representation

[3] This is solved in StackExchange, for example.

has dimension k and each one splits into an alternating representation of dimension 1 and an irreducible $(2,1,\dots,1)$-representation of dimension $k-1$ when we restrict it to the subgroup Σ_k.

As the only difference between $\mu^{\text{Alt}}_{k+1}(f)$ and $\mu^{\text{Alt}}_{k}\big(D^2(f),\pi\big)$ is the different groups acting, Σ_{k+1} and Σ_k as a subgroup, the result follows for these cases.

On the other hand, $P_k+1\equiv 0$ if, and only if, there is no $(2,1,\dots,1)$-isotype, so Item *(i)* follows.

Finally, Item *(ii)* is trivial, as the only two possible representations of Σ_2 are the trivial and the alternating one. QED

Remark 4.4.8. One can determine the difference between $\mu^{\text{Alt}}_{k+1}(f)$ and $\mu^{\text{Alt}}_{k}\big(D^2(f),\pi\big)$, for it depends on the number of repetitions of the $(2,1,\dots,1)$-representation. For example, one can compute it through the inner product between the characters of the whole representation of Σ_{k+1} and the $(2,1,\dots,1)$-representation (see Definition A.2.5 and Corollary A.2.10), which is nothing more than counting the number of fixed generators by P_{k+1} in a convenient basis.

Also, for $k=n$ and $k=n+1$, one is dealing with zero homology and the multiple point spaces are points, this ease the relation and we can say something more.

Theorem 4.4.9. *With the hypotheses of Theorem 4.4.7, for $k=n$,*

$$(n+1)\ \text{rank}\ H_0^{\text{Alt}_{n+1}}\left(D^{n+1}(f_t)\right)=\text{rank}\ H_0^{\text{Alt}_n}\left(D^{n+1}(f_t)\right).$$

Also, for $k=n+1$,

$$\frac{d(f)s(f)^2}{s(f)-1}\mu^{\text{Alt}}_{n+2}(f)=\mu^{\text{Alt}}_{n+1}\big(D^2(f),\pi\big).$$

Proof. Now, we are dealing with points and the zero homology. In particular, we can identify the elements in the homology with the 0-chains. Hence, one can find a basis of $H_0^{\text{Alt}_{n+1}}\left(D^{n+1}(f_t)\right)$ with elements of the form $\text{Alt}_{\mathbb{Z}}p$, where $p\in D^{n+1}(f_t)$ (recall Equation (2.1)).

The action of Σ_{n+1} on the orbit of p is the regular representation, so it decomposes into the alternating representation we are considering, n $(2,1,\dots,1)$-subrepresentations and more irreducible subrepresentations (see Example A.1.4 and Corollary A.2.10). The contributions to the alternating isotype of the representation of Σ_n come from: one alternating representation from each $(2,1,\dots,1)$-subrepresentation, each one of the alternating subrepresentation of Σ_{n+1} will be preserved in the subgroup, and there are no more contributions from other isotypes. This happens for every orbit of points in $D^{n+1}(f_t)$, proving the first statement.

To prove the second part, recall that $\mu^{\text{Alt}}_{n+2}(f)$ comes from the bottom row of the spectral sequence (see, for example, Table 4.1), and the argument is similar but, now, working with the multiple point space of the unfolding. Therefore, again, the alternating isotype of Σ_k is $k+1$ times bigger than the alternating isotype of Σ_{k+1}. Hence, if originally $\mu^{\mathrm{Alt}}_{n+2}(f)$ was

$$\left|\sum_{\ell=d(f)+1}^{s(f)} (-1)^l \binom{s(f)}{\ell}\right| = \binom{s(f)-1}{d(f)},$$

now, $\mu^{\mathrm{Alt}}_{n+1}\big(D^2(f),\pi\big)$ is

$$\begin{aligned}\left|\sum_{\ell=d(f)+1}^{s(f)} (-1)^\ell \ell \binom{s(f)}{\ell}\right| &= \frac{d(f)\,(d(f)+1)}{s(f)-1}\binom{s(f)}{d(f)+1} \\ &= \frac{d(f)s(f)^2}{s(f)-1}\binom{s(f)-1}{d(f)}.\end{aligned}$$

QED

Remark 4.4.10. Although these inequalities and equalities are enough for our purposes, one can specify the relation of $\mu^{\mathrm{Alt}}_{n+1}(f)$ and $\mu^{\mathrm{Alt}}_{n}\big(D^2(f),\pi\big)$ using the ideas of the second part of the proof of Theorem 4.4.9 and considering the exact sequence of the pair.

Also, one may ask what happens if the group acts by permutation of the elements of a base for some $k < n$ (this action could be not faithful). Regarding this, there are algorithms to compute the alternating part based in the same idea: looking for orbits and the relation between the actions. An upper bound is also possible with the same ideas.

There are some interesting corollaries of Theorems 4.4.7 and 4.4.9. For example, it could happen that there is not enough space in the homology group to fit a $(2,1,\dots,1)$-subrepresentation.

Corollary 4.4.11. *With the notation of Theorem 4.4.7, for $k = 1,\dots,n$, if*

$$\operatorname{rank}\Big(H_{n-1-k+2}\big(D^{k+1}(F), D^{k+1}(f_t)\big)\Big) - \mu^{\mathrm{Alt}}_{k+1}(f) < k,$$

then

$$\mu^{\mathrm{Alt}}_{k+1}(f) = \mu^{\mathrm{Alt}}_{k}\big(D^2(f),\pi\big).$$

Proof. The proof is based on the fact that a $(2,1,\dots,1)$-representation of Σ_k has dimension k and the ideas of Theorems 4.4.7 and 4.4.9. QED

Another example is an inequality involving the full Milnor number on both contexts.

Corollary 4.4.12. *For f as in Theorem 4.4.7, $\mu_I(f) \leq \mu_D(f)$. This holds with equality if, and only if, $H_{n-1}\big(D^2(f_t)\big)$ coincides with its alternating isotype and all the P_i are zero for all i, for P_i as in Theorem 4.4.7.*

Thence, there are some nice characterizations as well, in particular weak Mond's conjecture for $\mu_D(f)$.

Corollary 4.4.13 (see Theorem 3.1.22)**.** *For f as in Theorem 4.4.7, the following are equivalent:*

(i) f *is stable,*

(ii) $\mu_I(f) = 0$, *and*

(iii) $\mu_D(f) = 0$.

Proof. If f is stable, then $\mu_D(f) = \mu_I\big(D^2(f), \pi\big) = 0$ as so are all the alternating Milnor numbers. If $0 = \mu_D(f)$, then $\mu_I(f) = 0$ by Corollary 4.4.12, but if $\mu_I(f) = 0$, then f is stable by the weak Mond's conjecture for corank 1 (see Theorem 3.1.22). QED

4.4.2. Delving into other multiplicities

For an $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ of corank one, we have studied the homology of the projection of $D^2(f_t)$ onto $\mathbb{C}^n$ and we have compared it with the homology of the image of f_t, for f_t a stable perturbation of f. One can keep looking for relations between the multiple point spaces using the same ideas.

On one hand, we can reproduce the roles of the image of f_t and $D(f_t)$ easily:

$$\begin{array}{ccccccccc}
\cdots & \dashrightarrow & D^3(f_t) & \xrightarrow{\pi^3_2} & D^2(f_t) & \xrightarrow{\pi^2_1} & D^1(f_t) = \mathbb{C}^n & \xrightarrow{f_t} & \mathbb{C}^{n+1} \\
 & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow \\
\cdots & & D^4_3(f_t) & & D^3_2(f_t) & & D^2_1(f_t) = D(f_t) & & \mathrm{Im}(f_t)
\end{array},$$

where $\pi^k_{k-1} : D^k(\bullet) \to D^{k-1}(\bullet)$ is the projection that forgets the last entry, for $k = 2, \dots, d(f)$, and $D^k_{k-1}(\bullet)$ is the image of π^k_{k-1}.

On the other hand, although $\pi^k_{k-1} : D^k(f) \to D^{k-1}(f)$ has the problem that the target is an ICIS as well, the homology of $D^k_{k-1}(f_t)$ is well defined for any $\mathscr{A}$-finite $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ of corank 1. This is a consequence of the ICSS of, for example, Theorem 2.3.1 applied to $\pi^k_{k-1} : D^k(f_t) \to D^{k-1}(f_t)$. Hence, again by the iteration principle, the Betti numbers of $D^k_{k-1}(f_t)$ are determined by the Σ_{i+1}-alternated homologies of $\big(D^{k+i}(F), D^{k+i}(f_t)\big)$, for $i = 0, \dots, d(f) - k$.

Furthermore, $D^{k+i}(f_t)$ is the unique fiber, up to isomorphism, of $D^{k+i}(f)$ and the action of the permutations does not depend on the stable perturbation, so these Betti numbers will be well defined.

Finally, note that the homology will appear again in middle dimension for the pair of dimensions $(n, n+1)$, for the same reason it happens for $D(f_t)$.

Notation 4.4.14. We will simply write $\beta_k(f)$ to denote $\beta_{n-k+1}\big(D^k_{k-1}(f_t)\big)$.

We have compared $\beta_1(f) \coloneqq \mu_I(f)$ with $\beta_2(f) = \mu_D(f)$ and, similarly, we can compare $\beta_k(f)$ with $\beta_{k+1}(f)$. This is very easy if $d(f) < n+1$ or we have a mono-germ, because we can forget about the the homology of the pair and the unfolding by the exact sequence of the pair. As we were saying, by an ICSS as in Theorem 2.3.1 and the iteration principle, arranging the terms in a convenient way, we have

$$\beta_{k+1}(f) = \operatorname{rank} \bigoplus_{i\geq 2} H^{\mathrm{Alt}_i}\big(D^{k+i}(f_t)\big) \oplus H\big(D^{k+1}(f_t)\big) \tag{4.4}$$

and

$$\beta_k(f) = \operatorname{rank} \bigoplus_{i\geq 1} H^{\mathrm{Alt}_{i+1}}\big(D^{k+i}(f_t)\big) \oplus H\big(D^{k}(f_t)\big), \tag{4.5}$$

from where we have omitted the index of the homology. Therefore, subtracting Equation (4.5) from Equation (4.4) and using the ideas of Theorems 4.4.7 and 4.4.9, we get

$$\begin{aligned}\beta_{k+1}(f) - \beta_k(f) = \sum_{i\geq 2} \frac{\operatorname{rank}\ H\big(D^{k+i}(f_t)\big)^{(2,1,\dots,1)}}{i} &+ \mu\big(D^{k+1}(f)\big) \\ - \mu\big(D^{k}(f)\big) - \operatorname{rank}\ H^{\mathrm{Alt}_2}\big(D^{k+1}(f_t)\big)&,\end{aligned} \tag{4.6}$$

with $(2, 1, \dots, 1)$ partition of $i + 1$.

Remark 4.4.15. One can take this to a broader context as long as the first page of the spectral sequence collapses and the iteration principle works.

We put this in practice with some examples.

Example 4.4.16. For $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ as in Theorem 4.4.7 and taking $k = 1$ in Equation (4.6),

$$\begin{aligned}\mu_D(f) - \mu_I(f) &= \frac{\operatorname{rank}\ H_0\left(D^3(f_t)\right)^{(2,1)}}{2} + \mu\left(D^2(f)\right) - 0 - \mu_2^{\mathrm{Alt}}(f) \\ &= \frac{\operatorname{rank}\ H_0\left(D^3(f_t)\right)^{(2,1)}}{2} + \operatorname{rank} H_1\left(D^2(f_t)\right)^{(2)},\end{aligned}$$

where $H_1\big(D^2(f_t)\big)^{(2)}$ is the part of the homology that is fixed by the group Σ_2, i.e., the *trivial isotype* of Σ_2. The last equality is due to the fact that there are only two irreducible representations of Σ_2, the alternating and the trivial one.

Example 4.4.17. For $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ as in Theorem 4.4.7, and taking $k = 2$ in Equation (4.6),

$$\begin{aligned}\beta_3(f) - \mu_D(f) &= \operatorname{rank}\ H_0\left(D^3(f_t)\right) - \mu\left(D^2(f)\right) - \operatorname{rank}\ H_0^{\mathrm{Alt}_2}\left(D^3(f_t)\right) \\ &= -\mu\left(D^2(f)\right) + \operatorname{rank}\ H_0\left(D^3(f_t)\right)^{(2)},\end{aligned}$$

following the same notation as above.

Note that the triple points of f_t are strict in Examples 4.4.16 and 4.4.17, i.e., in $D^3(f_t)$ the points are the Σ_3-orbit of (a, b, c) with $a \neq b \neq c \neq a$ that come from transverse triple points (recall Example 1.2.6). Say we have T triple points, then:

- rank $H_0\left(D^3(f_t)\right)^{(2,1)} = 4T$, as it is the complement of the alternating and trivial isotype and both representations have dimension one[4] (observe, also, that one can apply the second part of Corollary A.2.10).
- rank $H_0\left(D^3(f_t)\right)^{(2)} = 3T$, as it is the trivial isotype of Σ_2 fixing the first entry (it has elements of the form $(a,b,c)+(a,c,b)$).
- rank $H_0^{\text{Alt}}\left(D^3(f_t)\right) = 3T$, similarly as the previous case.
- $\beta_3(f) = 6T$, as it is simply counting the elements of the orbits.

In conclusion, we have the following results, which were obtained also by Houston with similar invariants (see [Hou01, Theorems 2.7 and 2.8]).

Proposition 4.4.18. *For $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ $\mathscr{A}$-finite and being T the number of triple points of a stable perturbation of f,*

$$\mu_D(f) = \mu\left(D^2(f)\right) + 3T.$$

Proof. If f has corank 1, we are done by Proposition 3.1.16 or Proposition 4.2.20. If f does not have corank one, D^2 has dimension one and D^3 is zero dimensional, so the homology is in middle dimension and the same argument can be applied. QED

Similarly:

Proposition 4.4.19. *For $f:(\mathbb{C}^3,0)\to(\mathbb{C}^4,0)$ as in Theorem 4.4.7 and being Q the number of quadruple points of a stable perturbation f_t of f,*

$$\mu_D(f) = 4Q + \mu\big(D^2(f)\big) + \frac{\mu\big(D^3(f)\big) - \mu_3^T(f) + \mu_3^{\text{Alt}}(f)}{2},$$

where $\mu_3^T(f) \coloneqq \text{rank } H_1\big(D^3(f_t)\big)^{(3)}$ is the invariant homology by Σ_3 and $\mu_3^{\text{Alt}}(f)$ is defined as in Definition 4.2.19, i.e., the alternating homology by Σ_3.

Proof. By Equation (4.6) for $k=1$, we have that

$$\begin{aligned}\mu_D(f) - \mu_I(f) = &\frac{\text{rank } H\big(D^3(f_t)\big)^{(2,1)}}{2} + \frac{\text{rank } H\big(D^4(f_t)\big)^{(2,1,1)}}{3}\\ &+ \mu\big(D^2(f)\big) - 0 - \mu_2^{\text{Alt}}(f).\end{aligned}$$

Now, observe that Σ_3 has only three irreducible representations: the trivial representation, alternating representation and the $(2,1)$-representation. With this in mind,

$$\frac{\text{rank } H\big(D^3(f_t)\big)^{(2,1)}}{2} = \frac{\text{rank } H\big(D^3(f_t)\big) - \mu_3^T(f) - \mu_3^{\text{Alt}}(f)}{2}.$$

[4] The correct term is degree, see Definition A.1.2.

Finally, recall that $\mu_I(f) = \mu_2^{\mathrm{Alt}}(f) + \mu_3^{\mathrm{Alt}}(f) + \mu_4^{\mathrm{Alt}}(f)$ (see Propositions 3.1.16 and 4.2.20). Furthermore,

$$\frac{\operatorname{rank} H\big(D^4(f_t)\big)^{(2,1,1)}}{3} = 3Q,$$

as we have a regular representation of Σ_4 and we can apply the second part of Corollary A.2.10, and

$$\mu_4^{\mathrm{Alt}}(f) = Q.$$

The result follows from here. QED

4.4.3. A useful relation

There is another relation between μ_I and μ_D. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite of corank one and let $g : (\mathbb{C}^{n-1}, S') \to (\mathbb{C}^n, 0)$ be a transverse slice. By the Lê-Greuel type formula (recall Theorems 4.3.1 and 4.3.3), we know that

$$\mu_I(f) + \mu_I(g) = \#\Sigma(p|_{Z_s}) = \sum_{\mathcal{Q}} \#\Sigma\big(p|_{\mathcal{Q}(f_s)}\big),$$

where $p : \mathbb{C}^{n+1} \to \mathbb{C}$ is the generic projection which defines the transverse slice, Z_s is the image of a stable perturbation of f, $\mathcal{Q}$ runs trough all the stable types in the target and $\mathcal{Q}(f_s)$ denotes the points of f_s in the target that are of stable type $\mathcal{Q}$.

Using the same argument, we have that

$$\mu_D(f) + \mu_D(g) = \#\Sigma(p \circ f_s) = \sum_{\mathcal{Q}_S} \#\Sigma\big((p \circ f_s)|_{\mathcal{Q}_S(f_s)}\big),$$

where now $\mathcal{Q}_S$ runs trough all the stable types in the source.

If a stable type in the source $\mathcal{Q}_S$ corresponds to $\mathcal{Q}$ in the target, the restriction $f_s : \mathcal{Q}_S(f_s) \to \mathcal{Q}(f_s)$ is a local diffeomorphism and is r-to-one, where $r = r(\mathcal{Q})$ is the number of branches of the stable type $\mathcal{Q}$. Hence,

$$\#\Sigma((p \circ f_s)|_{\mathcal{Q}_S(f_s)}) = r(\mathcal{Q})\#\Sigma(p|_{\mathcal{Q}(f_s)}).$$

Therefore, if $\mu_I(f_t)$ and $\mu_I(g_t)$ are constant in a family, this implies that $\mu_D(f_t)$ and $\mu_D(g_t)$ are also constant, by upper semi-continuity.

Proposition 4.4.20. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite of corank one and let $g : (\mathbb{C}^{n-1}, S') \to (\mathbb{C}^n, 0)$ be a transverse slice. If $\mu_I(f_t)$ and $\mu_I(g_t)$ are constant in a family, then $\mu_D(f_t)$ and $\mu_D(g_t)$ are also constant.*

The previous argument could make the reader think that, when we deal with Whitney equisingularity, controlling the target is enough to control the source, or vice versa. This idea is wrong in general. We take care of the details to control the target and the source in Section 4.5, and the problem with the previous idea is that $\mu_D(g)$ is not what we need (see Corollary 4.5.4 and Theorem 4.5.7). See also Example 4.5.8 for an example where this idea fails.

4.5. Whitney equisingularity

In [Gaf93], Gaffney showed that a one-parameter family $f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is Whitney equisingular if, and only if, it is excellent (in Gaffney's sense) and all the polar multiplicities in the source and target are constant on t. The problem is that, for each d-dimensional stratum in the source or target, we need $d+1$ invariants, so the total number of invariants we need to control the Whitney equisingularity is huge.

In this section, we follow the approach of Teissier in [Tei82] for hypersurfaces with isolated singularities or Gaffney in [Gaf96] for ICIS to show that, in the corank one case, Whitney equisingularity can be characterized in terms of the $\mu_I^*(f_t)$ and $\mu_I^*\big(D^2(f_t), \pi\big)$ sequences, obtained by taking successive transverse slices of f_t.

In Section 4.3, we already made use of the stratification $\mathscr{S}$ by stable types of the image of a locally stable mapping $f : X \to Y$ between smooth manifolds X and Y with $\dim X = n$ and $\dim Y = n+1$ and with only corank one singularities. Since each stable type is determined by its Mather algebra $\mathcal{Q}$, we can denote by $\mathcal{Q}(f)$ the stratum of points $y \in f(X)$ such that the multi-germ of f at y has type $\mathcal{Q}$. Because f is stable, $\mathscr{S}$ is a *partial stratification* of f in the sense of [GWdPL76, Proposition 3.1]. It follows that we have an induced stratification $\mathscr{S}'$ on X, with strata $\mathcal{Q}_S(f) = f^{-1}\big(\mathcal{Q}(f)\big)$, such that $f : X \to Y$ is a Thom stratified map.

Suppose, now, that we have an $\mathscr{A}$-finite germ $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$ of corank one, where X is an n-dimensional ICIS. By the Mather-Gaffney criterion for germs on ICIS, Proposition 4.1.8, we can take a finite representative $f : X \to Y$, where Y is an open neighbourhood of 0 in $\mathbb{C}^{n+1}$ such that $f^{-1}(0) = S$ and $f : X - S \to Y - \{0\}$ is a locally stable mapping. The stratification by stable types on $f : X - S \to Y - \{0\}$ extends to $f : X \to Y$ just by adding S and $\{0\}$ as strata in the source and target, respectively, as in Definition 1.3.10 and its following comments. By shrinking the representative if necessary, we can always assume that f has no 0-stable singularities, so S and $\{0\}$ are in fact the only 0-dimensional strata.

Finally, we give a version of excellency, stratification by stable types and Whitney equisingularity for unfoldings of germs on ICIS.

Let $(\mathcal{X}, \pi, F, j)$ be a one-parameter unfolding of (X, f) which is origin preserving (that is, $S \subset X_t$ and $f_t(S) = 0$, for all t) so we can see the unfolding as a family of germs $f_t : (X_t, S) \to (\mathbb{C}^{n+1}, 0)$.

Definition 4.5.1 (cf. Definition 1.3.11)**.** We say that $(\mathcal{X}, \pi, F, j)$ is *excellent* if there exist a representative $F : \mathcal{X} \to Y \times U$, where Y and T are open neighbourhoods of the origin in $\mathbb{C}^{n+1}$ and $\mathbb{C}$ respectively, such that for all $t \in T$, $f_t^{-1}(0) = S$ and $f_t : X_t - S \to Y - \{0\}$ is a locally stable mapping with no 0-stable singularities.

When the unfolding is excellent, $F : \mathcal{X} - S \times \{0\} \to (Y - \{0\}) \times T$ is also stable, so we have a well defined stratification by stable types. This extends to $F : \mathcal{X} \to Y \times T$ just by adding $S \times T$ and $\{0\} \times T$ as strata in the source and target, respectively. These are, in fact, the only 1-dimensional strata (cf. Definition 1.3.10).

Definition 4.5.2 (cf. Definition 1.3.13)**.** We say that $(\mathcal{X}, \pi, F, j)$ is *Whitney equisingular* if $F : \mathcal{X} \to Y \times T$ is a Thom stratified map with the stratification by stable types.

Now, we recall the definition of polar multiplicities, following Gaffney in [Gaf93].

Definition 4.5.3. Let $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite of corank one. For each stable type $\mathcal{Q}$ such that $d = \dim \mathcal{Q}(f) > 0$ and for each $i = 0, \ldots, d-1$, the *ith-polar multiplicities* in the source and target are

$$m_i(f, \mathcal{Q}) = m_0\left(P_i\big(\overline{\mathcal{Q}(f)}\big)\right), \quad m_i(f, \mathcal{Q}_S) = m_0\left(P_i\big(\overline{\mathcal{Q}_S(f)}\big)\right),$$

where the bar means the Zariski closure and $P_i(Z)$ is the absolute polar variety of codimension i of Z in the sense of Lê and Teissier in [TT81, p. 462].

The *dth-stable multiplicities* are

$$m_d(f, \mathcal{Q}) = \deg\left(\pi : P_d\big(\overline{\mathcal{Q}(F)}, \pi\big) \to (\mathbb{C}^r, 0)\right),$$
$$m_d(f, \mathcal{Q}_S) = \deg\left(\pi : P_d\big(\overline{\mathcal{Q}_S(F)}, \pi\big) \to (\mathbb{C}^r, 0)\right),$$

where, now, $(\mathcal{X}, \pi, F, j)$ is an r-parameter versal unfolding of (X, f) and $P_d(\mathcal{Z}, \pi)$ is the relative polar variety of codimension d of a family $\pi : \mathcal{Z} \to \mathbb{C}^r$ (see [Tei82, Section IV.1]).

Finally, we denote by $c(f)$ the number of all 0-stable singularities that appear in a stable perturbation of (X, f).

It follows from the definition of relative polar variety that the top polar multiplicity $m_d(f, \mathcal{Q})$ is equal to the number of critical points of $p|_{\mathcal{Q}(f_s)}$, where $p : \mathbb{C}^{n+1} \to \mathbb{C}$ is a generic linear projection and (X_s, f_s) is a stable perturbation of (X, f). Since the 0-stable singularities are also critical points of p in the stratified sense, we get the following reformulation of Theorem 4.3.1:

Corollary 4.5.4. *For a corank* 1 *and $\mathscr{A}$-finite multi-germ $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$, X an* ICIS *of dimension* $\dim X = n > 1$*, let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear projection which defines a transverse slice $g : \big(Y, S\big) \to (\mathbb{C}^n, 0)$, where $Y = X \cap (p \circ f)^{-1}(0)$. Then,*

$$\mu_I(X, f) + \mu_I(Y, g) = \sum_{\dim \mathcal{Q}(f_s) = d > 0} m_d(f, \mathcal{Q}) + c(f).$$

Now, we define the μ_I^* and μ_D^*-sequences of a corank one map germ.

Definition 4.5.5. Consider $f : (X, S) \to (\mathbb{C}^{n+1}, 0)$ $\mathscr{A}$-finite of corank one, with X an ICIS of dimension $\dim X = n > 1$. We take a generic flag of vector subspaces

$$H_{(n-1)} \subset \cdots \subset H_{(1)} \subset H_{(0)} = \mathbb{C}^{n+1},$$

such that $H_{(i)}$ has codimension i. We put $X_{(i)} = X \cap f^{-1}\left(H_{(i)}\right)$ and $f_{(i)} = f|_{X_{(i)}}$ and define the *μ_I^*-sequence of* (X, f), or f, as

$$\mu_I^*(X, f) := \left(\mu_I(X, f), \mu_I\big(X_{(1)}, f_{(1)}\big), \ldots, \mu_I\big(X_{(n-1)}, f_{(n-1)}\big)\right).$$

Sometimes we do not consider the top image Milnor number $\mu_I(X, f)$ in the μ_I^*-sequence and, then, we denote it by $\tilde{\mu}_I^*(X, f)$.

It is well-known that, by generic, we mean a suitable Zariski open in a convenient space, and this definition does not depend on the generic flag we are taking. The details can be seen in [MNnB14, pp. 1380–1381].

In next lemma, we see that all polar multiplicities can be seen as top polar multiplicities of the corresponding transverse slices.

Lemma 4.5.6. *With the hypothesis and notation of Definition 4.5.5, suppose that $\mathcal{Q}$ is a stable type such that* $\dim \mathcal{Q}(f_s) = d > 0$, *for a stable perturbation* (X_s, f_s) *of* (X, f). *Then,*

$$m_{d-i}\left(f_{(i)}, \mathcal{Q}\right) = m_{d-i}\left(f, \mathcal{Q}\right), \ i = 1, \dots, d.$$

Proof. By induction, it is enough to prove that

$$m_{d-i}\left(f_{(1)}, \mathcal{Q}\right) = m_{d-i}\left(f, \mathcal{Q}\right), \ i = 1, \dots, d.$$

To see this, we first observe that $\mathcal{Q}\big(f_{(1)}\big) = \mathcal{Q}(f) \cap H_{(1)}$, so the equality for $i = 2, \dots, d$ follows directly from [TT81, Corollary 4.1.9].

For $i = 1$, we can see f as a stabilisation of $f_{(1)}$. If $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$ is the linear form such that $H_{(1)} = \ell^{-1}(0)$, this means that $f|_{\ell^{-1}(t)}$, with $t \neq 0$, is a stable perturbation of $f_{(1)}$. In particular, $m_{d-1}\big(f_{(1)}, \mathcal{Q}\big)$ is the number of critical points of a generic linear projection $p : \mathbb{C}^{n+1} \to \mathbb{C}$ restricted to $\mathcal{Q}\big(f|_{\ell^{-1}(t)}\big) = \mathcal{Q}(f) \cap \ell^{-1}(t)$. This number can be also seen as

$$\deg\Big(\ell : P_{d-1}\big(\overline{\mathcal{Q}(f)}, \ell\big) \to (\mathbb{C}, 0)\Big), \tag{4.7}$$

where $P_{d-1}\big(\overline{\mathcal{Q}(f)}, \ell\big)$ is the closure of the set of critical points of $(p, \ell)|_{\mathcal{Q}(f)}$.

On the other hand, $m_{d-i}\left(f, \mathcal{Q}\right) = m_0\big(P_{d-1}\big(\overline{\mathcal{Q}(f)}\big)\big)$. Since $P_{d-1}\big(\overline{\mathcal{Q}(f)}\big)$ is 1-dimensional and ℓ is generic, this is equal to

$$\deg\Big(\ell : P_{d-1}\big(\overline{\mathcal{Q}(f)}\big) \to (\mathbb{C}, 0)\Big), \tag{4.8}$$

where $P_{d-1}\big(\overline{\mathcal{Q}(f)}\big)$ is again the closure of the set of critical points of $(p, \ell)|_{\mathcal{Q}(f)}$. So, Equations (4.7) and (4.8) are equal. QED

We arrive to the main theorem, which characterises the Whitney equisingularity of a family of map germs in terms of the μ_I^*-sequences of f_t and $\big(D^2(f_t), \pi\big)$.

Theorem 4.5.7. *Let* $f_t \colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ *be a one-parameter family of $\mathscr{A}$-finite corank one map germs. Then, the family is Whitney equisingular if, and only if, the sequences* $\mu_I^*(f_t)$ *and* $\tilde{\mu}_I^*\big(D^2(f_t), \pi\big)$ *are constant on* t.

Proof. Suppose that the sequences $\mu_I^*(f_t)$ and $\tilde{\mu}_I^*\big(D^2(f_t), \pi\big)$ are constant. First of all, the constancy of $\mu_I(f_t)$ implies that the family is excellent (see Theorem 3.2.3). In fact, this holds not only for the family f_t, but also for all families $f_{t(i)}$, $i = 1, \dots, n-1$. In particular, all the numbers of 0-stable singularities $c\big(f_{t(i)}\big)$ are constant (see [Gaf93, Proposition 3.6]). By Gaffney's results in [Gaf93, Theorems 7.1 and 7.3], we need to proof that all polar invariants in the source and target are constant.

By Lemma 4.5.6, the constancy of the polar multiplicities follows from the constancy of the top polar multiplicities of all the transverse slices $f_{t(i)}$, with $i = 1, \dots, n-1$. Secondly, we apply recursively Corollary 4.5.4 on $f_{t(i)}$ for $i = 0, \dots, n-2$. For any i, we have

$$\mu_I\big(f_{t(i)}\big) + \mu_I\big(f_{t(i+1)}\big) = \sum_{\dim \mathcal{Q}(f_{t(i)})=d>0} m_d\big(f_{t(i)}, \mathcal{Q}\big) + c\big(f_{t(i)}\big).$$

The polar multiplicities $m_d\big(f_{t(i)}, \mathcal{Q}\big)$ are upper semi-continuous (see [Gaf93, Proposition 4.15]). Therefore, all $m_d\big(f_{t(i)}, \mathcal{Q}\big)$ must be constant.

For the polar multiplicities in the source, we follow the same argument, but this time applied to the family $\big(D^2(f_t), \pi\big)$. Observe that the polar multiplicities of $f_{t(i)}$ in the source coincide with the polar multiplicities of $\big(D^2(f_{t(i)}), \pi\big)$ in the target. Hence, we need to study $\mu_I^*\big(D^2(f_{t(i)}), \pi\big)$. Moreover, it follows from Proposition 4.4.20 that

$$\mu_D(f_t) + \mu_D\big(f_{t(1)}\big) = \sum_{\mathcal{Q}_S} r(\mathcal{Q}) \# \Sigma\big(p|_{\mathcal{Q}(f_s)}\big).$$

Since the right-hand side is constant, for all the members in the sum are either polar multiplicities in the target or numbers of 0-stable invariants, so is the left-hand side. Again, the upper semi-continuity of μ_D (see Corollary 4.1.13) implies that $\mu_D(f_t)$ is also constant. Hence, as $\mu_D(f_t) = \mu_I\big(D^2(f_t), \pi\big)$, it is enough to consider the reduced sequence $\tilde{\mu}_I^*\big(D^2(f_t), \pi\big)$.

Finally, the converse is easy. If the family f_t is Whitney equisingular, so are the families $f_{t(i)}$, for $i = 1, \dots, n-1$. By Thom's second isotopy lemma, they are topologically trivial and, hence, their image Milnor numbers are constant (see Corollary 3.1.13 and take into account that the family has to be good because it is Whitney equisingular). Analogously, the family $\big(D^2(f_t), \pi\big)$ must be also Whitney equisingular, which gives the constancy of the sequence $\tilde{\mu}_I^*\big(D^2(f_t), \pi\big)$. QED

The following example shows that $\tilde{\mu}_I^*\big(D^2(f_t), \pi\big)$, or even $\mu_I^*\big(D^2(f_t), \pi\big)$, is not enough to control Whitney equisingularity.

Example 4.5.8. Consider the one-parameter family $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ such that

$$f_t(x, y) = (x, y^4, x^5 y - 5x^3 y^3 + 4xy^5 + y^6 + ty^7).$$

This family is an example of topologically trivial family such that it is not Whitney equisingular, which was shown in [RS19, Example 5.5].

In this example, we also have constancy of $\mu\big(D(f_t)\big)$. Therefore, the source is Whitney regular as $D(f_t)$ is a family of plane curves. In particular, all the multiplicities in the source and $\mu_I^*\big(D^2(f_t), \pi\big)$ are constant. However, neither the polar multiplicities in the target nor $\mu_I^*(f_t)$ are constant because the family is not Whitney equisingular (and by Theorem 4.5.7).

As a corollary of Theorem 4.5.7, and closing the topic we have started with Example 4.5.8, the Whitney equisingularity of a family $f_t : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ is controlled by just two invariants in the target. In fact, in [MNnB14], it was shown that f_t (of any corank) is Whitney equisingular if, and only if, $\mu\big(D(f_t)\big)$ and $\mu\big(\operatorname{im}(f_{t(1)})\big)$ are constant, where μ is the usual Milnor number of a plane curve.

Corollary 4.5.9. *Let $f_t \colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ be a one-parameter family of $\mathscr{A}$-finite corank one map germs. Then, the family is Whitney equisingular if, and only if, $\mu_I(f_t)$ and $\mu_I\big(f_{t(1)}\big)$ are constant on t.*

The proof of Theorem 4.5.7 allows us to state a partial result when only the sequence $\mu_I^*(f_t)$ is constant. We say that the family f_t is *Whitney equisingular in the target* if there exists a representative of the unfolding $F : \mathcal{X} \to Y \times T$ as in Definition 4.5.2 such that the stratification by stable types on $Y \times T$ is a Whitney stratification. Hence:

Proposition 4.5.10. *Let $f_t \colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a one-parameter family of $\mathscr{A}$-finite corank one map-germs. Then, the family is Whitney equisingular in the target if, and only if, the sequence $\mu_I^*(f_t)$ is constant on t.*

Chapter 5

Monodromies without fixed points

> For $\varepsilon \leq \varepsilon_0$ the space $S_\varepsilon - K$ is a smooth fiber bundle over $\mathbb{S}^1$, with projection mapping $\phi(z) = f(z)/|f(z)|$.
>
> John Milnor, *Singular points of complex hypersurfaces* [Mil68]

This chapter contains the results of the joint paper with Nuño-Ballesteros and Lê Dũng Tràng, [GCTNB21].

The main result of these works proves that, in a general context, there is a *geometric local monodromy* of a germ $f : (X, x) \to (\mathbb{C}, 0)$ without fixed points. This is a generalization of a theorem of Lê Dũng Tràng in [Trá75], stated for a smooth source. Also, this generalization was already stated in Tibar's PhD thesis (see [Tib92]) and in his paper [Tib93]. We will see the technical details to prove this here (and in [GCTNB21]).

We also show some applications, for germs $f : (X, x) \to (\mathbb{C}, 0)$ where X has maximal rectified homotopical depth at x in particular. To be more precise, we show that, given a family of such functions with isolated critical points and constant total Milnor number, the family has no coalescing of (non-trivial) singularities.

This last result was our original motivation, having in mind the setting of singularities of map germs, as we explain in Section 5.1.

5.1. Motivation

Given a one-parameter family $f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1})$, one of the conditions to be excellent is having an isolated instability *uniformly*. This means that the instabilities do not *split* or *coalesce* along the family. Also, we have seen in Theorem 3.2.3 that a corank one unfolding that has constant image Milnor number is excellent. In addition, we have seen in Theorem 3.1.7 that the image Milnor number is conservative, something that is

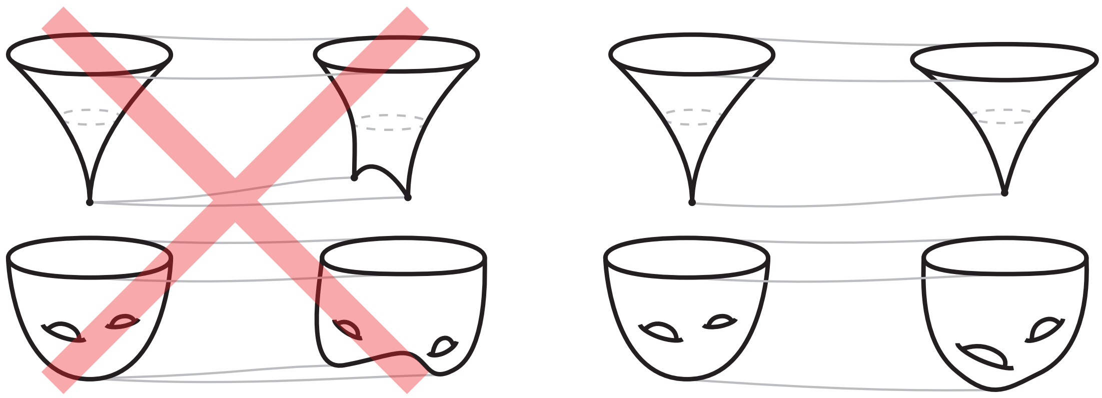


Figure 5.1: Representations of the non-coalescing theorem for hypersurfaces with isolated singularities: families of hypersurfaces (above) and their Milnor fibers (below).

expressed as

$$\mu_I(f) = \beta_n\big(\operatorname{im}(f_t)\big) + \sum_{y\in\operatorname{im}(f_t)} \mu_I(f_t; y).$$

If we put these two results together, an obvious question arises.

Q_f: If the *total* image Milnor number is constant, do the instabilities coalesce?

To be more precise, if we have an $\mathscr{A}$-finite germ f and we consider a one-parameter unfolding given by f_t, could

$$\mu_I(f) = \sum_{y\in\operatorname{im}(f_t)} \mu_I(f_t; y)$$

imply that, actually, $\mu_I(f) = \mu_I(f_t; y_t)$?

In other terms, taking into account the conservation of the image Milnor number, if a family f_t is not excellent because we do not have isolated instabilities uniformly, does homology appear in the image? Equivalently, is $\beta_n\big(\operatorname{im}(f_t)\big) > 0$?

Indeed, we know that this is true for hypersurfaces with isolated singularities and their *total Milnor number* (see Figure 5.1). There is a proof of this non-coalescing result using a theorem of A'Campo in [Trá73b, Theorems A and B], where Lê Dũng Tràng uses an argument that involves the local geometric monodromy of the complex functions defining the hypersurfaces.

On the other hand, we can define a local geometric monodromy using a stabilisation of an $\mathscr{A}$-finite germ and the projection to the parameter space. More precisely, if we have an $\mathscr{A}$-finite germ f and a stabilisation $F = (f_s, s)$, the disentanglement of f coincides with the generic fiber $\pi^{-1}(s)$ of the projection

$$\begin{aligned} \pi : \operatorname{im}(F) &\to \mathbb{C} \\ (y, t) &\mapsto t \end{aligned}$$

This led Nuño-Ballesteros and the author to work on a generalization of the proof for hypersurfaces, together with Lê Dũng Tràng, to see if a general context includes images of germs in a convenient way.

$Q_{\mathfrak{X}}$: If the *total Milnor number* is constant along a family $\mathfrak{X} = \{(X_t, t)\}_t$, do the *singularities* coalesce?

Unfortunately, as we will see in Example 5.7.3, this general proof does not apply for our desired and evasive disentanglements.

5.2. Local monodromy

The adjective *local* in *local monodromy* makes reference to the Milnor-Lê fibration of a germ $f : (X, x) \to (\mathbb{C}, 0)$, where we can take a *monodromy*. In this section, we introduce these two concepts: *Milnor-Lê fibration* and *monodromy*.

5.2.1. Milnor-Lê fibrations

The well-known fibration theorem of Milnor, see [Mil68, Theorem 4.8], states that, for any non-constant germ $f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$, one can associate a smooth locally trivial fibration

$$\varphi_\varepsilon : S_\varepsilon(x) - f^{-1}(0) \longrightarrow \mathbb{S}^1,$$

for $1 \gg \varepsilon > 0$, induced by $f/|f|$, where $S_\varepsilon(x)$ is the sphere centered at x with radius ε and $\mathbb{S}^1$ is the circle of radius 1 of $\mathbb{C}$ centered at the origin. This is usually called the *Milnor fibration*.

Milnor also showed in [Mil68, Theorem 5.11] that the fibers of φ_ε are diffeomorphic to $\mathring{B}_\varepsilon(x) \cap f^{-1}(c)$, for $1 \gg |c| > 0$. Indeed, it is implicit in [Mil68, Section 11] that, when we have isolated critical point, the fibration given by

$$\psi_{\varepsilon,\eta} : \mathring{B}_\varepsilon(x) \cap f^{-1}\big(\mathring{D}_\eta - \{0\}\big) \longrightarrow \mathring{D}_\eta - \{0\}$$

is isomorphic to the Milnor fibration on $\mathbb{S}^1$, where $\mathring{B}_\varepsilon(x)$ is the open ball centered at the point x with radius ε. This fact was also reflected in his work in [Mil66].

Milnor's work was extended by Hamm in [Ham71], for germs $f : (X, x) \to (\mathbb{C}, 0)$ defined on a smooth X such that $X - f^{-1}(0)$ is non-singular.

Another milestone was [Trá77, Theorem 1.1], where Lê Dũng Tràng proved that, given a map germ $f : (X, x) \to (\mathbb{C}, 0)$ where X is analytic, the mapping

$$\tilde{\psi}_{\varepsilon,\eta} : \mathring{B}_\varepsilon(x) \cap X \cap f^{-1}\big(\mathring{D}_\eta - \{0\}\big) \longrightarrow \mathring{D}_\eta - \{0\}$$

induced by f is a topological fibration (proved equivalent to the Milnor fibration by Cisneros-Molina, Seade and Snoussi in [CMSS09]). After this theorem, the last fibration is usually called *Milnor-Lê fibration*.

This topic has been a very fruitful area of research, the reader is refereed to [Sea19] or [CMTS20, Chapter 6] to see more information about these kind of fibrations and related research.

5.2.2. Construction of the monodromy

Every time we have a fiber bundle over $\mathbb{S}^1$ we should think about its *monodromy* because it provides useful information of the fibration, and the ones above are not an exception. More precisely, they lead to a notion of *monodromy associated to f at x*.

We introduce the construction of a *geometric monodromy* in three different ways, but the reader should pay special attention to the last one because it is the one we use later.

Method I: Local triviality of a fibration

Roughly speaking, if we have a fiber bundle $\varphi : X \to \mathbb{S}^1$, we can build a *geometric monodromy* by taking the fiber $F := \varphi^{-1}(x_0)$ *to give a loop around* $\mathbb{S}^1$. This is mathematically materialized with the following steps:

1. Consider a point x_0 in $\mathbb{S}^1$ and a neighbourhood I_0 of x_0 where the fibration φ is trivial. Then, take a point $x_1 \neq x_0$ in I_0.

2. As there is a diffeomorphism ψ where $\varphi^{-1}(I_0) \xrightarrow[\sim]{\psi} F \times I_0$, the map
$$\begin{aligned} \tilde{h}_{x_0,x_1} : F \times \{x_0\} &\longrightarrow F \times \{x_1\} \\ (z, x_0) &\longmapsto (z, x_1) \end{aligned}$$
can be pushed to the total space X, giving $h_{x_0,x_1} := \psi^{-1} \circ \tilde{h}_{x_0,x_1} \circ \psi$ (see Figure 5.2).

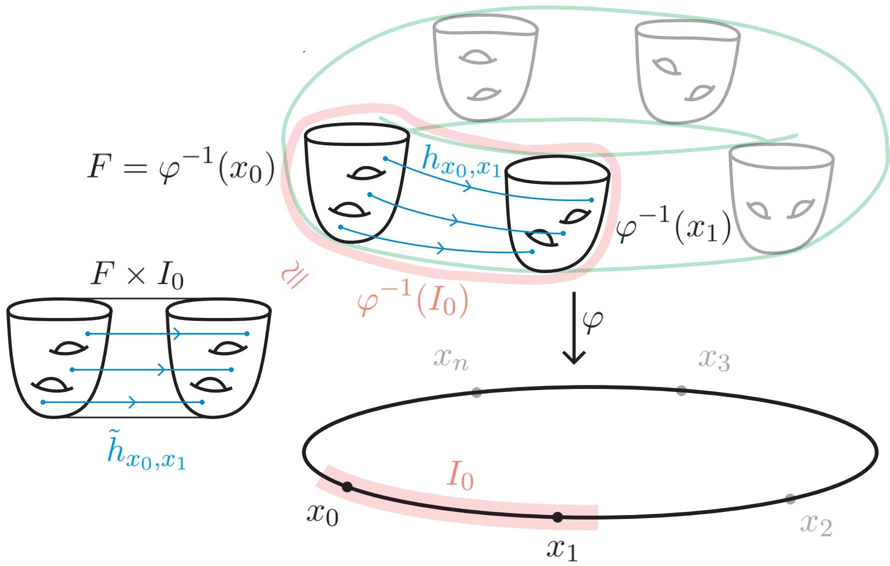


Figure 5.2: Representation of the first two steps.

3. We can repeat this process starting with x_i and picking a point x_{i+1} in a neighbourhood of x_i, always in the *same direction*. This gives the mappings $h_{x_i,x_{i+1}}$ in X.

4. As $\mathbb{S}^1$ is compact, the neighbourhoods can be taken in such a way that eventually we reach an interval I_n that contains the point x_0. Then, as one can see in Figure 5.3, we can consider $x_{n+1} = x_0$ and a *geometric monodromy* h is the composition

$$h \coloneqq h_{x_n,x_0} \circ \cdots \circ h_{x_0,x_1}.$$

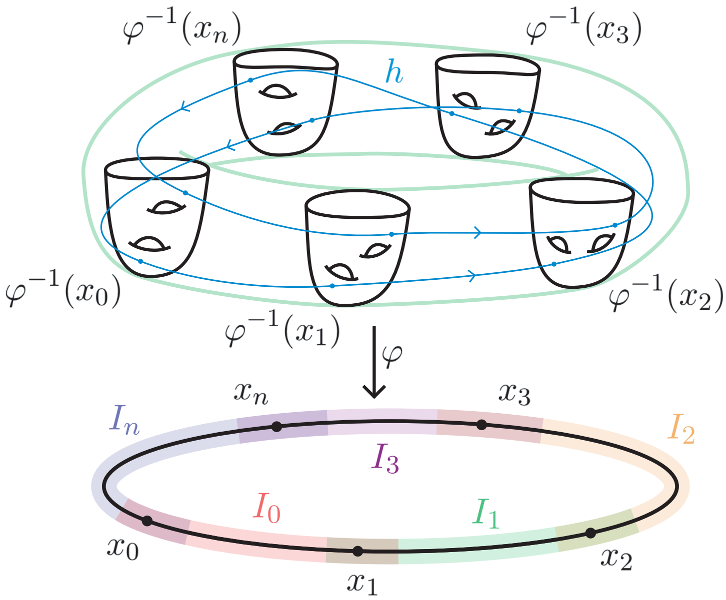


Figure 5.3: Representation of the last two steps

Method II: Difference between the product and the fibration

As we were saying, the *monodromy* furnishes a lot of information of a locally trivial fibration. Indeed, if we have a fiber bundle $\varphi : X \to \mathbb{S}^1$ with fiber $F \coloneqq \varphi^{-1}(x_0)$, it is a *geometric monodromy* what measures the difference between X and $F \times \mathbb{S}^1$: φ is C^0-equivalent to the fibration

$$\tilde{\pi} : {}^{F \times I}\big/_{\sim} \longrightarrow \mathbb{S}^1,$$

with $I = [0, 2\pi]$ and the relation $(x, 0) \sim \big(h(x), 2\pi\big)$ for some homeomorphism h and $\tilde{\pi}\big([x, t]\big) = t$. So, in fact, we can take as definition of *geometric monodromy* a homeomorphism h that appears in this setting.

Method III: Vector fields

Finally, let $\varphi : X \to \mathbb{S}^1$ be a proper stratified submersion. In particular, it is a locally trivial fibration such that the trivialisations are stratified homeomorphisms. One can build on X a *stratified vector field* v which lifts the unit vector field tangent to $\mathbb{S}^1$ (we will discuss how at the end of Section 5.4). The integration of this vector field defines an homeomorphism $h : F \to F$ of a fiber $F := \varphi^{-1}(x_0)$ of φ onto itself that we call *a geometric monodromy of* φ.

A geometric monodromy is not uniquely defined, but one can prove that its isotopy class is unique. Therefore, there is an isomorphism induced by a geometric monodromy of φ on the homology (or cohomology) of the fiber F called *the monodromy of* φ.

In the case of the Milnor-Lê fibration one often uses the terminology of *local geometric monodromy* and *local monodromy of* f *at the point* x.

We turn, now, to the importance of the monodromy of a fiber bundle.

In [Trá75], Lê Dũng Tràng gave a proof of the fact that, for any germ of complex analytic function

$$f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$$

having a critical point at x, there is a local geometric monodromy of f at x without fixed points.

By a well-known theorem of Lefschetz (see, for example, [Hat02, p. 179]), this result implies that the local monodromy of f at x has Lefschetz number equal to 0. In fact, in [A'C73, Theorem 1 bis], A'Campo showed that the Lefschetz number is zero in a more general situation, with heavy mathematical machinery, and attributed the proof of this theorem to Deligne. Having Lefschetz number equal to zero leads to interesting properties, as we will see in Section 5.8.

Let (X, x) be any germ of complex analytic space. Here, we give the following generalization of Lê's theorem:

Theorem 5.2.1. *Let* $f\colon (X, x) \to (\mathbb{C}, 0)$ *be a germ of complex analytic function such that* $f \in \mathfrak{m}^2_{X,x}$. *Then, there is a local geometric monodromy of* f *at* x *which does not fix any point.*

In particular, this proves:

Theorem 5.2.2 (see [A'C73, Theorem 1 bis])**.** *Let* $f\colon (X, x) \to (\mathbb{C}, 0)$ *be a germ of complex analytic function such that* $f \in \mathfrak{m}^2_{X,x}$. *Then, the local monodromy of* f *at* x *has Lefschetz number equal to* 0.

As in [Trá75], the proof of Theorem 5.2.1 uses the notion of *relative polar curve*, which is due essentially to Thom. The first step, when $X = \mathbb{C}^{n+1}$, is choosing a sufficiently small open neighbourhood U of x. For almost all linear function $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$, one has that the critical space of the restriction $(\ell, f)|_{U - f^{-1}(0)}$ is either always empty or a non-singular curve. When it is non-empty, we call the closure of the critical space of $(\ell, f)|_{U - f^{-1}(0)}$ the *relative polar curve* $\Gamma_\ell(f, x)$ *of* f *at* x *with respect to* ℓ.

The remarkable property of the relative polar curve is that, when f has a critical point at x, its image by $(\ell, f)|_U$ is empty or a curve that Thom called the *Cerf's diagram*, which is tangent to the axis of values of ℓ (see for example [TNBS20, Proposition 6.6.5]).

We show in Section 5.3 how to adapt this construction to the case that X is singular at x by taking a Whitney stratification. In addition, the condition that $f \in \mathfrak{m}_{X,x}^2$ is used here in order to prove that the Cerf's diagram is tangent to the ℓ-axis, replacing the hypothesis for the smooth case of having a critical point at x.

5.3. Relative polar curves

Let $f : (X, x) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function. We still call $f : X \to \mathbb{C}$ a representative of this germ. Let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of a sufficiently small representative X of (X, x) such that x is in the closure $\overline{S}_\alpha$ of all the strata S_α and the set of indices A is finite.

Using [TN17, Lemma 21], one can prove that there is a non-empty open Zariski subset Ω_α of the affine functions such that, for every ℓ in Ω_α, $\ell(x) = 0$ and the critical locus C_α of the restriction $(\ell, f)|_{S_\alpha - f^{-1}(0)}$ is either always empty or a non-singular curve. Then, the closure Γ_α of C_α in X is either empty or a reduced curve (see Figure 5.4).

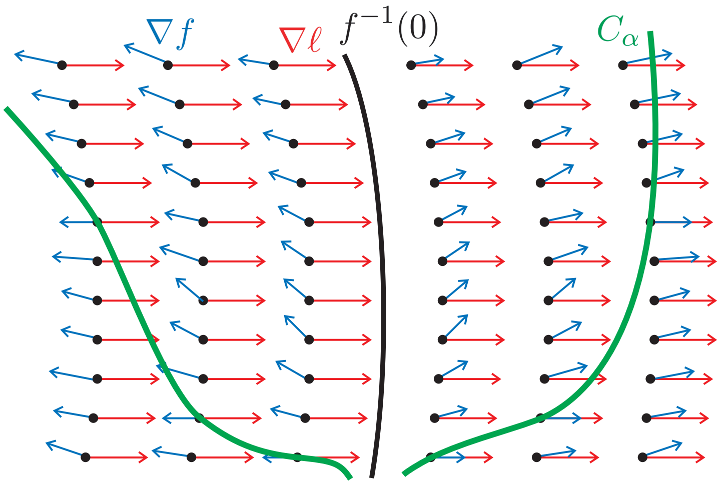


Figure 5.4: The polar curve appears wherever the gradients of f and ℓ are colinear.

Definition 5.3.1. For $\ell \in \cap_{\alpha \in A} \Omega_\alpha$, the union $\cup_{\alpha \in A} \Gamma_\alpha$ is either empty or a reduced curve. This curve is called the *relative polar curve* $\Gamma_\ell(f, \mathcal{S}, x)$ *of* f *at* x *relatively to* ℓ *and the stratification* $\mathcal{S}$ *of* X.

Remark 5.3.2. Notice that, if the stratum S_α has dimension one, the whole stratum S_α is critical and Γ_α is the closure $\overline{S}_\alpha$. In this case, if S_α is connected, Γ_α is a branch of the curve $\Gamma_\ell(f, \mathcal{S}, x)$ at x, i.e., an analytically irreducible curve at x.

Using [TN17, Lemma 21], we can also show that one can choose the sets Ω_α such that the restriction $(\ell, f)|_{\Gamma_\alpha}$ is finite for any $\alpha \in A$. Then, a theorem of Remmert implies that the image of Γ_α by (ℓ, f) is a curve, Δ_α, for any $\alpha \in A$ (see, for example, [BBH+98, p. 5]).

Definition 5.3.3. The union $\cup_{\alpha \in A} \Delta_\alpha$ is either empty or a reduced curve. This curve is called the *Cerf's diagram* $\Delta_\ell(f, \mathcal{S}, x)$ *of* f *at* x *relatively to* ℓ *and the stratification* $\mathcal{S}$.

Note. When the stratification $\mathcal{S}$ is fixed, we shall speak of the *relative polar curve* $\Gamma_\ell(f, x)$ and the *Cerf's diagram*[1] $\Delta_\ell(f, x)$ without mentioning the stratification $\mathcal{S}$. But the reader must be aware that the notion of polar curve and Cerf's diagram depends on the choice of the stratification.

We shall go back and forth between the case $(\mathbb{C}^{n+1}, x)$ and the general case of germs of reduced analytic spaces (X, x) and compare them to generalize what we have in [Trá75]. For example, if $(X, x) = (\mathbb{C}^{n+1}, x)$, we can consider a Whitney stratification which has only one stratum.

In [Trá75, p. 418], we have seen that the emptiness of $\Gamma_\ell(f, x)$ means that the Milnor fiber of f at x is diffeomorphic to the product of the Milnor fiber of $f|_{\{\ell=0\}}$ at x with an open disc, hence, the local geometric monodromy of f at x is induced by the product of the local geometric monodromy of $f|_{\{\ell=0\}}$ at x and the identity of the open disc.

Also, for a germ of complex analytic function $f : (X, x) \to (\mathbb{C}, 0)$, in general, we may suppose that the hyperplane $\{\ell = 0\}$ is transverse to all the strata of the Whitney stratification $\mathcal{S}$ and it induces a Whitney stratification on $X \cap \{\ell = 0\}$. Then, using the same arguments of [TN17, Remark 24], we can prove the following:

Proposition 5.3.4. *If, for a general linear form* ℓ *at* x*, the relative polar curve* $\Gamma_\ell(f, x)$ *is empty, there is a stratified homeomorphism between the Milnor fiber of* f *at* x *and the product of an open disc with the Milnor fiber of the restriction* $f|_{X \cap \{\ell=0\}}$ *at* x.

The proof of this proposition is based on the techniques Mather used to prove the Thom-Mather first isotopy lemma, see [Mat12, GWdPL76]. We will outline these techniques in Section 5.4 and use them later.

Now, observe that, when $(X, x) = (\mathbb{C}^{n+1}, x)$, the point x is a critical point of f if and only if $f \in \mathfrak{m}^2_{\mathbb{C}^{n+1},x}$. In the case of a germ of complex analytic function on (X, x), the hypothesis $f \in \mathfrak{m}^2_{X,x}$ replaces the condition that f is critical at x. In fact, a key result for the proof of Theorem 5.2.1 is:

Proposition 5.3.5. *For a sufficiently general linear form* ℓ*, if* $f \in \mathfrak{m}^2_{X,x}$*, the Cerf's diagram* $\Delta_\ell(f, x)$ *is tangent at the point* $(0, 0)$ *to the first axis, the image by* (ℓ, f) *of* $\{f = 0\}$.

Proof. Of course, we have a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ on a sufficiently small representative of the germ (X, x). We may assume that x is in the closure of all the strata.

It is enough to prove the proposition for the image Δ_α of Γ_α by (ℓ, f), for each $\alpha \in A$.

In [Trá75, Section 2], it was considered that ℓ is a coordinate of $\mathbb{C}^{n+1}$ to compare easily the growth of f and ℓ along a component of the Cerf's diagram. We are going to

[1] Observe that the notation used for the Cerf's diagram is not the same notation we use for the discriminant of a map.

give a similar proof for the $(\mathbb{C}^{n+1}, x)$ case for any general linear form, and generalize it twice to reach our current context.

Suppose that $(X, x) = (\mathbb{C}^{n+1}, x)$, for our purpose ℓ can be expressed as

$$\ell(v) = \langle v, a\rangle = \sum_{i=1}^{n+1} v_i \overline{a_i},$$

and we can assume that $\|a\| = 1$. Let us define H as the kernel of ℓ and, then, any vector of $\mathbb{C}^{n+1}$ can be written as a sum of a vector of H and a multiple of the vector a (note that a is the unitary normal of H).

Now, we can take a parametrization $p(t)$ of Γ_α and compare the growths of f and ℓ there. Using De l'Hôpital's rule and identifying ℓ with its differential we have

$$\lim_{t\to 0} \frac{(f\circ p)(t)}{\big(\ell\circ p\big)(t)} = \lim_{t\to 0} \frac{(f\circ p)'(t)}{(\ell\circ p)'(t)} = \lim_{t\to 0} \frac{D_{p(t)} f\big(p'(t)\big)}{\ell\big(p'(t)\big)}.$$

Furthermore, we can decompose $p'(t)$ as the sum of a vector of H, say $p'_H(t)$, and λa, hence,

$$\lim_{t\to 0} \frac{D_{p(t)} f\big(p'(t)\big)}{\ell\big(p'(t)\big)} = \lim_{t\to 0} \frac{D_{p(t)} f\big(p'_H(t)\big) + \lambda D_{p(t)} f(a)}{\ell\big(p'_H(t)\big) + \lambda \ell(a)}.$$

Finally, we know that Df and ℓ are colinear along $p(t)$, and we have assumed $\ell(a) = \|a\| = 1$, therefore

$$\begin{aligned}\lim_{t\to 0} \frac{D_{p(t)} f\big(p'_H(t)\big) + \lambda D_{p(t)} f(a)}{\ell\big(p'_H(t)\big) + \lambda \ell(a)} &= \lim_{t\to 0} \frac{D_{p(t)} f\,(a)}{\ell(a)} \\ &= D_x f(a).\end{aligned}$$

At this point, we see where the condition of $f \in \mathfrak{m}^2_{X,x}$ appears, because this last term is zero in that case. This proves the tangency of the statement in this context.

If we want the same result on $(X, x) \subset (\mathbb{C}^N, x)$, (X, x) regular at x, the main problem is that ℓ is defined in X, and we cannot work with such a vector a and space H. What we can do is to extend ℓ to the ambient space and work on the tangent bundle of X. Hence, we can choose a linear function $L : \mathbb{C}^N \to \mathbb{C}$ such that $L|_X = \ell$ and H' as the kernel of L. By genericity, $T_x X$ is not contained in H' so $H' \cap T_x X$ is a hyperplane of $T_x X$, say H. This happens, nearby x, for every tangent space along a parametrization of Γ_α (in this case there is only one stratum), so we can reproduce the computations we did before.

Lastly, if (X, x) is general, we can still extend ℓ but we cannot work with the tangent bundle of X any more (e.g., if (X, x) is a Whitney umbrella at x even x is a stratum by itself). To avoid this complication, we shall find a convenient hyperplane of $\mathbb{C}^N$ for the role of H' and, then, work with the extension of f when needed. From now on, we will work with a stratum S_α, or its adherence, but for the sake of the similarity with the previous cases we will call Y the closure $\overline{S_\alpha}$.

Therefore, our first step is to find an hyperplane to work with. For this purpose, consider the (projective) conormal space $C(Y)$ of Y in $\mathbb{C}^N$, this is given by the closure in $Y \times \check{\mathbb{P}}^{N-1}$ of the space

$$\left\{(q, H') \,|\, (q, H') \in Y^{reg} \times \check{\mathbb{P}}^{N-1} : T_q Y^{reg} \subset H'\right\},$$

together with the conormal map $\nu : C(Y) \to Y$. It is a classical fact (see [Tei82, II.4.1]) that $\dim \nu^{-1}(x) \leq N-2$ or, being more specific, there is a hyperplane H' outside $\nu^{-1}(x)$ and, by continuity, outside every fiber of ν over a neighbourhood of x in Y.

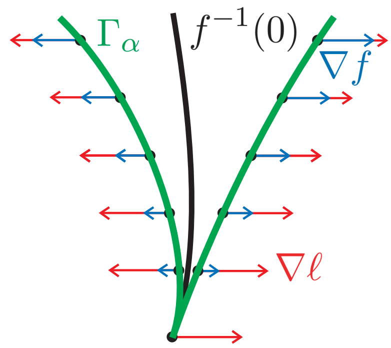
Therefore, consider a linear form $L : \mathbb{C}^N \to \mathbb{C}$ with such a hyperplane H' as kernel and define ℓ to be $L|_Y$. Furthermore, since $f \in \mathfrak{m}^2_{Y,x}$, we can take an extension $F : (\mathbb{C}^N, x) \to (\mathbb{C}, 0)$ of f, such that $F \in \mathfrak{m}^2_{\mathbb{C}^N,x}$.

Finally, as we have done before, consider a parametrization $p(t)$ of a branch of Γ_α and compare the growths of $f\big(p(t)\big)$ and $\ell\big(p(t)\big)$. To do so, define a_t to be the unitary normal of the hyperplane $H_t := H' \cap T_{p(t)}Y$ in $T_{p(t)}Y$, well defined by the previous election of H', and let a_0 be its limit. We can, now, proceed as before and finish the computation with F, i.e.,

$$\begin{aligned}\lim_{t\to 0} \frac{(f\circ p)(t)}{(\ell \circ p)(t)} &= \lim_{t\to 0} D_{p(t)} f(a_t)\\ &= \lim_{t\to 0} D_{p(t)} F(a_t)\\ &= D_x F(a_0)\\ &= 0.\end{aligned}$$

Note that the election of H', for S_α, was made in an open set. Since we have only a finite number of strata to which x is adherent, we can take a common H' for every S_α and repeat the computation. QED

Note. The proof of this result is nothing more than a technical and proper way of expressing a simple idea: considering that $D(\ell, f) = (\nabla\ell, \nabla f)^T$, the images of $D(\ell, f)$ take always the form $(r, *)$ and the limit over the points on a (branch of a) polar curve Γ_α, which was parametrized by $p(t)$ in the proof, has image $(r, 0)$.



Observe that, in the case of a general (X, x), one has to take ℓ generic enough to assure that the argument works. In other words, the generic ℓ that gave the definition of relative polar curve in Definition 5.3.1 could be not generic enough. This is where the conormal space $C(Y)$ plays its role.

Associated with the Cerf's diagram we have the *carousel* (see Figure 5.5), a construction which again appears in [Trá75, p. 418]. This is a vector field ω over a small enough solid torus $D \times \partial D_\eta$ centered at the origin in $\mathbb{C} \times \mathbb{C}$ such that:

(i) its projection onto the second component gives a tangent vector field over ∂D_η of length η and positive direction (called in [Trá75] the *unitary vector field* of ∂D_η),

(ii) its restriction to $\{0\} \times \partial D_\eta$ is indeed the unitary vector field,

(iii) for every component of the Cerf's diagram with reduced equation $\delta_\alpha = 0$, ω is tangent to every $\delta_\alpha = \varepsilon$ with $\varepsilon \in \mathbb{C}$ small enough, and

(iv) the only integral curve that is closed after a loop in ∂D_η is $\{0\} \times \partial D_\eta$.

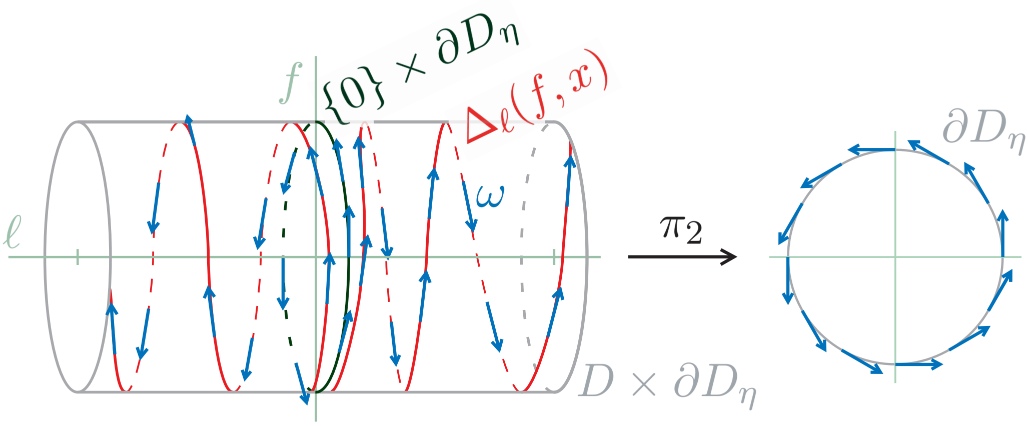


Figure 5.5: Representation of a carousel ω.

We have a carousel ω by Proposition 5.3.5 and Lemma 5.7.1, and we can use techniques of stratification theory to lift ω and obtain a *stratified vector field* on X which is *globally integrable*. As the carousel projects to a unit vector field tangent to $\mathbb{S}^1$ by Item (i), the integral curves of this vector field define a local geometric monodromy of f at x and of its restriction to $X \cap \{\ell = 0\}$ (recall Method III to construct a geometric monodromy). By the condition given in Item (iv), the fixed points of the monodromy of f can appear only on $X \cap \{\ell = 0\}$. Thus, the proof of Theorem 5.2.1 will follow by induction on the dimension of X at x (see Figure 5.6).

We cover the techniques to lift ω in the next section.

5.4. Lifting vector fields

The construction of the Milnor-Lê fibration of a complex analytic function $f : (X, x) \to (\mathbb{C}, 0)$ when X is a complex analytic space is a consequence of the Thom-Mather first isotopy lemma. The strategy to prove Theorem 5.2.1 is to take a generic linear form ℓ on the ambient space of (X, x) and consider the map $\Phi = (\ell, f) : (X, x) \to (\mathbb{C}^2, 0)$. We want to trivialize this map in such a way that its composition with the projection onto the second component $\pi_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ gives the Milnor-Lê fibration of f and its restriction to $\big(X \cap \ell^{-1}(0), x\big)$ gives the Milnor-Lê fibration of the restriction $f : \big(X \cap \ell^{-1}(0), x\big) \to (\mathbb{C}, 0)$. This would allow us to use an induction process, as in [Trá75].

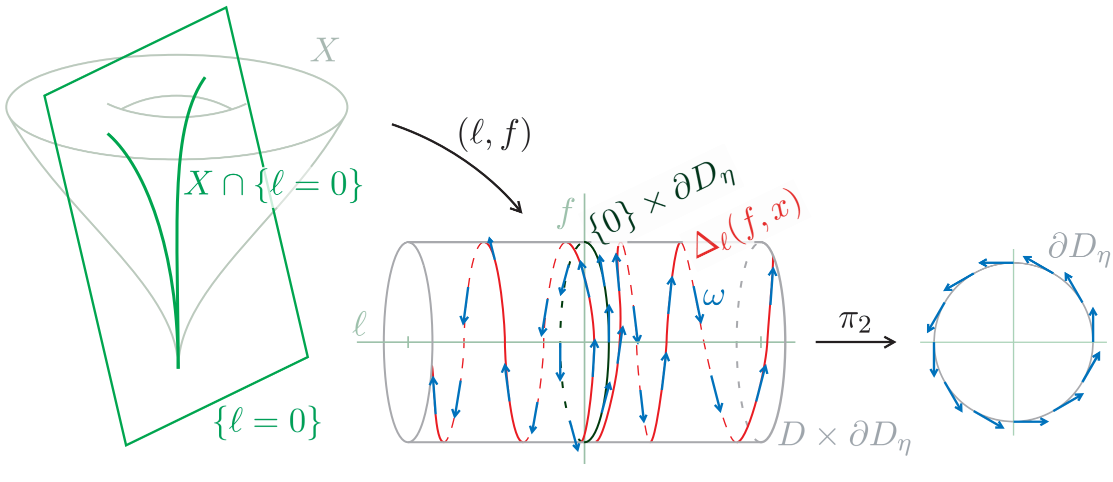


Figure 5.6: Ingredients of the proof of Theorem 5.2.1.

One could think that this could be done just by using the Thom-Mather second isotopy lemma. Unfortunately, this seems not possible and we are forced to use some of the ingredients in the proof of the isotopy lemmas, such as *controlled tube systems* or *controlled stratified vector fields*, in order to construct a lifting of the vector field which fits into our problem.

For the sake of completeness, we include in this section all the definitions and main results that we need for that purpose. However, the impatient[2] reader can jump to Definition 5.4.9 to understand Corollary 5.4.13, which is what we actually need.

Instead of the original proof of the isotopy lemmas by Mather in [Mat12], we follow the notations and statements of [GWdPL76, Chapter II], where the reader can find more details and the proofs of all the results.

We recall that a *stratified vector field* on a stratified set X of a smooth manifold N is a map $\xi : X \to TN$ tangent to each stratum S_α of X and smooth on S_α, but ξ might not be continuous. We now give the definitions of *weakly controlled tube system* and *weakly controlled stratified vector field*.

Definition 5.4.1 (see [GWdPL76, Definition II.1.4])**.** If X is a submanifold of N, a tube at X is a quadruple $T = (E, \pi, \rho, e)$ where $\pi : E \to X$ is a smooth vector bundle, $\rho : E \to \mathbb{R}$ is a quadratic function of a Riemann metric on E that vanishes on the zero section and $e : \big(E, \zeta(X)\big) \to (N, X)$ a germ along $\zeta(X)$ of a local diffeomorphism, commuting with the zero section $\zeta : X \to E$ so that $e \circ \zeta$ along X is the inclusion $X \subset N$.

If X is a Whitney stratified subset of a manifold N, a *tube system for the stratification* consists of a tube for every strata.

[2] And sane.

Definition 5.4.2 (see [GWdPL76, Definition II.2.5]). A tube system $\mathcal{T} = \{T_\alpha\}_{\alpha\in A}$, with $T_\alpha = (E_\alpha, \pi_\alpha, \rho_\alpha, e_\alpha)$, for a Whitney stratification $\{S_\alpha\}_{\alpha\in A}$ of some subset X of a manifold N is *weakly controlled* if the relation

$$\left(\pi_\alpha \circ e_\alpha^{-1}\right) \circ \left(\pi_{\alpha'} \circ e_{\alpha'}^{-1}\right) = \left(\pi_\alpha \circ e_\alpha^{-1}\right), \qquad \alpha, \alpha' \in A,$$

holds for every pair of tubes $(T_\alpha, T_{\alpha'})$ where the composition makes sense.

Remark 5.4.3. The notion of weakly controlled tube system for a stratification is not a strange thing to ask. Actually, any given Whitney stratification admits a weakly controlled tube system (see [GWdPL76, Corollary II.2.7]).

Definition 5.4.4 (see [GWdPL76, Definition II.3.1]). If we have a tube system for a Whitney stratification of X and ξ is a stratified vector field on X, we shall say that ξ is a *weakly controlled vector field* if

$$D\left(\pi_\alpha \circ e_\alpha^{-1}\right) \circ \xi = \xi \circ \left(\pi_\alpha \circ e_\alpha^{-1}\right)$$

holds for every tube, using the notation of Definition 5.4.2.

Next, we give the control conditions relative to a stratified mapping (recall Definition 1.3.6).

Definition 5.4.5 (see [GWdPL76, Definition II.2.5]). Let $f : N \to N'$ be a smooth map and $X \subset N$ and $X' \subset N'$ two stratified sets such that $f(X) \subset X'$ and the induced map $f| : X \to X'$ is a stratified map. Assume also that we have a tube system $\mathcal{T} = \{T_\alpha\}_{\alpha\in A}$ for the stratification $\{S_\alpha\}_{\alpha\in A}$ of X and a tube system $\mathcal{T}' = \{T'_\beta\}_{\beta\in B}$ for the stratification $\{S'_\beta\}_{\beta\in B}$ of X'. Then, we say that $\mathcal{T}$ is *controlled* over $\mathcal{T}'$ if

(i) $\mathcal{T}$ is weakly controlled,

(ii) $f \circ \left(\pi_\alpha \circ e_\alpha^{-1}\right) = \left(\pi_\beta \circ e_\beta^{-1}\right) \circ f$, for every S_α mapping into S'_β, and

(iii) $\left(\rho_\alpha \circ e_\alpha^{-1}\right) \circ \left(\pi_{\alpha'} \circ e_{\alpha'}^{-1}\right) = \left(\rho_\alpha \circ e_\alpha^{-1}\right)$ holds for every pair $(T_\alpha, T_{\alpha'})$ such that $f(S_\alpha \cup S_{\alpha'}) \subseteq S'_\beta$ for some S'_β in X'.

In fact, the Thom condition (see Definition 1.3.7) ensures the existence of a controlled tube system as follows:

Theorem 5.4.6 (see [GWdPL76, Theorem II.2.6]). *Let N, N', X, X', f as in Definition 5.4.5 and assume $f| : X \to X'$ is a Thom map. Then, each weakly controlled tube system $\mathcal{T}'$ of X' has a tube system $\mathcal{T}$ of X controlled over $\mathcal{T}'$.*

We also have control conditions relative to a stratified mapping for stratified vector fields.

Definition 5.4.7 (see [GWdPL76, Definition II.3.1]). Let $X, X', \mathcal{T}$, and $\mathcal{T}'$ be as in Definition 5.4.5. Assume that we have ξ and ξ' stratified vector fields on X and X', respectively. Then, we say that ξ is *controlled* over ξ' if *(i)* ξ is weakly controlled,

(ii) $D\big(\rho_\alpha \circ e_\alpha^{-1}\big) \circ \xi = 0$ holds for every T_α, and

(iii) $D\big(\rho_\alpha \circ e_\alpha^{-1}\big) \circ \xi|_{f^{-1}S'_\beta} = 0$ for every S_α mapping into S'_β.

Again, the Thom condition is the key point to lift any weakly controlled vector field in the target to a controlled vector field in the source:

Theorem 5.4.8 (see [GWdPL76, Theorem II.3.2]). *Let N, N', X, X', f as in Definition 5.4.5 and assume $f| : X \to X'$ is a Thom map. Let $\mathcal{T}$ and $\mathcal{T}'$ be tube systems of the stratifications of X and X', respectively, such that $\mathcal{T}$ is controlled over $\mathcal{T}'$. Then, any weakly controlled vector field ξ' on X' lifts to a stratified vector field ξ which is controlled over ξ'.*

The last ingredient is about integrability of stratified vector fields. Specifically, if we have a stratified vector field ξ on X and we integrate it on every stratum S_α we have a smooth flow $\theta_\alpha : D_\alpha \to S_\alpha$, where $D_\alpha \subseteq \mathbb{R} \times S_\alpha$ is the maximal domain of the integration, which contains $\{0\} \times S_\alpha$. Setting D as the union of every D_α, we obtain a map $\theta : D \to X$ that is not necessarily continuous.

Definition 5.4.9 (see [GWdPL76, Definition II.4.3]). With the notation of the above paragraph, if θ is continuous on a neighbourhood of $\{0\} \times X$ we say that ξ is *locally integrable.* Furthermore, if $D = \mathbb{R} \times X$, we say that ξ is *globally integrable.*

It is here where the control conditions over the vector fields play their role:

Theorem 5.4.10 (see [GWdPL76, Theorem II.4.6]). *Let N, N', X, X', f as in Definition 5.4.5. Assume also that X is locally closed in N. If ξ and ξ' are stratified vector fields on X and X', respectively, and ξ is controlled over ξ' with respect to some tube system $\mathcal{T}$ of X, then ξ is locally integrable if ξ' is so.*

Theorem 5.4.11 (see [GWdPL76, Lemma II.4.8]). *Let N, N', X, X', f as in Definition 5.4.5. Assume also $f| : X \to X'$ is proper. If ξ and ξ' are stratified vector fields on X and X', respectively, and ξ is locally integrable, then ξ is globally integrable if ξ' is so.*

So, to summarize, if we combine Theorems 5.4.6, 5.4.8, 5.4.10 and 5.4.11 we get:

Corollary 5.4.12. *Let N, N', X, X', f as in Definition 5.4.5 and assume $f| : X \to X'$ is a Thom proper map. If we have a weakly controlled tube system with a weakly controlled vector field ξ' on X' such that it is globally integrable, it lifts to a globally integrable vector field on X.*

Corollary 5.4.13. *Let $f : N \to N'$ a smooth map and let $X \subset N$ be a Whitney stratified subset such that $f| : X \to N'$ is a proper stratified submersion. If ξ' is a globally integrable smooth vector field on N', it lifts to a globally integrable vector field on X.*

Corollary 5.4.13 is a consequence of Corollary 5.4.12 since the Thom condition is satisfied in this case (see [GWdPL76, Corollary II.3.3]). We also remark that these two corollaries, among other things, are used in [GWdPL76] to prove the Thom-Mather isotopy lemmas.

Finally, we show how Corollary 5.4.13 can be used to construct a local geometric monodromy of a function as we said in Method III.

Recall that, given a complex analytic function $f\colon (X,x) \to (\mathbb{C},0)$, there exist ε and η with $0<\eta \ll \varepsilon \ll 1$ such that

$$f\colon X \cap B_\varepsilon \cap f^{-1}(\partial D_\eta) \to \partial D_\eta \tag{5.1}$$

is a proper stratified submersion, for some Whitney stratification on the source and the trivial stratification on ∂D_η (see the proof of [Trá77, Theorem 1.1]). By the Thom-Mather first isotopy lemma, Equation (5.1) is a locally trivial C^0-fibration with fiber F.

In fact, we have something more: we take on ∂D_η the vector field of constant length η and positive direction and, by Corollary 5.4.13, this vector field can be lifted to a stratified vector field ξ on the source, which is globally integrable, and its flow gives a geometric monodromy.

In the next section, we show that, instead of a Euclidean ball B_ε, we can take a convenient *polydisc*, which is better to proceed with the induction hypothesis we announced at the end of Section 5.3. Furthermore, we will see in Section 5.6 that $\Phi = (\ell, f)$ is a Thom map, so we can factorize the lifting of the vector field on ∂D_η by Φ (recall Figure 5.6).

5.5. Privileged polydiscs

In [Trá75], instead of a usual Milnor ball B for a function $f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$, it is considered a *privileged polydisc* $\Delta = D_1 \times \cdots \times D_{n+1}$ with respect to some generic choice of coordinates $z_1, \ldots, z_{n+1}$ in $\mathbb{C}^{n+1}$. Here, we show how to adapt this notion to the case of a function $f : (X,x) \to (\mathbb{C},0)$ on a complex analytic set X.

Assume that $\dim(X,x) = n+1$ and that (X,x) is embedded in $(\mathbb{C}^N, 0)$. We take a representative $f : X \to \mathbb{C}$ and a Whitney stratification in X and $\mathbb{C}$ such that $f : X \to \mathbb{C}$ is a stratified function. We say that $z_1, \ldots, z_N$ are *generic coordinates* if, for each $i = 0, \ldots, n$, the $(N-i)$-plane H^i through the origin given by $\{z_1 = \cdots = z_i = 0\}$ is transverse to all the strata of X except, perhaps, the stratum $\{x\}$.

We consider the set $X^i = \pi_i(X \cap H^i) \subset \mathbb{C}^{N-i}$, where π_i is the projection onto the last $N-i$ coordinates, with the induced stratification and the function $f^i : X^i \to \mathbb{C}$ given by

$$f^i(z_{i+1}, \ldots, z_N) = f(0, \ldots, 0, z_{i+1}, \ldots, z_N).$$

A *polydisc* centered at 0 in $\mathbb{C}^N$ is a set of the form $\Delta = D_1 \times \cdots \times D_n \times B$, where $D_1, \ldots, D_n \subset \mathbb{C}$ are discs and $B \subset \mathbb{C}^{N-n}$ is a ball centered at x. We also denote by $\Delta^i = D_{i+1} \times \cdots \times D_n \times B$ the corresponding polydisc in $\mathbb{C}^{N-i}$. Each polydisc Δ^i is considered with the obvious Whitney stratification given by taking all combinations of products of interiors and boundaries on the discs and the ball (see [Trá75, p. 411]).

Observe that a polydisc has a ball in the product of its definition. This ball is necessary to have control on the codimension of X. More precisely, as we will work with the sets X^i and $X^i \cap \Delta^i$, we need to stop taking sections as soon as X^i is a curve and, at that point, we take a ball that completes the product structure we want to find (the so-called polydiscs).

Definition 5.5.1. We say that Δ is a *privileged polydisc* if, for any smaller polydisc $\Delta' \subset \Delta$ centered at 0 in $\mathbb{C}^N$, all the strata of $(\Delta')^i$ are transverse to all the strata of X^i, for all $i = 0, \dots, n$.

For each privileged polydisc Δ, $X^i \cap \Delta^i$ has an induced Whitney stratification by transversality. Also, the function $f^i : X^i \cap \Delta^i \to \mathbb{C}$ has isolated critical value at the origin in $\mathbb{C}$ by the curve selection lemma, so $(f^i)^{-1}(b)$ is transverse to all the strata of $X^i \cap \Delta^i$ for all $b \in \mathbb{C}^*$ small enough. In particular, there exists $\eta > 0$ small enough such that

$$f^i : X^i \cap \Delta^i \cap (f^i)^{-1}(\partial D_\eta) \longrightarrow \partial D_\eta$$

is a proper stratified submersion and, hence, a locally C^0-trivial fibration homotopic to a Milnor fibration with a homotopy which preserves the fibres. This follows from the Thom-Mather first isotopy lemma and that privileged polydiscs are good neighbourhoods relatively to $\{f = 0\}$ in Prill's sense (see [Pri67]), see the end of [Trá75, Section 1] for more details. In fact, this is the original definition of privileged polydisc in [Trá75] in the case $X = \mathbb{C}^{n+1}$.

The existence of privileged polydiscs is proved in the next lemma.

Lemma 5.5.2. *Any small enough polydisc is privileged.*

Proof. We show, by induction on $i = 0, \dots, n$, that f^i has a privileged polydisc Δ^i.

The case $i = n$ is obvious since a privileged polydisc is nothing but a Milnor ball. Assume f^i has a privileged polydisc Δ^i. We shall find a disc D_ε such that $D_\varepsilon \times \Delta^i$ is a privileged polydisc for f^{i-1}. To find it, we use the function $\rho : \mathbb{C}^{N-i+1} \to \mathbb{R}$ given by $\rho(z) = |z_i|$.

By the curve selection lemma, we can find $\varepsilon > 0$ such that, for any $0 < \varepsilon' \leq \varepsilon$, $\partial D_{\varepsilon'} \times \mathbb{C}^{N-i}$ is transverse to each stratum of X^{i-1}.

Consider the polydisc $D_{\varepsilon'} \times \left(\Delta^i\right)'$, for a polydisc $\left(\Delta^i\right)'$ contained in Δ^i and $\varepsilon' \leq \varepsilon$. We have two types of strata: $\mathring{D}_{\varepsilon'} \times R_\alpha$ and $\partial D_{\varepsilon'} \times R_\alpha$, for some stratum R_α of $\left(\Delta^i\right)'$. On the other hand, if we consider a stratum S_β of X^{i-1}, and we take the hyperplane section to get X^i, it gives the stratum S'_β of X^i.

By induction hypothesis, R_α is transverse to S'_β, that is,

$$T_z R_\alpha + T_z S'_\beta = \mathbb{C}^{N-i}, \tag{5.2}$$

for all $z \in R_\alpha \cap S'_\beta$. This obviously implies that

$$\mathbb{C} \times T_z R_\alpha + T_{(t,z)} S_\beta = \mathbb{C} \times \mathbb{C}^{N-i},$$

which gives the transversality between $\mathring{D}_{\varepsilon'} \times R_\alpha$ and S_β at (t,z).

Moreover, the choice of ε implies that

$$T_t\partial D_{\varepsilon'} \times \mathbb{C}^{N-i} + T_{(t,z)}S_\beta = \mathbb{C} \times \mathbb{C}^{N-i},$$

for all $(t,z) \in (\partial D_{\varepsilon'} \times \mathbb{C}^{N-i}) \cap (V \times S_\beta)$. Therefore, any vector $(u,v) \in \mathbb{C} \times \mathbb{C}^{N-i}$ can be written as $(u,v) = (u_1,v_1) + (u_2,v_2)$, for some $(u_1,v_1) \in T_t\partial D_{\varepsilon'} \times \mathbb{C}^{N-i}$ and $(u_2,v_2) \in T_{(t,z)}S_\beta$. If $z \in R_\alpha$, we also have, by Equation (5.2), that the previous v_1 can be written as $w_1 + w_2$, with $w_1 \in T_zR_\alpha$ and $w_2 \in T_zS'_\beta$. We get

$$(u,v) = (u_1,w_1) + (0,w_2) + (u_2,v_2),$$

with $(u_1,w_1) \in T_t\partial D_{\varepsilon'} \times T_zR_\alpha$ and $(0,w_2)+(u_2,v_2) \in T_{(t,z)}S_\beta$. This shows that $\partial D_{\varepsilon'} \times R_\alpha$ is also transverse to S_β. QED

Remark 5.5.3. We see in the proof of Lemma 5.5.2 that the choice of the radius of each disc of Δ is independent of the radii of the other discs. The reason of this independence is that we were asking that $\partial D_{\varepsilon'} \times \mathbb{C}^{N-i}$ has to be transverse to each stratum of X^{i-1} at any point instead of being transverse only at points on $X^{i-1} \cap \mathbb{C} \times \Delta^i$, which would have given $D_{\varepsilon'}$ a relation with Δ^i that restricts it.

5.6. Thom condition for Φ

Now, we show that the mapping $\Phi = (\ell, f) : (X,x) \to (\mathbb{C}^2, 0)$, for a generic linear form ℓ, can be stratified in such a way that it satisfies the Thom condition.

First, we recall the fact that a function always satisfies the Thom condition (see [BMM94, Theorem 4.2.1], compare to [GWdPL76, Corollary II.3.3] or the work of Hironaka in [Hir77]):

Theorem 5.6.1. *Let X be a complex analytic subspace of an open set of $\mathbb{C}^N$, $f : (X,x) \to (\mathbb{C},0)$ be a germ of complex analytic mapping and complex stratifications $\mathcal{S}$ and $\mathcal{T}$ respectively in the source and the target that stratify a representative of the germ f. If $\mathcal{S}$ is Whitney regular, the stratification of f has the Thom property.*

We consider the mapping $\Phi = (\ell, f) : (X,x) \to (\mathbb{C}^2, 0)$, where ℓ is a generic linear form. We take a small enough representative $\Phi : X \to W$, where W is an open neighbourhood of 0 in $\mathbb{C}^2$, such that the Cerf's diagram $\Delta_\ell(f,x)$ is a closed analytic subset of W. Since the Whitney stratification $\mathcal{S} = \{S_\alpha\}$ of X is analytic, the set $\Sigma(\Phi)$ of critical points of Φ in the stratified sense is either empty or it is analytic of dimension ≤ 1, by the genericity of ℓ. We remark that $\Sigma(\Phi)$ contains the relative polar curve $\Gamma = \Gamma_\ell(f,x)$, although $\Sigma(\Phi)$ may have other components contained in $f^{-1}(0)$.

The image $\mathcal{D} = \Phi\big(\Sigma(\Phi)\big)$[3], if not empty, contains $\Delta_\ell(f,x)$, although it may also contain the axis $\{v = 0\}$. We consider in W the Whitney stratification $\mathcal{T}$ given by the

[3] We use a different notation to denote the discriminant of Φ to avoid confusion.

strata $W-\mathcal{D}, \mathcal{D}-\{0\}$ and $\{0\}$. In order to have a stratified mapping, we have to change the stratification in X. We define $\mathcal{S}'$ as the family of sets of the form $S_\alpha \cap \Phi^{-1}(T) \cap \Sigma(\Phi)$ and $S_\alpha \cap \Phi^{-1}(T) - \Sigma(\Phi)$, where $S_\alpha \in \mathcal{S}$ and $T \in \mathcal{T}$.

With the purpose of preserving the Whitney regularity, we need the following lemma:

Lemma 5.6.2. *Let $\varphi : V \to W$ be a smooth mapping, where $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are open subsets. Assume that $\mathcal{S}$ is a Whitney stratification of a subset $X \subset V$ such that for all $S \in \mathcal{S}$, $\varphi|_S : S \to W$ is a submersion and that $\mathcal{T}$ is a Whitney stratification of W. Then,*

$$\mathcal{S}' = \{S \cap \varphi^{-1}(T) : \ S \in \mathcal{S}, \ T \in \mathcal{T}\}$$

is also a Whitney stratification of X.

Proof. Take a pair of strata $A = S \cap \varphi^{-1}(T)$ and $B = S' \cap \varphi^{-1}(T')$, with $S, S' \in \mathcal{S}$ and $T, T' \in \mathcal{T}$. We factorize φ as the composition

$$V \xrightarrow{\ i\ } G(\varphi) \xrightarrow{\ \pi_2\ } W \ ,$$

where $G(\varphi) \subset V \times W$ is the graph of φ, i is the diffeomorphism given by $i(v) = \big(v, \varphi(v)\big)$ and $\pi_2(v, w) = w$. Therefore, B is Whitney regular over A in V if, and only if, $i(B)$ is Whitney regular over $i(A)$ in $G(\varphi)$ or, equivalently, in $V \times W$ (recall that Whitney regularity is invariant under diffeomorphisms: Remark 1.3.4 and Example 1.3.5).

To prove this, observe that we can write $i(A)$ and $i(B)$ in the form

$$i(A) = i(S) \cap (S \times T), \quad i(B) = i(S') \cap (S' \times T'). \tag{5.3}$$

Moreover, $i(S')$ is Whitney regular over $i(S)$ and $S' \times T'$ is Whitney regular over $S \times T$.

Let $\{x_n\}$ and $\{y_n\}$ be sequences in $i(A)$ and $i(B)$, respectively, both converging to $x \in i(A)$. We also assume that $\overline{x_n y_n}$ converges to a line L and that $T_{y_n} i(B)$ converges to a plane E in the corresponding Grassmannians of $\mathbb{R}^n \times \mathbb{R}^m$. We have to show that $L \subset E$.

By taking subsequences if necessary, we can assume that $T_{y_n} i(S')$ converges to a plane E_1 and that $T_{y_n}(S' \times T')$ converges to another plane $E_2 \times E_3$, again in the corresponding Grassmannians of $\mathbb{R}^n \times \mathbb{R}^m$. Since $i(S')$ is Whitney regular over $i(S)$ and $S' \times T'$ is Whitney regular over $S \times T$, we have $L \subset E_1 \cap (E_2 \times E_3)$.

From Equation (5.3), it follows that $E \subset E_1 \cap (E_2 \times E_3)$. Furthermore, $\varphi|_{S'} : S' \to W$ is a submersion, which factors as the composition

$$S' \xrightarrow{\ i\ } i(S') \xrightarrow{\ \pi_2\ } W \ .$$

This implies, by construction, that $T_{y_n} i(S')$ and $T_{y_n}(S' \times T')$ are transverse in $\big(T_{\pi_1(y_n)} S'\big) \times \mathbb{R}^m$ and, also, that E_1 and $E_2 \times E_3$ are transverse in $E_2 \times \mathbb{R}^m$. Thus, $\dim E = \dim E_1 \cap (E_2 \times E_3)$ and, hence, $E = E_1 \cap (E_2 \times E_3) \supset L$. QED

Theorem 5.6.3. *We can choose the representative $\Phi : X \to W$ small enough such that it is a Thom map with the stratifications $\mathcal{S}'$ and $\mathcal{T}$.*

Proof. Let us begin seeing that the pair $(\mathcal{S}', \mathcal{T})$ is a stratification of $\Phi : X \to W$. Hence, the first thing to do is showing that the sets of $\mathcal{S}'$ are submanifolds.

Let $S_\alpha \in \mathcal{S}$ and $T \in \mathcal{T}$. The set $S_\alpha \cap \Phi^{-1}(T) - \Sigma(\Phi)$ is a submanifold of $S_\alpha - \Sigma(\Phi)$, since $\Phi|_{S_\alpha - \Sigma(\Phi)}$ is a submersion. The set $S_\alpha \cap \Phi^{-1}(T) \cap \Sigma(\Phi)$ is either the point $\{x\}$ or has dimension 1 and its closure is analytic. By the curve selection lemma, we can take a smaller representative such that $S_\alpha \cap \Phi^{-1}(T) \cap \Sigma(\Phi)$ is smooth.

By construction, Φ maps strata of $\mathcal{S}'$ onto strata of $\mathcal{T}$. We have to prove that Φ maps these strata submersively. Recall that $\mathcal{T}$ is given by the strata $W - \mathcal{D}, \mathcal{D} - \{0\}$ and $\{0\}$, so the only non-trivial case is when we consider a stratum of the form $S_\alpha \cap \Phi^{-1}\big(\mathcal{D} - \{0\}\big) \cap \Sigma(\Phi)$, mapped by Φ into $\mathcal{D} - \{0\}$. The two strata in the source and the target have dimension 1, and Φ is holomorphic and finite-to-one on $\Sigma(\Phi)$, so necessarily Φ is a local diffeomorphism.

The next step is to show that $\mathcal{S}'$ is a Whitney stratification. The case of a pair of strata of $\mathcal{S}'$ contained in $X - \Sigma(\Phi)$ follows directly from Lemma 5.6.2. The case of a pair of strata of $\mathcal{S}'$ contained in $\Sigma(\Phi)$ is trivial, since one of them must be the stratum $\{x\}$. Therefore, we only need to consider the case of $A, B \in \mathcal{S}'$ such that $B \subset X - \Sigma(\Phi)$ and $A \subset \Sigma(\Phi)$ has dimension 1. In this case, the set of points in A such that B is not Whitney regular over A at x is analytic and proper. Again, by the curve selection lemma, we can take a smaller representative such that B is Whitney regular over A.

Finally, it only remains to show that Φ satisfies the Thom condition. To do this, consider a pair of strata $A, B \in \mathcal{S}'$ such that $A \subseteq \overline{B}$.

If $A \subset \Sigma(\Phi)$, the induced map $\Phi : A \to T$ is a local diffeomorphism, where $T \in \mathcal{T}$, so the Thom condition is satisfied trivially. Otherwise, $A \subset X - \Sigma(\Phi)$ and also $B \subset X - \Sigma(\Phi)$, so we can assume that $A = S_\alpha \cap \Phi^{-1}(T) - \Sigma(\Phi)$ and $B = S_\beta \cap \Phi^{-1}(T') - \Sigma(\Phi)$, for some $S_\alpha, S_\beta \in \mathcal{S}$ and $T, T' \in \mathcal{T}$.

We take a sequence $\{x_n\}$ in B converging to a point x in A. Moreover, to ease the notation, we will simply write F^g_y to refer to the set $g^{-1}\big(g(y)\big)$ for any mapping g and point y. The Thom condition holds if

$$\lim_n T_{x_n} F^{\Phi|_B}_{x_n} \supseteq T_x F^{\Phi|_A}_x, \tag{5.4}$$

as we showed in Remark 1.3.8. Since $x \in A = S_\alpha \cap \Phi^{-1}(T) - \Sigma(\Phi)$ and $x_n \in B = S_\beta \cap \Phi^{-1}(T') - \Sigma(\Phi)$, we have

$$F^{\Phi|_B}_{x_n} = F^{\Phi|_{S_\beta - \Sigma(\Phi)}}_{x_n} \text{ and } F^{\Phi|_A}_x = F^{\Phi|_{S_\alpha - \Sigma(\Phi)}}_x,$$

therefore Equation (5.4) can be rewritten as

$$\lim_n T_{x_n} F^{\Phi|_{S_\beta}}_{x_n} \supseteq T_x F^{\Phi|_{S_\alpha}}_x. \tag{5.5}$$

Now, we use the fact that $\Phi = (\ell, f)$. Since $\ell^{-1}(0)$ is transverse to S_β, Equation (5.5) is equivalent to

$$\left(\lim_n T_{x_n} F^{f|_{S_\beta}}_{x_n}\right) \cap T_x F^\ell_x \supseteq T_x F^{f|_{S_\alpha}}_x \cap T_x F^\ell_x. \tag{5.6}$$

By Theorem 5.6.1, f is a Thom map with the stratification $\mathcal{S}$. Thus,

$$\lim_n T_{x_n} F_{x_n}^{f|_{S_\beta}} \supseteq T_x F_x^{f|_{S_\alpha}},$$

which implies Equation (5.6). QED

Remark 5.6.4. With the notation of Theorem 5.6.3, if $f \in \mathfrak{m}_{X,x}^2$, $\Delta_\ell(f,x)$ is tangent to the axis $\{v=0\}$ in $\mathbb{C}^2$ (see Proposition 5.3.5). Moreover, we can add $W \cap \{u=0\} - \{0\}$ as a new stratum in $\mathcal{T}$ and the corresponding strata in $\mathcal{S}'$ so that $\Phi : X \to W$ is still a Thom mapping with the new stratifications.

5.7. Proof of the main theorem

In this section we give the proof of Theorem 5.2.1. The proof is by induction on the dimension of (X,x). To do this, we need Lê Dũng Tràng's carousel construction in [Trá75]. We also refer to [TNBS20, Section 6.8] for a detailed construction of the carousel for a general plane curve $(C,0)$.

In our case, we apply this construction for the Cerf's diagram $C = \Delta_\ell(f,x)$ of a holomorphic function $f : (X,x) \to (\mathbb{C},0)$ with respect to a generic linear form ℓ. The key point here is that C is tangent to the axis $\{v=0\}$ at the origin if $f \in \mathfrak{m}_{X,x}^2$, where u, v are the coordinates of the plane $\mathbb{C}^2$ (see Proposition 5.3.5).

Lemma 5.7.1 (see [Trá75, p. 418]). *Let $(C,0)$ be a plane curve which is tangent to the axis $\{v=0\}$. There exist discs D and D_η centered at the origin in $\mathbb{C}$ and a smooth vector field ω on the solid torus $D \times \partial D_\eta$ such that:*

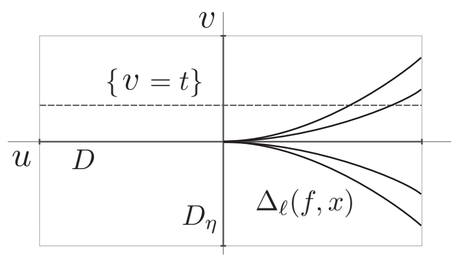
(i) the projection onto the second component of ω gives the unit tangent vector field over ∂D_η (i.e., the tangent field of length η in the positive direction),

(ii) the restriction to $\{0\} \times \partial D_\eta$ is indeed the unit vector field,

(iii) the vector field ω is tangent to $(D_\rho \times \partial D_\eta) \cap \{\delta = \varepsilon\}$ for all $\varepsilon \in \mathbb{C}$ small enough, where $\delta = 0$ is a reduced equation of C, and

(iv) the only integral curve that is closed after a loop in ∂D_η is $\{0\} \times \partial D_\eta$.

The discs D and D_η in Lemma 5.7.1 are chosen small enough so that there is a disc D_1 such that $D \subset D_1$, $(D_1 \times \{0\}) \cap C = \{0\}$ and $\{v = t\}$ intersects the curve C in $(D \times \{0\}, C)_0$ points in $D \times D_\eta$, for $\eta \geq |t| > 0$, where $(\bullet, \bullet)_0$ is the local intersection number at 0.



The geometrical meaning of the carousel is the following: we first take a representative C of the plane curve on some open neighbourhood W of the origin in the plane $\mathbb{C}^2$. Let

L be the intersection of W with the axis $\{u = 0\}$. We consider W with the Whitney stratification given by the strata $W - (C \cup L)$, $C - \{0\}$, $L - \{0\}$ and $\{0\}$ and the function germ $\pi_2 : (W, 0) \to (\mathbb{C}, 0)$ given by $\pi_2(u, v) = v$. The choice of D and D_η is made so that

$$\pi_2 : D \times \partial D_\eta \to \partial D_\eta$$

is a proper stratified submersion with the induced stratification in $D \times \partial D_\eta$. By conditions in Items *(i)* to *(iii)* in Lemma 5.7.1, ω is a stratified vector field on $D \times \partial D_\eta$ which is a lifting of the unit tangent vector field on ∂D_η. Hence, its flow provides a local geometric monodromy $h : D \times \{t\} \to D \times \{t\}$ for some $t \in \partial D_\eta$, which preserves the point $(0, t)$ and the finite set $C \cap \big(D \times \{t\}\big)$. By the condition given in Item *(iv)*, the only fixed point of h is $(0, t)$. See also the previous representations in Figures 5.5 and 5.6.

Now we can give the proof of our main result:

Proof of Theorem 5.2.1. Assume that $(X, x) \subset (\mathbb{C}^N, x)$. We take a privileged polydisc Δ in $\mathbb{C}^N$ at x and a small disc D_η in $\mathbb{C}$ at 0 such that the restriction

$$f : X \cap \Delta \cap f^{-1}(\partial D_\eta) \to \partial D_\eta$$

is a proper stratified submersion. We claim that there exists a stratified vector field ξ on $X \cap \Delta \cap f^{-1}(\partial D_\eta)$ which is a lifting of the unit vector field θ on ∂D_η and whose flow provides a local geometric monodromy with no fixed points. We prove this by induction on the dimension of X at x.

Assume, first, that $\dim(X, x) = 1$. Let $X_1, \ldots, X_r$ be the analytic branches of X at x. Then, $X \cap \Delta \cap f^{-1}(\partial D_\eta)$ is the disjoint union of all the sets $X_i \cap \Delta \cap f^{-1}(\partial D_\eta)$, $i = 1, \ldots, r$. Hence, it is enough to show the claim in the case that X is irreducible at x. Let $norm : \tilde{X} \to X$ be the normalization of X at x. Since $f \in \mathfrak{m}^2_{X,x}$, we can take an analytic extension $\overline{f} : (\mathbb{C}^N, x) \to (\mathbb{C}, 0)$ such that $F \in \mathfrak{m}^2_N$. After a reparametrization, we can assume that $\tilde{X}$ is an open neighbourhood of 0 in $\mathbb{C}$, $\{0\} = norm^{-1}(x)$ and $F \circ norm(s) = s^k$, for some $k \geq 2$. In this case, θ lifts in a unique way by the map $F \circ norm$ and has a local geometric monodromy with no fixed points. But $norm$ induces a diffeomorphism on $\tilde{X} - \{0\}$ onto $X - \{x\}$, so we have also a unique lifting on $X \cap \Delta \cap f^{-1}(\partial D_\eta)$ whose geometric monodromy has no fixed points.

Now, we assume the claim is true when $\dim(X, x) = n$ and prove it in the case that $\dim(X, x) = n + 1$. Let $\ell : \mathbb{C}^N \to \mathbb{C}$ be a generic linear form and consider the map $\Phi = (\ell, f)$. We have a commutative diagram as follows:

$$\begin{array}{ccccc}
X \cap \Delta \cap \Phi^{-1}(D \times \partial D_\eta) & \xrightarrow{\Phi} & D \times \partial D_\eta & \xrightarrow{\pi_2} & \partial D_\eta \\
\uparrow & & \uparrow & \nearrow_{\pi_2} & \\
X \cap \Delta \cap \ell^{-1}(0) \cap f^{-1}(\partial D_\eta) & \xrightarrow{(0,f)} & \{0\} \times \partial D_\eta & &
\end{array},$$

where the vertical arrows are the inclusions and π_2 is the projection onto the second component. Here, we choose the polydiscs Δ and $D \times D_\eta$ small enough such that Φ is a

Thom proper map (see Lemma 5.5.2, Theorem 5.6.3, and Remark 5.6.4). The stratification in $D \times \partial D_\eta$ is given by the strata $D \times \partial D_\eta - (C \cup L)$, $(D \times \partial D_\eta) \cap C$ and L, where $L = \{0\} \times \partial D_\eta$ and $C = \Delta_\ell(f, x)$ is the Cerf's diagram.

By induction hypothesis, there exists a stratified vector field ξ_1 on $X \cap \Delta \cap \ell^{-1}(0) \cap f^{-1}(\partial D_\eta)$ which is a lifting of θ and whose geometric monodromy has no fixed points. If C is empty, the claim is obvious by Proposition 5.3.4, so we can assume that C is not empty.

By the carousel construction in Lemma 5.7.1, there exists a stratified vector field ω on $D \times \partial D_\eta$ which satisfies the conditions of Items *(i)* to *(iv)*. Since ω is a lifting of θ, it is globally integrable by Theorem 5.4.11. Moreover, ω is not zero along L and $(D \times \partial D_\eta) \cap C$, so we can use the flow of ω to construct a weakly controlled tube system $\mathcal{T}'$ of $D \times \partial D_\eta$ such that ω is weakly controlled. By Corollary 5.4.12, ω lifts to a stratified vector field ξ on $X \cap \Delta \cap \Phi^{-1}(D \times \partial D_\eta)$ which is globally integrable. Moreover, by using a partition of unity, we can construct ξ in such a way that it coincides with ξ_1 on $X \cap \Delta \cap \ell^{-1}(0) \cap f^{-1}(\partial D_\eta)$.

Let $F = X \cap \Delta \cap f^{-1}(t)$, with $t \in \partial D_\eta$, and consider the geometric monodromy $h : F \to F$ induced by ξ. On one hand, ξ is an extension of ξ_1, so $h\big(F \cap \ell^{-1}(0)\big) = F \cap \ell^{-1}(0)$ and h has no fixed points on $F \cap \ell^{-1}(0)$. On the other hand, Item *(iv)* of Lemma 5.7.1 implies that h does not have fixed points on $F - \ell^{-1}(0)$ either. This completes the proof.
QED

The proof relied on the hypothesis of f being in $\mathfrak{m}^2_{X,x}$ and, actually, this hypothesis is necessary, even if f has critical point at x in the stratified sense. Here we give a couple of examples which illustrate this fact.

Example 5.7.2. Let $(C, 0)$ be the ordinary triple point singularity in $(\mathbb{C}^3, 0)$. This is equal to the union of the three coordinate axis in $\mathbb{C}^3$ (see Figure 5.7) and the defining equations are given by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}.$$

This gives to $(C, 0)$ a structure of isolated determinantal singularity in the sense of [NnBOOT13]. According also to [NnBOOT13], we can construct a determinantal smoothing of $(C, 0)$ by taking the 2×2-minors of $M_t = M + tA$, where A is a generic (2×3)-matrix with coefficients in $\mathbb{C}$ and $t \in \mathbb{C}$.

In fact, let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $(X, 0)$ be the surface in $(\mathbb{C}^3 \times \mathbb{C}, 0)$ defined as the zero set of the 2×2-minors of M_t. The projection $f : (X, 0) \to (\mathbb{C}, 0)$, $f(x, y, z, t) = t$, provides a flat deformation whose special fibre is $(C, 0)$ and whose generic fibre $F = f^{-1}(t)$, for $t \neq 0$, is a smooth curve (see Figure 5.7). We can see F as a kind of *determinantal Milnor fibre* of $(C, 0)$.

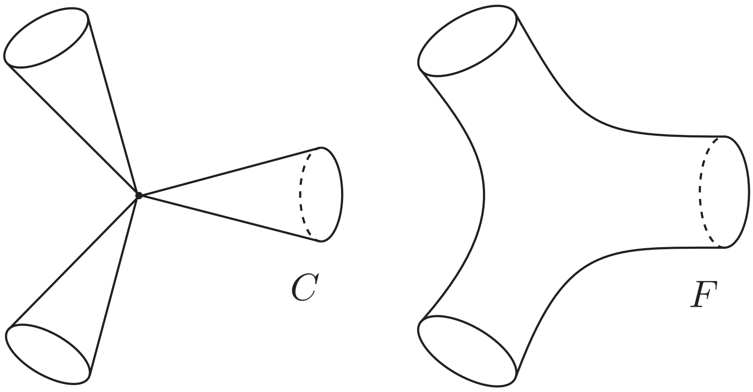


Figure 5.7: Special fiber (left) and generic fiber (right) of Example 5.7.2.

Finally, to find a geometric monodromy, we can consider the family of morphisms in X given by $h_\theta(x, y, z, t) = e^{i\theta}(x, y, z, t)$. As X is homogeneous, this gives well-defined morphisms between fibers of the fibration given by f and $h_{2\pi} = \mathrm{id}$ is a local geometric monodromy. Furthermore, F is diffeomorphic to a disk with two holes, so $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^2$ and the Lefschetz number is -1. In particular, any local geometric monodromy must have a fixed point. In addition, one can also confirm the computation of the monodromy in [BG80, p. 279].

This is not a counterexample of Theorem 5.2.1: a simple computation shows that $f \notin \mathfrak{m}^2_{X,x}$ in this example.

Example 5.7.3. Consider the A_4 plane curve singularity $(C, 0)$ whose equation in $(\mathbb{C}^2, 0)$ is $x^5 - y^2 = 0$. The monodromy of the classical Milnor fibre of $(C, 0)$ is well known and we will not discuss it. Instead, we see $(C, 0)$ as the image of the map germ $g_0\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ given by $g_0(s) = (s^2, s^5)$, which has an isolated instability at the origin.

Since we are in the range of Mather's nice dimensions, we can take a stabilisation $G = (g_t, t)$ (recall Remark 1.2.15). Observe that the disentanglement of g, say F, can also be seen as the generic fibre of the function $f : (X, 0) \to (\mathbb{C}, 0)$ where $(X, 0)$ is the image of G in $(\mathbb{C}^2 \times \mathbb{C}, 0)$ and $f(x, y, t) = t$. Hence, we are interested in the local monodromy of f at the origin.

In our case, we take $g_t(s) = (s^2, s^5 + ts)$. It is easy to see that, for $t \neq 0$, g_t is an immersion with two transverse double points $p = g_t(a_1) = g_t(a_2)$ and $q = g_t(b_1) = g_t(b_2)$ where a_1, a_2, b_1, b_2 are the four roots of $s^4 + t = 0$, with $a_1 = -a_2$ and $b_1 = -b_2$. Hence, g_t defines a stabilisation of g_0. Observe that the number of double points coincides with the delta invariant $\delta(C, 0) = 2$. The disentanglement F is the image of g_t and is homeomorphic to the quotient of a closed 2-disk D_t under the relations $a_1 \sim a_2$ and $b_1 \sim b_2$ (see Figure 5.8). Thus, F has the homotopy type of $\mathbb{S}^1 \vee \mathbb{S}^1$, so $\mu_I(g_0) = 2$.

The locally C^0-trivial fibration is the restriction $f : X \cap (B \times S^1_\eta) \to S^1_\eta$, with the obvious notation, for a small enough $\eta > 0$.

In order to construct a geometric monodromy $h\colon F \to F$, it is enough to find a one-parameter group of stratified homeomorphisms $h_\theta\colon X \cap (B \times S^1_\eta) \to X \cap (B \times S^1_\eta)$, with

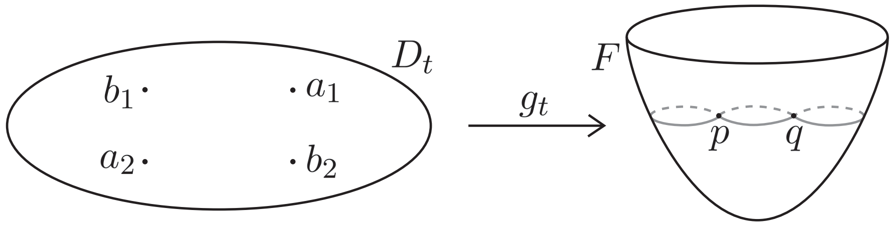


Figure 5.8: The mapping g_t and the double points, a_1, b_1, a_2 and b_2.

$\theta \in \mathbb{R}$, which make the following diagram commutative

$$\begin{array}{ccc} X \cap (B \times S^1_\eta) & \xrightarrow{f} & S^1_\eta \\ {\scriptstyle h_\theta}\downarrow & & \downarrow{\scriptstyle r_\theta} \\ X \cap (B \times S^1_\eta) & \xrightarrow{f} & S^1_\eta \end{array},$$

where $r_\theta(t) = e^{i\theta}t$. In this situation, $h : F \to F$ is obtained as the restriction of $h_{2\pi}$.

In order to do this, since $(C, 0)$ is weighted homogeneous with weights $(5, 2)$, it is better to consider the (non-Euclidean) ball B given by $|x|^5 + |y|^2 \leq 1$ instead of a Euclidean ball in $\mathbb{C}^2$. Thus, $C \cap B = g_0(D)$, where D is the disk in $\mathbb{C}$ given by $|s|^{10} \leq 1/2$. For $t \neq 0$, $F = g_t(D_t)$, where now $D_t = g_t^{-1}(B)$ is the disk in $\mathbb{C}$ given by

$$|s|^{10} + |s|^2|s^4 + t|^2 \leq 1.$$

Given a point $(x, y, t) \in X$, we have $(x, y, t) = G(s, t)$ for some $s \in \mathbb{C}$. We define $h_\theta \colon X \to X$ as

$$h_\theta\big(G(s, t)\big) = G\big(e^{\frac{i\theta}{4}}s, e^{i\theta}t\big).$$

Now, we have to check that, indeed, this gives a group of stratified homeomorphisms. We consider in X the stratification given by $\{X - Y, Y\}$, where Y is the double point curve with equations $x^2 + t = 0$, $y = 0$. Since G is an embedding on $X - Y$, h_θ is well defined and is a diffeomorphism on $X - Y$. When $(x, y, t) \in Y$, we have $G(s, t) = (s^2, 0, t)$, with $s^2 = x$ and $s^4 + t = 0$. It follows that

$$h_\theta(x, 0, t) = G\big(e^{\frac{i\theta}{4}}s, e^{i\theta}t\big) = \big(e^{\frac{i\theta}{2}}s^2, 0, e^{i\theta}t\big) = \big(e^{\frac{i\theta}{2}}x, 0, e^{i\theta}t\big),$$

and $\big(e^{i\theta/2}x\big)^2 + e^{i\theta}t = e^{i\theta}(x^2 + t) = 0$. Thus, h_θ is also well defined on Y, $h(Y) = Y$ and the restriction $h \colon Y \to Y$ is a diffeomorphism. It is also clear that $h_\theta \colon X \to X$ and its inverse are both continuous, so it is a stratified homeomorphism. It only remains to show that $h_\theta\big(X \cap (B \times S^1_\eta)\big) = X \cap (B \times S^1_\eta)$, because we have to work with a specific representative, but this a consequence of the equality:

$$\big|e^{\frac{i\theta}{4}}s\big|^{10} + \big|e^{\frac{i\theta}{4}}s\big|^2\big|\big(e^{\frac{i\theta}{4}}s\big)^4 + e^{i\theta}t\big|^2 = |s|^{10} + |s|^2|s^4 + t|^2.$$

The geometric monodromy $h\colon F\to F$ is now the restriction of $h_{2\pi}$, which gives $h\big(g_t(s)\big)=g_t\big(e^{i\pi/2}s\big)$, that is, it is obtained by a $\pi/2$-rotation in the disk D_t.

To finish, we compute $h_*\colon H_1(F;\mathbb{Z})\to H_1(F;\mathbb{Z})$. We recall that F is homeomorphic to the quotient of D_t under the relations $a_1\sim a_2$ and $b_1\sim b_2$. The four points a_1,a_2,b_1,b_2 are on a square contained in the interior of D_t and centered at the origin, which is obviously invariant under the $\pi/2$-rotation. We denote by a,b,c,d the four edges of the square as in Figure 5.9 (cf. Figure 5.10).

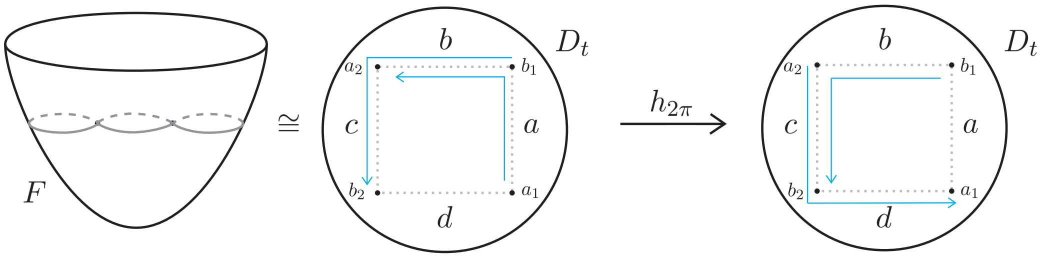


Figure 5.9: Monodromy of the fiber F.

We take the cycles $a+b$ and $c+d$ as a basis of $H_1(F;\mathbb{Z})$. Obviously, $h_*(a+b)=b+c$ and $h_*(b+c)=c+d=-(a+b)$ so the matrix of h_* with respect to this basis is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Lefschetz number is 1 and, hence, any local geometric monodromy must have a fixed point. In fact, in our construction, there is exactly one fixed point: the origin of the disk D_t, which is invariant under the $\pi/2$-rotation. As in Example 5.7.2, it is not difficult to check that $f\notin\mathfrak{m}^2_{X,x}$.

Remark 5.7.4. Example 5.7.3 is the example we were talking about in Section 5.1. In the next section we show that Theorem 5.2.2 can be used to prove a general non-coalescing theorem (Theorem 5.8.8). However, as this example shows, Theorem 5.2.2 is not true for the setting of hypersurfaces parametrised as the image of an $\mathscr{A}$-finite map germ.

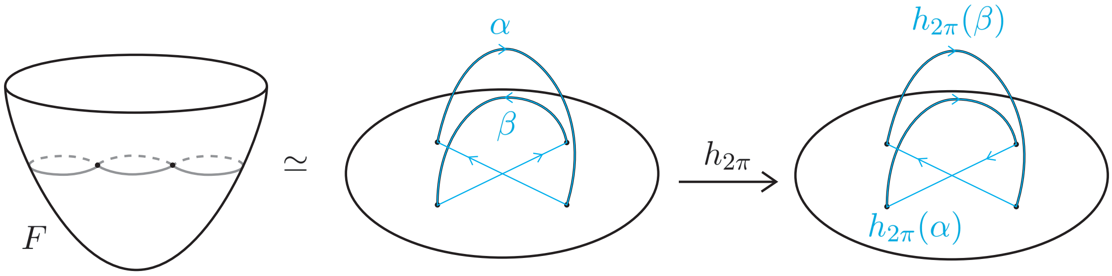


Figure 5.10: Another representation of h_* from Example 5.7.3.

5.8. Applications

The first application of Theorem 5.2.1 is the following corollary, which shows that any hypersurface (with possibly non-isolated singularities) (X, x) in $\mathbb{C}^{n+1}$ with smooth topological type must be smooth. We recall that two germs of complex spaces (X, x) and (Y, y) in $\mathbb{C}^{n+1}$ have the same *topological type* if there exists a homeomorphism $\varphi : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}^{n+1}, y)$ such that $\varphi(X, x) = (Y, y)$.

Corollary 5.8.1. *Let (X, x) be a germ of hypersurface in $\mathbb{C}^{n+1}$. If (X, x) has the topological type of a smooth hypersurface, then (X, x) is smooth.*

Proof. If (X, x) has the topological type of a smooth hypersurface, its Milnor fibre is contractible by [Trá73a, Proposition, p. 261]. This implies that (X, x) is smooth by [A'C73, Theorem 3]. Observe that Theorem 3 of [A'C73] is a consequence of Theorem 5.2.1: Let $f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$ be holomorphic which gives a reduced equation of (X, x). The Lefschetz number of the local monodromy of f is 1 and, hence, $f \notin \mathfrak{m}^2_{\mathbb{C}^{n+1},x}$, by Theorem 5.2.1. QED

This corollary is related to Zariski's multiplicity conjecture (see [Zar71]) which claims that two hypersurfaces in $\mathbb{C}^{n+1}$ with the same topological type have the same multiplicity. Since a hypersurface is smooth if, and only if, it has multiplicity 1, Corollary 5.8.1 is just a particular case of the conjecture. Zariski showed the conjecture for plane curves but it remains still open in higher dimensions. Another related result is Mumford's theorem in [Mum61], which states that, if X is a normal surface and X is a topological manifold at $x \in X$, then X is smooth at x.

Our second application is a non-coalescing theorem for families of functions defined on spaces with the *Milnor property*. In [Trá73b, Theorems A and B], Lê Dũng Tràng showed an interesting application of A'Campo's theorem (see also [Gab74, Laz73b]). Let $\{H_t\}_{t\in\mathbb{C}}$ be an analytic family of hypersurfaces defined on some open subset $U \subset \mathbb{C}^n$ with only isolated singularities. Take B a Milnor ball for H_0 around a singular point $x_0 \in H_0$ and assume, for all t small enough, that the sum of the Milnor numbers of all the singular points of H_t in B is constant, that is,

$$\sum_{x \in H_t \cap B} \mu(H_t, x) = \mu(H_0, x_0).$$

Then, $H_t \cap B$ contains a unique singular point x of H_t (recall Figure 5.1).

The purpose of the remaining of the section is to prove an adapted version of this result in a more general context, namely, for *Milnor spaces* in the sense of [HT20]:

Definition 5.8.2. A *Milnor space* is a reduced complex space X such that, at each point $x \in X$, the *rectified homotopical depth* $\operatorname{rhd}(X, x)$ is equal to $\dim(X, x)$.

We refer to [HT20, Section 9.4.2] for the definition of the rectified homotopical depth and basic properties of Milnor spaces. In general, $\operatorname{rhd}(X, x) \leq \dim(X, x)$, so Milnor spaces are those whose rectified homotopical depth is maximal at any point. Some important properties are the following:

1. any smooth space X is a Milnor space,
2. any Milnor space X is equidimensional,
3. if X is a Milnor space and Y is a hypersurface in X (i.e. Y has codimension one and is defined locally in X by one equation), then Y is also a Milnor space.

As a consequence, any local complete intersection X (not necessarily with isolated singularities) is a Milnor space. Our setting is motivated by the following theorem due to Hamm and Lê Dũng Tràng:

Theorem 5.8.3 (see [HT20, Theorem 9.5.4])**.** *Let (X, x) be a germ of Milnor space and assume that $f\colon (X, x) \to (\mathbb{C}, 0)$ has isolated critical point in the stratified sense. Then, the general fibre F of f has the homotopy type of a bouquet of spheres of dimension* $\dim(X, x) - 1$.

Then, using the Lefschetz number and Theorem 5.2.2, it is really easy to prove the following.

Corollary 5.8.4. *With the hypothesis and notation of Theorem 5.8.3, if $f \in \mathfrak{m}^2_{X,x}$, the trace of the induced map $h_*\colon H_{n-1}(F;\mathbb{Z}) \to H_{n-1}(F;\mathbb{Z})$ by the monodromy $h : F \to F$ is $(-1)^n$, where $n = \dim(X, x)$.*

Definition 5.8.5. With the hypothesis and notation of Theorem 5.8.3, the number of spheres of F is called the *Milnor number* of f and is denoted by $\mu(f)$. We say that the critical point is *non-trivial* if $\mu(f) > 0$.

We want to generalize Lê's non-coalescing theorem for families of hypersurfaces $\{H_t\}_{t\in\mathbb{C}}$. As we want to generalize it in the setting of fibers inside Milnor spaces, we obviously need a convenient concept of *family of Milnor spaces* and its corresponding *family of complex functions that give the equations of the fibers*. This is covered in Definition 5.8.6.

Consider a germ of complex analytic space $(X_0, 0)$. Let $f_0\colon (X_0, x_0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function. Let X_0 be a small representative of $(X_0, 0)$ and let $\mathcal{S}$ be a Whitney stratification of X_0. We assume that a representative f has an isolated critical point in the stratified sense at 0. We define:

Definition 5.8.6. A *stratified deformation* of (X_0, x_0) is a flat deformation $\pi\colon (\mathfrak{X}, x_0) \to (\mathbb{C}, 0)$, where $\mathfrak{X}$ is an analytic space with an analytic Whitney stratification such that, for a representative π:

1. $\pi^{-1}(0) = X_0$ as analytic spaces,
2. π has isolated critical points in the stratified sense,
3. the stratification of X_0 coincides with the induced stratification by $\mathfrak{X}$ on $\pi^{-1}(0)$.

Given a stratified deformation $\pi\colon (\mathfrak{X}, x_0) \to (\mathbb{C}, 0)$, a *stratified unfolding* is a holomorphic mapping $\mathcal{F}\colon (\mathfrak{X}, x_0) \to (\mathbb{C} \times \mathbb{C}, 0)$ such that $p_1 \circ \mathcal{F}|_{X_0} = f_0$ and $p_2 \circ \mathcal{F} = \pi$, where $p_i : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $i = 1, 2$, is the ith projection.

We can always assume that $\mathfrak{X}$ is embedded in $\mathbb{C}^N \times \mathbb{C}$ and choose coordinates in such a way that $\pi(x, t) = t$. So, we can write the stratified unfolding as $\mathcal{F}(x, t) = \big(f_t(x), t\big)$. For each $t \in \mathbb{C}$, we have a function $f_t\colon X_t \to \mathbb{C}$, where $X_t = \pi^{-1}(t)$. Here, we consider in X_t the stratification induced by $\mathfrak{X}$ and denote by $\Sigma(f_t)$ the set of stratified critical points of f_t.

Example 5.8.7. We consider the function $f_0 : (X_0, 0) \to (\mathbb{C}, 0)$, where X_0 is the surface in $\mathbb{C}^3$ given by $z^2 - y(x^2 + y)^2 = 0$ and $f_0(x, y, z) = x$. The stratification in X_0 is $\big\{\{0\}, C_0 - \{0\}, X_0\big\}$, where C_0 is the curve $z = x^2 + y = 0$. It is easy to see that f_0 has only one critical point in the stratified sense at the origin and that $\mu(f_0) = 1$ (see Figure 5.11).

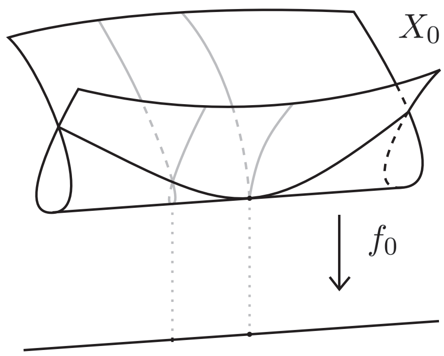


Figure 5.11: The function f_0 with a critical point.

Now, we define a stratified deformation $\pi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ and a stratified unfolding $\mathcal{F}\colon (\mathfrak{X}, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ as follows: $\mathfrak{X}$ is the hypersurface in $\mathbb{C}^3 \times \mathbb{C}$ with equation $z^2 - y(x^2 + y + t)^2 = 0$, $\pi(x, y, z, t) = t$ and $\mathcal{F}(x, y, z, t) = (x, t)$. The stratification in $\mathfrak{X}$ is $\big\{\{0\}, \mathcal{D} - \{0\}, \mathcal{C} - \mathcal{D}, \mathfrak{X} - \mathcal{C}\big\}$, where $\mathcal{D}$ is the curve $z = y = x^2 + t = 0$ and $\mathcal{C}$ is the surface $z = x^2 + y + t = 0$. Again, it is not difficult to check that all the conditions of Definition 5.8.6 hold.

For $t \neq 0$, $f_t : X_t \to \mathbb{C}$ has two critical points in the stratified sense at $(\pm\sqrt{-t}, 0, 0)$, which are the points in $D_t := X_t \cap \mathcal{D}$. We see that f_t has also Milnor number 1 at each critical point $(\pm\sqrt{-t}, 0, 0)$ (see Figure 5.12).

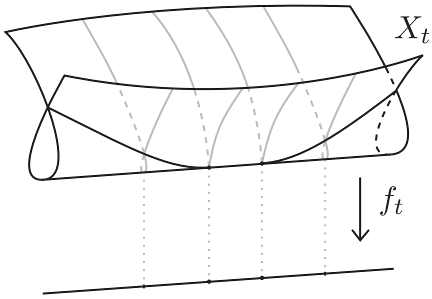


Figure 5.12: The function f_t with two critical points.

The following theorem could seem very restrictive due to the length of the hypotheses. On the contrary, its statement only says that, with a *general notion of family of ambient spaces* ($\mathfrak{X}$) and a *general notion of equation of the fibers* ($\mathcal{F}$), if it happens what we have proven in some cases (Theorem 5.2.1 or Corollary 5.8.4), then we have a non-coalescing result.

Theorem 5.8.8. *Let $f_0\colon (X_0, x_0) \to (\mathbb{C}, 0)$ be a function with a non-trivial isolated critical point and let $\mathcal{F}\colon (\mathfrak{X}, x_0) \to (\mathbb{C} \times \mathbb{C}, 0)$ be a stratified unfolding of f_0 such that $\mathfrak{X}$ is a Milnor space. We set $Y_t = f_t^{-1}(0)$ and assume that for any $x \in \Sigma(f_t) \cap Y_t$, the trace of the local monodromy of f_t at x in dimension $n-1$ is $(-1)^n$, where* $\dim(X_0, x_0) = n$*. Let B_0 be a Milnor ball for f_0 at x_0 and assume that for any $t \in \mathbb{C}$ small enough,*

$$\sum_{x \in \Sigma(f_t) \cap Y_t \cap B_0} \mu_x(f_t) = \mu_{x_0}(f_0), \tag{5.7}$$

where $\mu_x(f_t)$ is the Milnor number of f_t at x. Then, $Y_t \cap B_0$ contains a unique non-trivial critical point x of f_t.

Proof. Denote by $\Sigma(\mathcal{F})$ the set of stratified critical points of $\mathcal{F}$ and assume that $\dim(X_0, x_0) = n > 2$. It follows that $(x, t) \in \Sigma(\mathcal{F})$ if and only if $x \in \Sigma(f_t)$. Since the stratification of $\mathfrak{X}$ is analytic, $\Sigma(\mathcal{F})$ is also analytic. On one hand,

$$\dim \Sigma(\mathcal{F}) \cap \{t = 0\} = \dim \Sigma(f_0) = 0$$

and, thus, $\dim \Sigma(\mathcal{F}) \leq 1$. On the other hand, by Equation (5.7),

$$\Sigma(\mathcal{F}) \cap \{t = t_0\} = \Sigma(f_{t_0}) \neq \varnothing$$

for $t_0 \neq 0$, so $\dim \Sigma(\mathcal{F}) = 1$. Moreover, $\mathcal{F}^{-1}(0) \cap \Sigma(\mathcal{F}) = \{0\}$, hence, its image $\Delta = \mathcal{F}\big(\Sigma(\mathcal{F})\big)$ is also analytic of dimension 1 in $(\mathbb{C} \times \mathbb{C}, 0)$ by Remmert's proper mapping theorem (see, for example, [BBH+98, p. 5]).

We fix a small enough open polydisc $D_\eta \times D_\rho$ in $\mathbb{C} \times \mathbb{C}$ such that the restriction

$$\mathcal{F} : (B_0 \times D_\rho) - \mathcal{F}^{-1}(\Delta) \to (D_\eta \times D_\rho) - \Delta \tag{5.8}$$

is a proper stratified submersion and such that $\Delta \cap \big(D_\eta \times \{0\}\big) = \{0\}$. By the Thom-Mather first isotopy lemma, Equation (5.8) is a locally C^0-trivial fibration. Given $s \in D_\eta - \{0\}$, we have $(s, 0) \in \big(D_\eta \times D_\rho\big) - \Delta$ and, hence, the fibre

$$\mathcal{F}^{-1}(s,0) \cap (B_0 \times D_\rho) = \big(f_0^{-1}(s) \cap B_0\big) \times \{0\}$$

coincides with the general fibre of f_0.

Let $t \in D_\rho$ and assume that $\Sigma(f_t) \cap f_t^{-1}(0) \cap B_0 = \{x_1, \dots, x_k\}$. For each $i = 1, \dots, k$, we take a Milnor ball B_i for f_t at x_i such that B_i is contained in the interior of B_0 and $B_i \cap B_j = \emptyset$ if $i \neq j$. Now, we choose $0 < \eta' < \eta$ such that $(s,t) \notin \Delta$ for all s with $0 < |s| < \eta'$.

Fix a point $s \in D_{\eta'}$ and consider the loop $\gamma(\theta) = se^{i\theta}$, $\theta \in [0, 2\pi]$. This loop induces a geometric monodromy $h : f_t^{-1}(s) \cap B_0 \to f_t^{-1}(s) \cap B_0$ which coincides, up to isotopy, with the geometric monodromy of f_0 at x_0. Moreover, by adding the boundaries of the balls B_i as strata in the domain of Equation (5.8), we can assume that (recall Method III of constructing a geometric monodromy):

1. $h\big(f_t^{-1}(s) \cap B_i\big) = f_t^{-1}(s) \cap B_i$ and $h_i = h|_{f_t^{-1}(s) \cap B_i}$ is the monodromy of f_t at x_i, for each $1 = 1, \dots, k$;

2. h is the identity outside the interior of $B_1 \cup \dots \cup B_k$.

Let

$$U = f_t^{-1}(s) \cap \Big(B_0 - \cup_{i=1}^k \mathring{B}_i\Big)$$

and

$$V = f_t^{-1}(s) \cap \Big(\cup_{i=1}^k B_i\Big).$$

By considering the Mayer-Vietoris sequence of the pair (U, V), we get a diagram whose rows are exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{n-1}(U \cap V) & \longrightarrow & H_{n-1}(U) \oplus H_{n-1}(V) & \longrightarrow & H_{n-1}(U \cup V) & \longrightarrow & \\
 & & \downarrow \text{id} & & \downarrow \text{id} \oplus \big(\oplus_{i=1}^k (h_i)_*\big) & & \downarrow h_* & & \\
0 & \longrightarrow & H_{n-1}(U \cap V) & \longrightarrow & H_{n-1}(U) \oplus H_{n-1}(V) & \longrightarrow & H_{n-1}(U \cup V) & \longrightarrow & \\
 & & & & & & & & \\
 & \longrightarrow & H_{n-2}(U \cap V) & \longrightarrow & H_{n-2}(U) & \longrightarrow & 0 & & \\
 & & \downarrow \text{id} & & \downarrow \text{id} & & & & \\
 & \longrightarrow & H_{n-2}(U \cap V) & \longrightarrow & H_{n-2}(U) & \longrightarrow & 0 & &
\end{array} \tag{5.9}$$

By the exactness in one the rows of the sequence we get

$$a - \left(b + \sum_{i=1}^{k} \mu_{x_i}(f_t)\right) + \mu_{x_0}(f_0) - c + d = 0,$$

where

$$\begin{aligned} a &= \operatorname{rank} H_{n-1}(U \cap V), \\ b &= \operatorname{rank} H_{n-1}(U), \\ c &= \operatorname{rank} H_{n-2}(U \cap V), \text{ and} \\ d &= \operatorname{rank} H_{n-2}(U). \end{aligned}$$

Our hypothesis implies that

$$a - b - c + d = 0.$$

Finally, we use the fact that the trace is additive, which gives

$$a - \left(b + \sum_{i=1}^{k} \operatorname{tr}\left((h_i)_*\right) \right) + \operatorname{tr}(h_*) - c + d = 0$$

and, hence,

$$\sum_{i=1}^{k} \operatorname{tr}\left((h_i)_*\right) = \operatorname{tr}(h_*).$$

Again by hypothesis, $\operatorname{tr}\left((h_i)_*\right) = \operatorname{tr}(h_*) = (-1)^n$ for all $i = 1, \dots, k$, so, necessarily, $k = 1$.

We can use the same ideas if $n = 1$ or $n = 2$, with a diagram similar to the one of Equation (5.9). QED

Observe that the hypothesis of having trace equal to $(-1)^n$ at any point can be relaxed to having trace $k \neq 0$ that does not depend on the point. Also, the hypothesis of $\mathfrak{X}$ being a Milnor space is given to assure that the generic fibers of f_t have homology only in middle dimension (by Theorem 5.8.3). One can prove something similar if, in general, the non-trivial homology is sparse.

Remark 5.8.9. The proof of Theorem 5.8.8 is just an adaptation of the proof given in [Trá73b] for the case $X = \mathbb{C}^n$. A similar argument appears also in the paper [CNnBOOT21], where it is showed that any family of ICIS with constant total Milnor number has no coalescence of singularities.

Chapter 6

Developing a new technique

> For the pure geometrician himself this faculty is necessary: it is by logic that we prove, but by intuition that we discover.
>
> Henri Poincaré, *Science and method*

In this chapter, we cover a joint work in progress with Mond.

Here, we are trying to use the symmetric structure of the multiple point spaces to translate hard problems of the Thom-Mather theory into elementary problems. We try to do this with a new technique we are improving. As a first milestone, we have been able to generalize Lemma 3.1.18. Recall that (the first part of) Lemma 3.1.18 says that, if you consider an $\mathscr{A}$-finite germ of corank one that *admits a one-parameter stable unfolding* $F(x,t) = \big(f_t(x), t\big)$, then $D^k(f)$ is singular if, and only if, $H^{\mathrm{Alt}}_{n-k+1}\big(D^k(f_t)\big) \neq 0$. The generalization consists of dropping the hypothesis of admitting a one-parameter stable unfolding. For the mono-germ case, we also show that the image Milnor number is constant in a family if, and only if, the double point Milnor number is constant.

This chapter is strongly based on the fundamentals of representation theory and we use it fluently. If the reader is not familiar with this theory we recommend reading Appendix A.

6.1. Introduction

6.1.1. General idea

We have seen in Section 4.4.2 that the symmetric structure of the multiple point spaces $D^k(f)$ is very useful, and not only its alternating isotype. To be more precise, we have seen that the $(2,1,\dots,1)$-isotype is useful as well. Nevertheless, it is suspicious that we cannot use *all the symmetric structure* of $D^k(f)$ in an elegant way.

Observe that the first time we use something related to the alternating isotype in this text is in Lemma 3.1.18 and, to prove it, we use a theorem of Wall that extracts the alternating structure of a whole object, see Theorem 3.1.17. Indeed, if we look at the proof of Theorem 3.1.17 in [Wal80], we see that it is based in an interesting equality that relates an action of a group and the fixed points by the elements of the group.

Let G be a finite group acting on a finite simplicial complex M such that fixed simplices as sets are point-wise fixed. If M^g denotes the fixed points of M by g, then

$$\chi_G(M)(g) = \chi_{Top}(M^g), \tag{6.1}$$

where χ_G denotes the *character of the group* G[1] and χ_{Top} denotes the usual Euler-Poincaré characteristic. More precisely:

$$\chi_G(M)(g) \coloneqq \sum_i (-1)^i \operatorname{tr} g_* : H_i(M,\mathbb{C}) \to H_i(M,\mathbb{C}).$$

Equation (6.1) can be proven using standard arguments (see, for example, [Hat02, Theorem 2.44] and Wall's comments in [Wal80, p. 172]). Considering the exact sequences

$$0 \to B_n \to Z_n \to H_n \to 0 \qquad \text{and} \qquad 0 \to Z_n \to C_n \to B_{n-1};$$

where Z_n, B_n, C_n and H_n are the cycles, boundaries, chains and homology of M with coefficients in $\mathbb{C}$, respectively; and that these sequences split because they are free abelian groups; one has

$$\operatorname{tr} g|_{B_n} + \operatorname{tr} g_*|_{H_n} = \operatorname{tr} g|_{Z_n} \qquad \text{and} \qquad \operatorname{tr} g|_{Z_n} + \operatorname{tr} g|_{B_{n-1}} = \operatorname{tr} g|_{C_n}.$$

Substituting the first equation into the second one and taking an alternating sum yields

$$\sum_n (-1)^n \operatorname{tr} g_*|_{H_n} = \sum_n (-1)^n \operatorname{tr} g|_{C_n}.$$

Observe that the left-hand side of the equation is $\chi_G(M)(g)$ and the right-hand side is $\chi_{Top}(M^g)$, as the trace of $g|_{C_n}$ coincides with the number of fixed n-simplexes by g.

If we use Equation (6.1) with the multiple point spaces $D^k(f_t)$ and the group Σ_k, we obtain a relation between the action of Σ_k in $D^k(f_t)$ and the sets $D^k(f_t)^\sigma$, with $\sigma \in \Sigma_k$[2].

6.1.2. Equations

Let us examine what we have said little by little. Consider a corank one germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ that is $\mathscr{A}$-finite, a stable perturbation f_t and $\sigma \in \Sigma_k$ of cycle type $(r_1, \dots, r_m)$ with $\alpha_i = \#\{j : r_j = i\}$ (as we have done in Example 4.2.7). Then:

[1] Wall calls this the *equivariant Euler characteristic*, but this name is not standard.

[2] This equation was used before in this context, which was found by the author finishing this text, see [HK99, p. 336]. However, the way we are going to use it now is something new.

- $D^k(f)$ is empty or an ICIS with fiber $D^k(f_t)$ and dimension $p-k(p-n)$, if this is non-negative (see Theorem 2.4.4),
- $D^k(f)^\sigma$ is empty or an ICIS with fiber $D^k(f_t)^\sigma$ and dimension $p-k(p-n)-k+\sum_i \alpha_i$, if this is non-negative (see Lemma 4.2.10), and
- $\chi_{\Sigma_k}\big(D^k(f_t)\big)(\sigma)=\chi_{Top}\big(D^k(f_t)^\sigma\big)$, by Equation (6.1).

Definition 6.1.1. We will say that the *expected dimension* of $D^k(f)$, or $D^k(f_t)$, is $d_k:=p-k(p-n)$, and the *expected dimension* of $D^k(f)^\sigma$, or $D^k(f_t)^\sigma$, is $d_k^\sigma:=p-k(p-n)-k+\sum_i \alpha_i$.

Moreover, we can decompose the action of Σ_k using its character table, i.e., we can decompose $\chi_{\Sigma_k}\big(D^k(f_t)\big)(\sigma)$ considering the different irreducible representations of Σ_k and the number of times they appear. For example, if $d_2>0$, we have that

$$\begin{aligned}\chi_{\Sigma_2}\big(D^2(f_t)\big)(\sigma)&=1+(-1)^{d_2}\operatorname{tr}\sigma_*:H_{d_2}\big(D^2(f_t)\big)\to H_{d_2}\big(D^2(f_t)\big)\\&=1+(-1)^{d_2}\big(\chi_T(\sigma)T+\chi_{\mathrm{Alt}}(\sigma)A\big),\end{aligned}\tag{6.2}$$

where χ_T and χ_{Alt} are the characters of the trivial and the alternating representations, respectively, and T and A are the number of times the trivial and alternating representations appear repeated in the representation of Σ_2. In particular,

$$A=\operatorname{rank}H_{d_2}^{\mathrm{Alt}}\big(D^2(f_t)\big)=\mu_2^{\mathrm{Alt}}(f).$$

Furthermore, if $d_2^\sigma>0$, we arrive to the equation

$$\chi_T(\sigma)T+\chi_{\mathrm{Alt}}(\sigma)A=(-1)^{2-\sum_i\alpha_i}\mu\big(D^2(f)^\sigma\big)\tag{6.3}$$

using Equation (6.1) and simplifying it.

Another example could be computing this with $D^3(f_t)$ if d_3 and d_3^σ are positive:

$$\chi_T(\sigma)T+\chi_{\mathrm{Alt}}(\sigma)A+\chi_S(\sigma)S=(-1)^{3-\sum_i\alpha_i}\mu\big(D^3(f)^\sigma\big),\tag{6.4}$$

with the new character of the standard representation χ_S and the number of times it appears repeated in the representation of Σ_3.

However, if $d_k^\sigma\leq 0$, we will have to take into account that the first term of $\chi_{\Sigma_k}\big(D^k(f_t)\big)(\sigma)$ does not cancel out. We will study this in the next section.

6.2. Putting it in practice

For simplicity, in this section we are going to fix some notation and assume some things:

- $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ is a corank one germ that is unstable but $\mathscr{A}$-finite,
- f_t is a stable perturbation of f,

- $D^k(f) \neq \varnothing$ and it is non-smooth until otherwise stated,
- $\sigma \in \Sigma_k$ has cycle type $(r_1, \dots, r_m)$ with $\alpha_i = \# \{j : r_j = i\}$,
- d_k and d_k^σ are the expected dimensions, defined as in Definition 6.1.1.

As this is a new approach, let us begin with some examples to develop the ideas we have used in Equations (6.3) and (6.4) and introduce some occurrences in this setting.

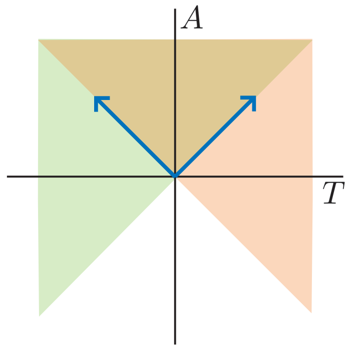
Example 6.2.1. If $n > 2$ and $k = 2$, we have that $d_2 > 1$ and $d_2^{(1\ 2)} = d_2 - 1 > 0$, so the equations we could get using id_{Σ_2} and $(1\ 2)$ are

$$\left.\begin{array}{lcl} T + A & = & \beta_{d_2} D^2(f_t) \\ T - A & = & -\beta_{d_2 - 1} D^2(f_t)^{(1\,2)} \end{array}\right\}.$$

Indeed, these equations are correct: the ones of the character of Σ_2 and the Euler-Poincaré characteristic cancel out because $D^2(f_t)^{(1\ 2)} \neq \varnothing$, for it is the Milnor fiber of a non-empty ICIS $D^2(f)^{(1\ 2)}$ that has to contain the point $(0, 0)$ (the only instability has to be fixed by any permutation).

As both $\beta_{d_2} D^2(f_t)$ and $\beta_{d_2-1} D^2(f_t)^{(1\ 2)}$ are non-negative, we arrive to

$$\left.\begin{array}{lcl} T + A & \geq & 0 \\ T - A & \leq & 0 \end{array}\right\}.$$



Obviously, we deduce that A has to be positive if the solution is not $T = A = 0$. Precisely, this is what we are assuming because we have said that $D^2(f)$ is non-smooth, so $T + A = \beta_{d_2} D^2(f_t) > 0$.

Example 6.2.2. If $n = 2$ and $k = 2$, we have that $d_2 = 1$ and $d_2^{(1\ 2)} = 0$, so the equations we get using id_{Σ_2} and $(1\ 2)$ are

$$\left.\begin{array}{lcl} T + A & = & \beta_1 D^2(f_t) \\ T - A & = & -\beta_0 D^2(f_t)^{(1\,2)} + 1 \end{array}\right\}.$$

Moreover, $D^2(f_t)^{(1\ 2)}$ is non-empty because it is the Milnor fiber of the ICIS $D^2(f)^{(1\,2)}$, as before. Hence, considering that $\beta_1 D^2(f_t) \geq 0$ and $\beta_0 D^2(f_t)^{(1\ 2)} \geq 1$ we arrive to

$$\left.\begin{array}{lcl} T + A & \geq & 0 \\ T - A & \leq & 0 \end{array}\right\}.$$

Once more, we conclude the same as in Example 6.2.1.

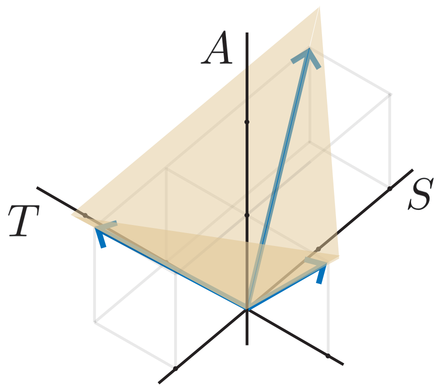
Example 6.2.3. If $n > 4$ and $k = 3$, we have that $d_3 \geq 3$, $d_3^{(1\,2)} = d_3 - 1 \geq 0$ and $d_3^{(1\,2\,3)} = d_3 - 2 \geq 0$, so the equations we could get are

$$\left.\begin{array}{lcl} T + A + 2S & = & \beta_{d_3} D^3(f_t) \\ T - A & = & -\beta_{d_3 - 1} D^3(f_t)^{(1\,2)} \\ T + A - S & = & \beta_{d_3 - 2} D^3(f_t)^{(1\,2\,3)} \end{array}\right\}.$$

Again, this equations are correct because $D^3(f_t)^{(1\ 2)}, D^3(f_t)^{(1\ 2\ 3)} \neq \varnothing$.

Also, observe that $\beta_{d_3-1}D^3(f_t)^{(1\ 2)}$ and $\beta_{d_3-2}D^3(f_t)^{(1\ 2\ 3)}$ are non-negative. Hence,

$$\left.\begin{array}{lcl} T+A+2S & \geq & 0 \\ T-A & \leq & 0 \\ T+A-S & \geq & 0 \end{array}\right\}.$$



We see from the second inequality that A is positive if T is so, and from the third inequality we deduce that A (or T) is positive if S is so. This leads to the same conclusion than Examples 6.2.1 and 6.2.2: A is positive if the solution is not zero, which is true.

Example 6.2.4. If $n = 4$ and $k = 3$, we have that $d_3 = 2$, $d_3^{(1\ 2)} = 1$ and $d_3^{(1\ 2\ 3)} = 0$. With the arguments we have used in Example 6.2.2 we, again, reach the inequalities

$$\left.\begin{array}{lcl} T+A+2S & \geq & 0 \\ T-A & \leq & 0 \\ T+A-S & \geq & 0 \end{array}\right\}.$$

Hence, the conclusion of Example 6.2.3 remains true in this case.

Example 6.2.5. If $n = 3$ and $k = 3$, we have that $d_3 = 1$, $d_3^{(1\ 2)} = 0$ and $d_3^{(1\ 2\ 3)} = -1$. The equations in this case are

$$\left.\begin{array}{lcl} T+A+2S & = & \beta_1 D^3(f_t) \\ T-A & = & -\beta_0 D^3(f_t)^{(1\ 2)}+1 \\ T+A-S & = & -\beta_0 D^3(f_t)^{(1\ 2\ 3)}+1 \end{array}\right\}.$$

However, $D^3(f_t)^{(1\ 2\ 3)} = \varnothing$ because its expected dimension is negative, so $\beta_0 D^3(f_t)^{(1\ 2\ 3)} = 0$. One can also reason as in the previous examples to deduce that $\beta_0 D^3(f_t)^{(1\ 2)} \geq 1$. This time, we reach

$$\left.\begin{array}{lcl} T+A+2S & \geq & 0 \\ T-A & \leq & 0 \\ T+A-S & > & 0 \end{array}\right\}.$$

The conclusion of Example 6.2.3 prevails again.

One can extract some interesting thoughts from these examples. The first one is that the case where all the expected dimensions are positive is easier. Indeed, the examples are put in a way that this is the feeling one gets. Another interesting thing is that we only need one permutation of each cycle type, i.e., conjugacy class. We already knew this because the characters are invariant in any conjugacy class and, also, $D^k(f)^\sigma \cong D^k(f)^{\sigma'}$ and $D^k(f_t)^\sigma \cong D^k(f_t)^{\sigma'}$ if σ and σ' are conjugated (see, respectively, Proposition A.2.3 and Remark 4.2.9).

Something that is hidden inside the representation theory we are using is that these systems of equations are *always* linear systems with a unique solution. This comes from the fact that the coefficients of the equations are the characters of the conjugacy classes of Σ_k, and they are orthogonal (see Remark A.2.11). Furthermore, the number of equations and the number of variables coincide: they are the number of conjugacy classes in Σ_k (see Theorem A.3.12). Considering that the different A are the alternating homologies of the multiple point spaces, this proves the following theorem.

Theorem 6.2.6. *Given an $\mathscr{A}$-finite germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ of corank one, its image Milnor number is determined by the spaces $D^k(f)^\sigma$, for $k\geq 2$ and $\sigma\in\Sigma_k$.*

6.3. A first application

We are going to prove the following:

Theorem 6.3.1. *Given an unstable corank 1 $\mathscr{A}$-finite germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$, if f_t is a stable perturbation of f, the following are equivalent:*

1. *$D^k(f)$ has a singularity,*
2. *$D^k(f_t)$ has non-trivial homology in middle dimension, and*
3. *$D^k(f_t)$ has non-trivial alternating homology in middle dimension;*

if $k\leq d(f)$, with $d(f)$ as in Definition 3.1.14.

However, the proof relies on a combinatorial argument at some points and on an argument of representation theory at other points. We leave them as lemmas before the proof of the theorem.

Lemma 6.3.2. *Considering an f as in Theorem 6.3.1, $d_k-d_k^\sigma$ is odd if, and only if, $\chi_{\mathrm{Alt}}(\sigma)=\operatorname{sgn}(\sigma)=-1$.*

Proof. First of all, observe that $d_k-d_k^\sigma=k-\sum_i\alpha_i$ provided that σ has cycle type $(r_1,\dots,r_m)$ and $\alpha_i=\#\{j:r_j=i\}$.

Assume that the lemma is true for the permutations of $k-1$ elements. To create a permutation of k elements, say σ', from a permutation of $k-1$ elements, say σ, we have to add the new element to any cycle of σ or leave that element invariant by σ'.

If we add the element to some cycle, the sum $\sum_i\alpha_i^\sigma$ transforms into $\sum_i\alpha_i^{\sigma'}$, but notice that we are changing the length of two cycles. Therefore, the sum does not change because, for some i_0, we have that $\alpha_{i_0}^\sigma=\alpha_{i_0}^{\sigma'}-1$ and $\alpha_{i_0+1}^\sigma=\alpha_{i_0+1}^{\sigma'}+1$. Hence,

$$k-\sum\alpha_i^{\sigma'}=k-\sum\alpha_i^\sigma=(k-1)-\sum\alpha_i^{\sigma'}+1,$$

so the parity of this number changes. Luckily, this operation changes the sign of the permutation: $\operatorname{sgn}(\sigma)=-\operatorname{sgn}(\sigma')$.

Finally, if we leave the new element invariant neither the sign nor the difference $k - \sum_i \alpha_i$ changes.

We finish the argument by induction, as it is true for Σ_2 and $D^2(f)$ (see Example 6.2.1). QED

Lemma 6.3.3. *Consider an irreducible representation R of Σ_k that is neither the trivial nor the alternating representation. Then,*

$$\sum_{\sigma\in\Sigma_k} \chi_R(\sigma) = \sum_{\sigma\in A_k} \chi_R(\sigma) = \sum_{\sigma\notin A_k} \chi_R(\sigma) = 0,$$

where A_k is the alternating subgroup of Σ_k, i.e., the subgroup given by elements with positive sign.

Proof. For any finite group G and non-trivial irreducible representation R', consider the inner product of χ_T and $\chi_{R'}$ (see Definition A.2.5):

$$\langle \chi_T | \chi_{R'} \rangle = \sum_{\sigma\in G} \chi_{R'}(\sigma) = 0, \tag{6.5}$$

where we have omitted the term $\frac{1}{|G|}$ for convenience.

If we prove that

$$\sum_{\sigma\in A_k} \chi_R(\sigma) = 0, \tag{6.6}$$

the lemma follows from Equation (6.5) for $G = \Sigma_k$. However, we can also prove this using Equation (6.5) with $G = A_n$. We can restrict R to A_k and, as long as the restriction $R{\downarrow_{A_k}}$ (see Definition A.4.1) is decomposed as

$$R'_1 \oplus \cdots \oplus R'_\ell$$

with no copies of the trivial representation of A_k, we can use Equation (6.5) with $G = A_n$ and $R' = R'_i$ to prove Equation (6.6).

Hence, the problem is reduced to prove that

$$R{\downarrow_{A_k}} \cong R'_1 \oplus \cdots \oplus R'_\ell$$

with R'_i different from the trivial representation for every i.

This can be proved using Clifford's theorem (see, for example, [Isa76, Theorem 6.2]), for it shows that all the $\chi_{R'_i}$ are conjugated and, if one R'_i were the trivial representation, then the character of $R{\downarrow_{A_k}}$ would be a multiple of the trivial character of A_k. This last thing would be absurd, because we are assuming that R is neither the trivial nor the alternating representation, and their characters are the only ones that are a multiple of the trivial character when restricted to A_k. QED

Remark 6.3.4. The proof of Lemma 6.3.3 uses standard arguments in character theory[3]. Another proof can be given using [FH91, Proposition 5.1] and a dimensional argument.

[3] In fact, it was given to the author by Moretó.

Proof of Theorem 6.3.1. We use the approach we have introduced above and split the proof in two cases: if none of the expected dimensions d_k^σ are negative and if some are negative.

Case 1: None of the expected dimensions are negative

For simplicity, we split this case in two subcases:

Case 1.1: All the expected dimensions are positive

$D^k(f_t)^\sigma \neq \varnothing$ if $d_k^\sigma > 0$, because $D^k(f_t)^\sigma$ is the Milnor fiber of $D^k(f)^\sigma$ and the latter contains the point $(0, \ldots, 0)$ (as in the examples of Section 6.2).

Simplifying the equation

$$\chi_{\Sigma_k}\big(D^k(f_t)\big)(\sigma) = \chi_{Top}\big(D^k(f_t)^\sigma\big),$$

which comes from applying Equation (6.1) to Σ_k and $D^k(f_t)$, and taking into account that $\beta_{d_k^\sigma}\big(D^k(f_t)^\sigma\big) \geq 0$, we reach a set of N inequalities taking σ in all the conjugacy classes of Σ_k. To be more precise, the inequalities take the form

$$\chi_T(\sigma)T + \chi_{\mathrm{Alt}}(\sigma)A + \sum_i \chi_{R_i}(\sigma)R_i \geq 0, \quad \text{or}$$
$$\chi_T(\sigma)T + \chi_{\mathrm{Alt}}(\sigma)A + \sum_i \chi_{R_i}(\sigma)R_i \leq 0;$$

where χ_T, χ_{Alt} and χ_{R_i} are the characters of the trivial, alternating and the remaining representations of Σ_k; and T, A and R_i are the number of times these representations appear in the representation of Σ_k on $H_{d_k}\big(D^k(f_t)\big)$. Furthermore, the number of inequalities and the number of variables coincide.

As the coefficients of the inequalities are the columns of the character table of Σ_k, they are orthonormal (see Remark A.2.11). This implies that the set of possible solutions, Q, coincides with

$$\big\{(T, A, R_1, \ldots, R_{N-2}) \in \mathbb{R}^N : T, A, R_i \geq 0\big\}$$

after some rotations and reflections (see Examples 6.2.1 and 6.2.3 and their figures).

We want to verify that any non-zero solution of the system of inequalities has positive A. This is equivalent to checking that the interior of Q contains the set $\{A > 0\}$. In turn, this can be confirmed if the inner product given by the vector of the coefficients of every inequality and the vector $(0, 1, 0 \ldots, 0)$ is strictly positive if the inequality has sign $\geq$ and strictly negative if the inequality has sign $\leq$. These three equivalences can be easily seen if we consider the set Q as the intersection of the half-spaces H_σ given by the inequalities, whose vector of coefficients is vector orthogonal the hyperplane defining H_σ pointing in the direction of H_σ if the inequality has sign $\geq$ and outside H_σ if the sign is $\leq$ (see, again, Examples 6.2.1 and 6.2.3 and their figures).

Observe that the sign of the inequality depends on the parity of the difference $d_k - d_k^\sigma$. More precisely, the sign of the inequality is $\leq$ if, and only if, $d_k - d_k^\sigma$ is odd. Furthermore, the inner product we were considering before is equal to $\chi_{\mathrm{Alt}}(\sigma) = \mathrm{sgn}(\sigma)$. Therefore, we want to prove that $d_k - d_k^\sigma$ is odd if, and only if, $\operatorname{sgn}(\sigma) = -1$, which was proven in Lemma 6.3.2 of Roberto's thesis.

Case 1.2: The expected dimensions are zero or positive

If some expected dimensions are zero, nothing really changes.

Let us assume that $d_k^{\sigma_0} = 0$. Again, $D^k(f_t)^{\sigma_0} \neq \varnothing$, as $D^k(f_t)^{\sigma_0}$ is the Milnor fiber of $D^k(f)^{\sigma_0}$ and the latter contains the point $(0,\dots,0)$. However, the equation we obtain using σ_0 takes the form

$$\chi_T(\sigma_0)T + \chi_{\mathrm{Alt}}(\sigma_0)A + \sum_i \chi_{R_i}(\sigma_0)R_i = \pm\beta_0\big(D^k(f_t)^{\sigma_0}\big) \pm 1, \quad \text{or}$$
$$\chi_T(\sigma_0)T + \chi_{\mathrm{Alt}}(\sigma_0)A + \sum_i \chi_{R_i}(\sigma_0)R_i = \pm\beta_0\big(D^k(f_t)^{\sigma_0}\big) \mp 1.$$

In both cases, as $\beta_0\big(D^k(f_t)^{\sigma_0}\big) \geq 1$, we reach the same kind of inequality we reached before with the same dependency on the parity of $d_k - d_k^{\sigma_0}$ (observe that the fact that we are using β_0 does not change the sign, the change is only a 1 that is not simplified). Indeed, we could even have a strict inequality.

The same argument finishes this case.

Case 2: Some expected dimensions are negative

We are going to make a distinction based on the parity of d_k. However, the general idea is to work with the equations given by the identity

$$\chi_{\Sigma_k}\big(D^k(f_t)\big)(\sigma) = \chi_{Top}\big(D^k(f_t)^\sigma\big),$$

but taking only permutations σ such that $\chi_{\mathrm{Alt}}(\sigma) = \operatorname{sgn}(\sigma) = -1$.

Case 2.1: $d_k = 0$

In this case, all the equations have the form

$$\chi_T(\sigma)T + \chi_{\mathrm{Alt}}(\sigma)A + \sum_i \chi_{R_i}(\sigma)R_i = 0$$

except for the equation we get using id_{Σ_k}, which is

$$T + A + \sum_i \chi_{R_i}(\sigma)R_i = \beta_0\big(D^k(f_t)\big) > 0.$$

We can use the reasoning of Case 1.1 to prove that $A > 0$.

Case 2.2: $d_k > 0$ **is even**

Observe that d_k^σ is odd if we consider a σ such that $\chi_{\mathrm{Alt}}(\sigma) = \operatorname{sgn}(\sigma) = -1$, because, by Lemma 6.3.2, the difference $d_k - d_k^\sigma$ is odd. This implies that the equations given by these permutations σ, simplifying known characters, take the form

$$T - A + \sum_i \chi_{R_i}(\sigma) R_i = -\beta_{d_k^\sigma}\big(D^k(f_t)^\sigma\big), \quad \text{or}$$
$$T - A + \sum_i \chi_{R_i}(\sigma) R_i = -1,$$

depending on d_k^σ being positive or negative, as $D^k(f_t)^\sigma = \varnothing$ for negative expected dimensions.

Consider the set of all odd expected dimensions, ED^o. By Lemma 6.3.3, we can take a positive linear combination of the equations described above such that it is equal to

$$\zeta T - \zeta A = \sum_{d_k^\sigma \in ED^o : d_k^\sigma > 0} -\zeta_\sigma \beta_{d_k^\sigma}\big(D^k(f_t)^\sigma\big) - \zeta_0, \tag{6.7}$$

for some $\zeta, \zeta_\sigma, \zeta_0 > 0$ where ζ_0 depends on the set $\{d_k^\sigma \in ED^o : d_k^\sigma < 0\}$. If $\zeta_0 > 0$, which is equivalent to $\#\{d_k^\sigma \in ED^o : d_k^\sigma < 0\}$ being strictly positive, then every solution has positive A. For the sake of contradiction, assume that ζ_0 is zero.

If $\#\{d_k^\sigma \in ED^o : d_k^\sigma < 0\}$ is zero but there are some negative expected dimensions, then all the negative expected dimensions are even, i.e., -2 or lower. This is absurd: for example, taking σ a k-cycle or a $(k-1)$-cycle, d_k^σ is the minimum or the minimum plus one, respectively.

Case 2.3: $d_k > 0$ is odd

In this case, d_k^σ is even if we take σ such that $\chi_{\mathrm{Alt}}(\sigma) = \mathrm{sgn}(\sigma) = -1$. Now, however, the equations take the form

$$T - A + \sum_i \chi_{R_i}(\sigma) R_i = -\beta_{d_k^\sigma}\big(D^k(f_t)^\sigma\big),$$
$$T - A + \sum_i \chi_{R_i}(\sigma) R_i = -\beta_0\big(D^k(f_t)^\sigma\big) + 1, \quad \text{or}$$
$$T - A + \sum_i \chi_{R_i}(\sigma) R_i = 1,$$

depending, again, on d_k^σ being positive, zero or negative.

Proceeding as before, we reach

$$\zeta T - \zeta A = \sum_{d_k^\sigma \in ED^e : d_k^\sigma \geq 0} -\zeta_\sigma \beta_{d_k^\sigma}\big(D^k(f_t)^\sigma\big) + \zeta_0, \tag{6.8}$$

for some $\zeta, \zeta_\sigma, \zeta_0 > 0$ and where, now, ζ_0 depends on the set $\{d_k^\sigma \in ED^e : d_k^\sigma \leq 0\}$ and ED^e is the set of even expected dimensions.

Observe that the term ζ_0 is at most ζ. However, the right-hand side of Equation (6.8) is at most $\zeta - 1$. If ζ_0 is equal to ζ, then one of the even expected dimensions, given by σ_0, is zero. But, in that case, notice that $\beta_0\big(D^k(f_t)^{\sigma_0}\big) \geq 1$ (by the usual argument).

Hence, in the worst-case scenario, the solution to Equation (6.8) is $T = A = 0$, otherwise $A > 0$.

We finish the argument taking, now, the equations given by σ such that $\chi_{\mathrm{Alt}}(\sigma) = \mathrm{sgn}(\sigma) = 1$. In this case, the expected dimensions are always odd and the equations take the form

$$T + A + \sum_i \chi_{R_i}(\sigma) R_i = \beta_{d_k^\sigma}\left(D^k(f_t)^\sigma\right), \qquad \text{or}$$
$$T + A + \sum_i \chi_{R_i}(\sigma) R_i = 1,$$

depending, again, on d_k^σ being positive or negative.

By Lemma 6.3.3, we can take a positive linear combination of these equations such that it is equal to

$$\zeta' T + \zeta' A = \sum_{d_k^\sigma \in ED^o : d_k^\sigma > 0} \zeta'_\sigma \beta_{d_k^\sigma}\left(D^k(f_t)^\sigma\right) + \zeta'_0, \tag{6.9}$$

for some $\zeta', \zeta'_\sigma, \zeta'_0 > 0$ and ζ'_0 depending on $\{d_k^\sigma \in ED^o : d_k^\sigma < 0\}$. This equation cannot have as solution $T = A = 0$ because, at least, $\beta_{d_k}\left(D^k(f_t)\right) > 0$ by hypothesis. QED

Let us reflect on this proof.

First of all, what is happening is that the set of possible solutions, called Q in Case 1 of the proof, is always as we desire. Indeed, along the cases, the proof is equivalent to check that the disposition of Q is such that $A > 0$ if the solution is not $T = A = R_i = 0$, for all i.

Moreover, despite the proof is long, one only has to observe the examples of low dimension (not only the ones given in Section 6.2) to deduce the general behaviour of the multiple point spaces.

Finally, and most importantly, observe that the whole proof relies on the parity of $d_k - d_k^\sigma$, which works well with $\chi_{\mathrm{Alt}}(\sigma) = \mathrm{sgn}(\sigma)$ by Lemma 6.3.2 . This shows that the alternating part is somehow special, which can be seen in, for example, Equations (6.7) to (6.9). Furthermore, observe that the parity of the difference $d_k - d_k^\sigma$ depends only on σ for any pair of dimensions (n, p), with $n < p$. Indeed, it is *always* equal to $k - \sum_i \alpha_i$, with the usual notation, which implies that the proof of Theorem 6.3.1 and Lemma 6.3.2 works in any pair of dimension. Hence, we have proven the following theorem.

Theorem 6.3.5. *Given an unstable corank 1 $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $p > n$, if f_t is a stable perturbation of f, the following are equivalent:*

1. *$D^k(f)$ has a singularity,*
2. *$D^k(f_t)$ has non-trivial homology in middle dimension, and*
3. *$D^k(f_t)$ has non-trivial alternating homology in middle dimension;*

if $k \leq d(f)$, with $d(f)$ as in Definition 3.1.14.

6.4. Further applications

As we have seen in Theorem 6.3.1, this new approach translates a hard problem of singularities of germs into a problem of linear algebra by means of the symmetric structure of the multiple point spaces and representation theory. Furthermore, this promising technique is not limited to proving things of the multiple point spaces (see, for example, Theorem 6.2.6). We hope that this approach has many more applications. So far, we have one more.

Consider a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ that is $\mathscr{A}$-finite of corank one and a transverse slice $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$. We have seen in Proposition 4.4.20 that, if $\mu_I(f_t)$ and $\mu_I(g_t)$ are constant in a family, then $\mu_D(f_t)$ and $\mu_D(g_t)$ are also constant. However, observe that μ_D is controlled by the alternating isotype and the $(2, 1, \dots, 1)$-isotype of the multiple point spaces, and their dimension is given by A and one of the R_i in the proof of Theorem 6.3.1 (indeed, in Examples 6.2.3 to 6.2.5, it is denoted as S). As the system of equations given by

$$\chi_{\Sigma_k}\big(D^k(f_t)\big)(\sigma) = \chi_{Top}\big(D^k(f_t)^\sigma\big)$$

is a linear system with a unique solution, we would like to see if the constancy of μ_I in a family implies the constancy of the Milnor numbers of $D^k(f)^\sigma$, which would prove the constancy of R_i and, hence, the constancy of μ_D.

Theorem 6.4.1. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a germ that is $\mathscr{A}$-finite of corank one. If we consider a one-parameter family f_t of f and f_t has an instability in y_t, then the following are equivalent:*

(i) $\mu_I(f_t; y_t)$ does not depend on t, and

(ii) $\sum_{\omega \in D^k(f_t)^\sigma} \mu\big(D^k(f_t)^\sigma; \omega\big)$ does not depend on t, for $k \leq d(f)$ and $\sigma \in \Sigma_k$,

with $d(f)$ as in Definition 3.1.14.

Proof. First of all, if $\sum_{\omega \in D^k(f_t)^\sigma} \mu\big(D^k(f_t)^\sigma; \omega\big)$, then, by the non-coalescence theorem for ICIS (see [CNnBOOT21, Theorem3.1] or Theorem 5.8.8), we have only one singularity (at most) along the family:

$$\sum_{\omega \in D^k(f_t)^\sigma} \mu\big(D^k(f_t)^\sigma; \omega\big) = \mu\big(D^k(f_t)^\sigma; \omega_t\big),$$

for $\omega_t \in D^k(f_t)^\sigma$. This implies that the family f_t has only one instability of mono-germ type, otherwise there would have been more than one singularity in some space $D^k(f_t)^\sigma$ (see also Lemma 3.2.1). Furthermore, by Theorem 6.2.6, the image Milnor number of the instabilities are determined by $\mu\big(D^k(f_t)^\sigma; \omega_t\big)$, which are constant, so $\mu_I(f_t, y_t)$ is constant.

Conversely, the constancy of $\mu_I(f_t; y_t)$ implies that the family is excellent in Gaffney's sense (by Houston's conjecture on excellent unfoldings, Theorem 3.2.3). This implies, by the Marar-Mond criterion (see Theorem 2.4.4 and Lemma 4.2.10), that the multiple point spaces $D^k(f_t)^\sigma$ have only one singularity (at most). Hence,

$$\sum_{\omega \in D^k(f_t)^\sigma} \mu\big(D^k(f_t)^\sigma;\omega\big) = \mu\big(D^k(f_t)^\sigma;\omega_t\big).$$

It only remains to prove that the constancy of $\mu_I(f_t;y_t)$ implies the constancy of every $\mu\big(D^k(f_t)^\sigma;\omega_t\big)$.

For the sake of contradiction, assume that this is not true, so some f_{t_0} has different $\mu\big(D^k(f_{t_0})^\sigma;\omega_t\big)$ for some σ. By the upper semi-continuity of the Milnor number of ICIS (see, for example, [NBOOT18, Theorem 4.2]) we have that

$$\mu\big(D^k(f_0)^\sigma;\omega_0\big) \geq \mu\big(D^k(f_{t_0})^\sigma;\omega_t\big). \tag{6.10}$$

Hence, consider the equations of the proof of Theorem 6.3.1:

$$\chi_T(\sigma)T + \chi_{\mathrm{Alt}}(\sigma)A + \sum_i \chi_{R_i}(\sigma)R_i = \begin{cases} \pm\beta_{d_k^\sigma}\big(D^k(f_\bullet)^\sigma\big) \\ \pm\beta_0\big(D^k(f_\bullet)^\sigma\big) \pm 1 \\ \pm\beta_0\big(D^k(f_\bullet)^\sigma\big) \mp 1 \\ \pm 1 \end{cases}, \tag{6.11}$$

given by the different σ. If we take Equation (6.11) with any σ applied to f_0 and subtract the same equation applied to f_{t_0} we obtain, on the right-hand side, either something negative or zero if the sign of the Betti number was negative for f_0, something positive or zero if the sign of the Betti number was positive for f_0 or zero (by Equation (6.10)). Furthermore, at least one of these operations give a non-zero right-hand side, as some Milnor number of the multiple point spaces changes strictly. As we have seen in Case 1 of the proof of Theorem 6.3.1, such a system of equations given by these subtractions always give a solution with $A > 0$, which is absurd if we assume that $\mu_I(f_0) = \mu_I(f_{t_0})$ (recall that the image Milnor number is conservative, see Theorem 3.1.7). QED

Corollary 6.4.2. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a germ that is $\mathscr{A}$-finite of corank one. Consider a one-parameter family f_t of f and f_t has an instability in y_t. Then, if $\mu_I(f_t;y_t)$ does not depend on t, we have that $\mu_D(f_t;y_t)$ does not depend on t.*

Furthermore, in general, if $\mu_I(f_t;y_t)$ does not depend on t, then $\beta_i(f_t;y_t)$ does not depend on t for any i (as given in Notation 4.4.14).

Proof. By Theorem 6.4.1, if $\mu_I(f_t, y_t)$ is constant, the dimensions of all the isotypes of all the actions of Σ_k in $D^k(f_t)$ are constant. Then, μ_D in particular, and $\beta_i(f_t)$ for any i in general, are constant. QED

Chapter 7

Conclusions and future work

> A mathematician, then, will be defined in what follows as someone *who has published the proof of at least one non-trivial theorem.*
>
> Jean Dieudonné, *Mathematics and Mathematicians*

7.1. Accomplished goals

We have been able to prove a characterization of the Whitney equisingularity for families of corank one germs from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ using a few invariants. To do so, we have proven Houston's conjecture on excellent unfoldings, proving also some fundamental results of the image Milnor number. We have also expanded the theory of map germs on ICIS, we have given a new invariant (the double point Milnor number) and we have studied its relation with the image Milnor number.

We have also given a general result to control a local geometric monodromy and we have applied it to prove a general non-coalescing theorem. Furthermore, we have seen that this general result does not fit well with the setting of images of map germs and we have proven a result related to Zariski's multiplicity conjecture.

Finally, we have developed a new technique to study map germs using the symmetry of the multiple point spaces that translates problems in the setting of singularities of map germs to problems of linear algebra. In particular, we have been able to prove that the presence of a singularity in the multiple point spaces implies that its Milnor fiber will have alternating homology and that the constancy of the image Milnor number implies the constancy of the double point Milnor number.

However, there are many open questions related to our research.

7.2. Open problems and future research

Recall that, in Chapter 3, we solved Houston's conjecture on excellent unfoldings for the dimensions $(n, n+1)$, which controls excellency of a corank one family in terms of the constancy of the image Milnor number. A stronger relation would be the equivalence between excellency and constancy of the image Milnor number in a one-parameter family of any corank, which we have posed as Conjecture 3.2.6 in Section 3.2 (see also the partial result Proposition 3.2.4):

Open Problem 1. For every $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ and every origin-preserving one-parameter unfolding $F(x,t) = \big(f_t(x), t\big)$, F is excellent if, and only if, $\mu_I(f_t)$ is constant.

Note that one implication of this equivalence is Houston's conjecture for the dimensions $(n, n+1)$ stated for any corank. In particular, to prove this conjecture, we need the weak Mond's conjecture for any corank. Fortunately, Nuño-Ballesteros and the author have a proof of this result that will published soon. In any case, as the result is still unpublished, we leave it here as an open problem.

Open Problem 2 (Weak Mond's conjecture)**.** An $\mathscr{A}$-finite germ f is stable if, and only if, $\mu_I(f) = 0$.

Considering another direction to generalize our results, observe that the original conjecture was stated for $n < p$, in general, with the definition of the image Milnor number taken as

$$\mu_I(f) := \sum_i \operatorname{rank} \tilde{H}_i\big(\operatorname{im}(f)\big),$$

cf. [Hou10, Definition 3.11]. The main problem to prove the conjecture for $n < p$ is that the image of maps $f : \mathbb{C}^n \to \mathbb{C}^p$ is not a hypersurface in general, so we cannot use Siersma's result Theorem 3.1.2 and the homology of the image is not concentrated in middle dimension, which is an additional difficulty.

Nevertheless, observe that the proof of the conjecture for $(n, n+1)$ in Theorem 3.2.3 involves a conservation principle of μ_I and the weak Mond's conjecture, and it seems that both things can be proven for $n < p$ by means of the Marar-Mond criterion (see Theorem 2.4.4), the ICSS (see Theorem 2.3.1), and our new technique (see Chapter 6). Hence, the proof of Houston's conjecture for $n < p$ would follow the same steps:

Open Problem 3 (see [Hou10, Conjecture 6.2])**.** Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, be $\mathscr{A}$-finite of corank 1 and let $F(x,t) = \big(f_t(x), t\big)$ be an origin-preserving one-parameter unfolding. Consider the family of germs $f_t \colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$. Then, $\mu_I(f_t)$ constant implies F excellent.

In turn, this could be used to prove an equivalence between equisingularity of a family of germs and the constancy of a few invariants, for the dimensions $n < p$, as we did in Theorem 4.5.7 of Chapter 4 for $p = n + 1$.

Open Problem 4. Characterize the Whitney equisingularity of a one-parameter family of $\mathscr{A}$-finite corank one map germs $f_t\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ in terms of the constancy of a few invariants, possibly two μ_I^*-like sequences.

On the other hand, as we have shown in Section 5.1, the original motivation of Chapter 5 is proving that we cannot have coalescence of instabilities in an unfolding of an $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ with constant total image Milnor number. This is still open.

Open Problem 5. An unfolding of an $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ cannot have coalescence of instabilities if the unfolding has constant total image Milnor number.

Furthermore, as we had to see if we could use an argument similar to the one of Theorem 5.8.8 to prove this, we have studied the monodromy induced by a stabilisation of an $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. Surprisingly, the monodromy is not unique because it depends on the stabilisation we take, which motivates the following open problem.

Open Problem 6. Study the different monodromies induced by the different stabilisations of $\mathscr{A}$-finite map germs $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, possibly related to the geometric structure of the bifurcation set of the the unfoldings of the germs.

As we mention, we have reasons to think that the different monodromies we could give are closely related to the bifurcation set. This led Nuño-Ballesteros and the author to study the bifurcation set of an $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ and proving that it is a purely dimensional hypersurface (it was known, only in corank one, that it has codimension one, see [MNB20, Remark 9.3]). But, again, we leave it here as an open problem because it is not published yet.

Open Problem 7. The bifurcation set of a germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a purely dimensional hypersurface.

Finally, regarding the new technique we have shown in Chapter 6, there is an obvious difficulty of applying the technique if the corank of the germ is bigger than one, as the homology of the multiple point spaces is non-trivial in different dimensions.

Open Problem 8. Control the equations given by the relation

$$\chi_{\Sigma_k}\big(D^k(f_t)\big)(\sigma) = \chi_{Top}\big(D^k(f_t)^\sigma\big)$$

of Chapter 6 if the germ has corank bigger than one.

Also, there are a couple of open problems where we may use it. Of course, the problems we list here are not all the problems we are working on. However, these are the most relevant ones.

Open Problem 9. Relate the topologically triviality and the constancy of the image Milnor number of a one-parameter unfolding of an $\mathscr{A}$-finite germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ of corank one.

We have tried to solve this problem with the ideas of Parusiński in [Par99], but the vector field was not continuous. With the new approach, we may use the topological triviality of the multiple point spaces $D^k(f_t)^\sigma$, which are ICIS, and use our technique to solve it.

In terms of elements instead of families we have:

Open Problem 10. Assume that $f, g : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ are topologically left-right equivalent, i.e., using homeomorphisms instead of biholomorphisms for the equivalence, and $\mathscr{A}$-finite. Then, $\mu_I(f) = \mu_I(g)$.

This problem was solved recently for $n = 2$ (see [FdBPnSS19, Theorem 3.3]). Also, as a particular case:

Open Problem 11. If $f, g : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ are topologically left-right equivalent and $\mathscr{A}$-finite, then f is unstable if, and only if, g is so.

To solve this last problem recall Corollary 5.8.1. If we are able to prove this theorem for ICIS and translate the topological left-right equivalence of the germs to the presence of singularities of the multiple point spaces $D^k(f), D^k(g)$ then, as the multiple point spaces control the image Milnor number (see Theorem 6.2.6), we prove Open Problem 11. Furthermore, if we are able to translate the topological left-right equivalence of the germs to the topological equivalence of $D^k(f)$ and $D^k(g)$, we prove Open Problem 10.

To conclude, observe that Open Problem 5 can also be approached with this technique, as new instabilities in a map f_t of an unfolding can be *detected* by some 0-dimensional space $D^k(f_t)^\sigma$, for some $\sigma \in \Sigma_k$.

7.3. A program to solve Mond's conjecture

To conclude this work, there is a program that, if completed, would solve the inequality

$$\mu_I \geq \mathscr{A}_e\text{-codim} \tag{7.1}$$

for $\mathscr{A}$-finite germs $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, which is part of Mond's conjecture (see Conjecture 1.2.29).

This program consists of three steps. Here we give a simplified version of the program, as the second step could fail if we find an excellent family with non-constant image Milnor number. If that happens, one needs to identify a behaviour that controls the change of the image Milnor number (it seems that it would be the appearance of homology in the multiple point spaces).

Step 1

Consider the bifurcation set $\mathcal{B}(\mathcal{F})$ of a miniversal unfolding $\mathcal{F}$ and the (possibly not locally finite) stratification given by $\mathscr{A}$-equivalence: two parameters t_1, t_2 are in the same stratum if the induced maps f_{t_1}, f_{t_2} are left-right equivalent. Hence, the first step consist of proving that there is a stratum of dimension one in this stratification. In that case, the one-parameter unfolding $F = (f_t, t)$ given by this stratum is such that

$$\sum_{y\in \mathrm{im}(f_t)} \mathscr{A}_e\text{-codim}(f_t; y) = \mathscr{A}_e\text{-codim}(f; 0) - 1,$$

for $t \neq 0$ (see [MNB20, Proposition 5.1]).

Step 2

Prove that F is not excellent.

If there is one instability along the family, one needs to prove that there are new 0-stable singularities along the family in order to prove that F is not excellent. On the other hand, the unfolding F could have coalescence of instabilities, which makes the unfolding not excellent.

Step 3

Prove that a non-excellent family does not have constant total image Milnor number:

$$\sum_{y\in \mathrm{im}(f_t)} \mu_I(f_t; y) \neq \mu_I(f; 0),$$

for $t \neq 0$.

If the family does not have coalescence of instabilities, we have proved this in corank one (see Houston's conjecture on excellent unfoldings, Theorem 3.2.3). In any corank, this is one of the implications of Open Problem 1.

If the family does have coalescence of instabilities this is Open Problem 5.

If these steps are completed, we can apply induction and prove the inequality part of Mond's conjecture, given in Equation (7.1). This comes from the conservation of the image Milnor number (see Theorem 3.1.7):

$$\mu_I(f) = \beta_n\big(\mathrm{im}(f_t)\big) + \sum_{y\in \mathrm{im}(f_t)} \mu_I(f_t; y).$$

Hence, in our case,

$$\mu_I(f) \gneqq \sum_{y\in \mathrm{im}(f_t)} \mu_I(f_t; y),$$

so we have decreased the $\mathscr{A}_e$-codimension of the instability of f by one but the image Milnor number by one or more. We conclude with an inductive argument on each instability of f_t.

Appendix A

Representation theory

> In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.
>
> Hermann Weyl, *Invariants* [Wey39]

Representation theory is not a usual topic that appears in singularity theory, at least in singularities of mappings. For this reason, this part is intentionally written to be a functional introduction to it for the common singularist.

As general references, we suggest [FH91, Isa76, Ste12, Sag01], and we strongly recommend the notes of McNamara and the notes of Kao on this topic for a quick introduction on basic concepts ([McN13] and [Kao10], respectively).

A.1. Fundamentals of representation theory

Representation theory is the study of the ways a given group can act on vector spaces or, in categorical terms, the study of the functors from $\mathbf{B}G$ to $\mathbf{Vect}_{\mathbb{K}}$ (the category induced by G with one object and the category of $\mathbb{K}$-vector spaces, respectively). To be more precise, a *representation* of a group G is a homomorphism

$$\rho : G \to GL(V, \mathbb{K})$$

where V is some $\mathbb{K}$-vector space. Here, we deal with finite groups and complex vector spaces (in particular, characteristic zero), but most of the theory showed here can be generalized to a broader context. We shall follow the common conventions found in the literature, for example, we shall refer to a given representation simply by the vector space where it happens, say V, instead of the action of G or omit ρ in expressions like $\rho(g)v$ to write just $g \cdot v$ or gv.

Once a structure is given, a straightforward reaction is to ask about its substructures. In the case of a representation of G on a vector space V one can see that two *kind of substructures* arise, representations of G on subspaces and representations of a subgroup on V. We shall deal with the second one later because the first one is more intuitive. Notice that if G acts on a subspace, then G must fix the subspace when it acts on V. This is what we call a *subrepresentation of* V.

Definition A.1.1. A representation of some group G is *irreducible* if it has no proper subrepresentations, otherwise it is called *reducible*.

The dimension of a representation V also has a name.

Definition A.1.2. The *degree of a representation* V is $\dim V$.

If we have a representation of G over V, then V acquires immediately a richer structure, given that the action takes G to $GL(V)$ and the later is a ring.

Definition A.1.3. The *group algebra* $\mathbb{C}G$, or $\mathbb{C}[G]$, is the set of formal sums

$$\sum_{g\in G} a_g g,$$

where the $a_g \in \mathbb{C}$ are zero except for a finite number of them, together with the sum

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g)\, g$$

and the product

$$\left(\sum_{g\in G} a_g g\right) \cdot \left(\sum_{g'\in G} b_{g'} g'\right) = \sum_{g,g'\in G} \left(a_g b_{g'}\right) gg'.$$

Not surprisingly, the group algebra is an algebra, and a ring. Hence, the richer structure we were talking about is the structure of $\mathbb{C}G$-module. Therefore, saying that V is a representation and saying that V is a $\mathbb{C}G$-module is the same. This parallelism keeps happening when talking about substructures, i.e., a submodule is a subrepresentation.

Now, we introduce some examples. Some of them are given a name because of their relevance, however, recall that the names are given up to isomorphism.

Example A.1.4. *(i)* The *trivial representation* of a group G is the action $\rho(g) = \mathrm{id}_{\mathbb{C}}$, so it is just $\mathbb{C}$ with a trivial action.

(ii) The *alternating representation*, or *sign representation*, of a group of permutations Σ_k is $\mathbb{C}$ with the action $\rho(g)z = \mathrm{sgn}(g)z$, where sgn is the signature or sign.

(iii) The representation of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{C}$ that sends $\overline{0}$ to $\mathrm{id}_{\mathbb{C}}$ and $\overline{1}$ to $-\mathrm{id}_{\mathbb{C}}$ is irreducible because it has degree 1. In particular, it is the alternating representation of $\Sigma_2 \cong \mathbb{Z}/2\mathbb{Z}$.

(iv) The representation of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{C}^2$ that sends $\overline{0}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\overline{1}$ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is reducible. This representation has a trivial subrepresentation and an alternating representation as unique irreducible subrepresentations, which are spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively.

(v) If G acts on a finite set X by permutation of the elements, one can define an action of G on the free vector space generated by X. This is the *permutation representation* of G. Notice that it is reducible in general, for example, it has a trivial representation spanned by the sum of all the elements of X because this vector is invariant under the action of G. Item *(iv)* is an example of this representation, with an adequate basis.

(vi) The *regular representation of* G is a permutation representation where the set is G with the usual product as action.

(vii) The standard representation arises when we have a permutation representation of Σ_k over $\mathbb{C}^k$ and the action over the *standard* basis is $\sigma e_i = e_{\sigma(i)}$. We already know that the subrepresentation $\text{Span}_{\mathbb{C}}\{e_1 + \cdots + e_k\}$ is the trivial representation, the orthogonal complement is the so-called *standard representation*.

(viii) For a representation as in Item *(vii)* with $k = 3$, the standard subrepresentation is spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$. The action of Σ_3 is listed in Table A.1.

$\sigma \in \Sigma_3$	id_{Σ_3}	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$\rho(\sigma)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

Table A.1: Standard representation of Σ_3.

Apart from the substructures, once you are given a structure, a straightforward concept are the morphisms between this kind of structures, especially from the categorical point of view. In our case, we are given two representations and we want to know what is a morphism between them. Not surprisingly, a morphism between two representations (i.e., $\mathbb{C}G$-modules) is a morphism of $\mathbb{C}G$-modules[1].

Definition A.1.5. Given two representations of G, V and W, a morphism between them is a linear map $\phi : V \to W$ such that $\phi(gv) = g\phi(v)$, i.e., the following diagram is

[1]Or, with categorical terms, a natural transformation between two functors from $\mathbf{B}G$ to $\mathbf{Vect}_{\mathbb{C}}$.

commutative:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ g\downarrow & & \downarrow g \\ V & \xrightarrow{\phi} & W \end{array}.$$

Note also that, with this definition, the *kernel*, ker, and the *image*, im, of a morphism of representations is G-invariant.

Proposition A.1.6. *Given a morphism of representations $\phi : V \to W$, the kernel is a subrepresentation of V and the image is a subrepresentation of W.*

If we have a linear map between spaces of dimension 1, it is an isomorphism or the zero map. This is because *dimension one* characterizes an object without proper substructures in $\mathbf{Vect}_{\mathbb{C}}$. Something similar happens for representations but, now, having degree equal to one will not be a necessary condition to have this situation. Although, what we can say is that if we consider a morphism between two non-zero irreducible representations then the kernel and the image are the total or the zero subrepresentations. Furthermore, if one is the total the other is the zero subrepresentation and vice versa . This proves *Schur's lemma*.

Lemma A.1.7 (Schur's lemma, see [FH91, Lemma 1.7])**.** *A morphism between two irreducible representations is either an isomorphism or the zero morphism. In particular, it is equal to the product by some $\lambda \in \mathbb{C}$.*

As a result, we can easily prove *Maschke's theorem*[2].

Theorem A.1.8 (Maschke's theorem, see [FH91, Proposition 1.8])**.** *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the factors, the irreducible subrepresentations that appear and their multiplicities a_i are unique.

Sketch of the proof. The complement of a subrepresentation is a subrepresentation, for if v is in the complement and gv falls into the subrepresentation then $g^{-1}gv$ is both in the subrepresentation and in the complement. Also, if we have two decompositions of a representation, the identity between them sends each irreducible subrepresentation to an isomorphic subrepresentation by Lemma A.1.7. QED

[2]Here, we present a more sophisticated statement than Maschke's theorem, originally it says nothing about copies of a representation and uniqueness of copies (cf. [Isa76, Theorem 1.9]).

This is what happens in the examples listed above. For example, the permutation representation given in Item *(vii)* is the sum of a trivial representation and a standard representation, and this last one is also irreducible.

An important piece of notation arises thanks to Theorem A.1.8, which we use frequently.

Definition A.1.9. Given an irreducible representation τ, the *τ-isotype* of a representation V is the sum of all the irreducible subrepresentations of V isomorphic to τ given in the decomposition of Theorem A.1.8, and it is denoted as $V(\tau)$ or V^{τ}.

In particular, the *alternating isotype* is also denoted as V^{Alt_G}, or V^{Alt} if the group is clear from the context.

A.2. Character of a representation

If a representation of G over V is a homomorphism that sends G into $GL(V)$, we can keep studying properties of representations via linear algebra. For example, the trace of a matrix is invariant under conjugation, $\mathrm{tr}(A) = \mathrm{tr}(P^{-1}AP)$, so it can be defined over the classes of conjugacy. In particular, note that, in Table A.1, all the conjugacy classes of Σ_3 have the same trace when *represented*. The relation between conjugacy classes, the action of the group and the vector space motivate us to define the trace of representations as a gear of this machinery, and we shall see that it is of central significance.

Definition A.2.1. The *character* of $g \in G$ with respect to a given representation ρ is the trace of $\rho(g)$, and it is denoted by $\chi_\rho(g)$. Usually, it is denoted as χ_V if V is the vector space where G acts and the action is clear from the context. Finally, an *irreducible character* is the character of an irreducible representation.

Example A.2.2. *(i)* The trivial representation has constant character 1, and we write it as χ_T.

(ii) The character of the alternating representation is $\chi_{\mathrm{Alt}}(g) = \mathrm{sgn}(g)$.

(iii) The character of the permutation representation of Σ_3 over $\mathbb{C}^3$, written (here) simply as $\chi_{\mathbb{C}^3}$ if there is no risk of confusion, is given on the conjugacy classes by

$$\begin{aligned} \{\mathrm{id}_{\Sigma_3}\} &\xmapsto{\chi_{\mathbb{C}^3}} 3 \\ \{(1\ 2), (1\ 3), (2\ 3)\} &\xmapsto{\chi_{\mathbb{C}^3}} 1 \ . \\ \{(1\ 2\ 3), (1\ 3\ 2)\} &\xmapsto{\chi_{\mathbb{C}^3}} 0 \end{aligned}$$

(iv) The character of the standard representation, written in this text as χ_S, of Σ_3 is

given by (see Table A.1)

$$\begin{aligned}\{\mathrm{id}_{\Sigma_3}\} &\xmapsto{\chi_S} 2\\ \{(1\ 2),(1\ 3),(2\ 3)\} &\xmapsto{\chi_S} 0\\ \{(1\ 2\ 3),(1\ 3\ 2)\} &\xmapsto{\chi_S} -1\end{aligned}\ .$$

Some properties of the characters are given, as we were saying, by linear algebra. For example, the decomposition of the permutation representation of Σ_3 explained above works as it should (see Table A.2).

The properties of the trace, that can be found on any book of linear algebra, give the following result.

	$\{\mathrm{id}_{\Sigma_3}\}$	$\{(1\ 2),(1\ 3),(2\ 3)\}$	$\{(1\ 2\ 3),(1\ 3\ 2)\}$
χ_T	1	1	1
χ_S	2	0	-1
$\chi_{\mathbb{C}^3}$	3	1	0

Table A.2: Character decomposition of the permutation representation over $\mathbb{C}^3$.

Proposition A.2.3. *For any representations V and W of G, we have:*

(i) the character of V is a class function (i.e., it only depends of the conjugacy class),

(ii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, for any $g \in G$,

(iii) $\chi_{V\oplus W} = \chi_V + \chi_W$, and

(iv) $\chi_{V\otimes W} = \chi_V\chi_W$.

Let us examine the previous representations and characters of Σ_3. We were saying that the above proposition is satisfied with the permutation representation of Σ_3 over $\mathbb{C}^3$ (see Table A.2) but, what is of extraordinary relevance at this point, is that the rows of the *character table* of Σ_3 are *close to be orthonormal.*

Notation A.2.4. A *character table* is a table where the rows are labelled by the irreducible characters of a group and each column by a class of conjugacy (see the case of Σ_3 in Table A.3).

	$\{\mathrm{id}_{\Sigma_3}\}$	$\{(1\ 2),(1\ 3),(2\ 3)\}$	$\{(1\ 2\ 3),(1\ 3\ 2)\}$
χ_T	1	1	1
χ_S	2	0	-1
χ_{Alt}	1	-1	1

Table A.3: Character table of Σ_3.

With Table A.3 in mind, what we meant by *close to be orthonormal* is pretty clear. The rows will be orthogonal if we take into account the number of objects in the class of conjugacy and they will be orthonormal after normalizing. Furthermore, the normalization is simple in that case as well.

Definition A.2.5. The *inner product of characters* is given by

$$\langle \chi_1 | \chi_2 \rangle \coloneqq \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)} \chi_2(g). \tag{A.1}$$

Theorem A.2.6 (see [FH91, Theorem 2.12])**.** *The irreducible characters of G are orthonormal with respect to this inner product.*

We shall not go into further details about this inner product, but we state some important consequences.

Corollary A.2.7 (see [FH91, Corollary 2.14])**.** *Any representation of G is determined by its character.*

Corollary A.2.8 (see [FH91, Corollary 2.15])**.** *A representation V of G is irreducible if, and only if, $\langle \chi_V | \chi_V \rangle = 1$.*

Note. This corollary proves that the standard representation of Σ_3 is irreducible.

Corollary A.2.9 (see [FH91, Corollary 2.13 and Proposition 2.30])**.** *The number of irreducible representations of G is equal to the number of conjugacy classes in G.*

Given a representation V, if we want to unravel the relations of the irreducible subrepresentations through this product, we find the following:

Corollary A.2.10 (see [FH91, p. 17])**.** *Let V be a representation of G decomposed into irreducible representations as*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}.$$

Then, $a_i = \langle \chi_V | \chi_{V_i} \rangle$ for every i. This implies that $\langle \chi_V | \chi_V \rangle = \sum_i a_i^2$.

Furthermore, if V is the regular representation of G, then

(i) every irreducible representation of G appears as a subrepresentation,

(ii) $a_i = \dim V_i$,

(iii) $\sum_i (\dim V_i)^2 = |G|$, and

(iv) for any $g \in G$ that is not the identity, $\sum_i \chi_{V_i}(g) \dim V_i = 0$.

Remark A.2.11. A trained eye could have seen in Table A.3 that the columns are also orthogonal, this is something that happens in general (see [FH91, Exercise 2.21] or consider an orthogonal matrix induced by a character table).

We stop the topic of character theory here. However, the computation of the characters of a group is an area of research nowadays. The interested reader should learn about the *Murnaghan-Nakayama rule* to compute *easily* characters or groups[3], in particular those of the symmetric groups, for example in [Sta99, Section 7.17] or [FH91, Problem 4.45]. Furthermore, there are striking things in this regard, such as the *complexity* of decomposing tensor products of irreducible representations (there are open problems concerning this and problems that are known *NP-hard*, see [BI08, IMW17]). Recall also the standard references to learn more about these topics, such as [Isa76].

A.3. Specht modules

We are going to study the irreducible representations of Σ_k and give them constructively from the conjugacy classes. A basic known fact is that the conjugacy classes of Σ_k are given by the *cycle shape* of permutations, i.e., the lengths of the cycles that appear in a cycle decomposition of a permutation. The cycle shapes of permutations of k elements are bijectively identified with the *partitions* of k.

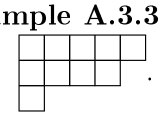
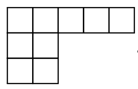
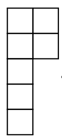
Definition A.3.1. A *partition* of a positive integer k is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots)$ that sum to k and are in a non-increasing order. If λ is a partition of k, we shall write $\lambda \vdash k$.

The construction of the irreducible representations of Σ_k requires, although each one is simple, a gobbledegook of definitions. We try to present them in a more schematic way for the sake of clarity.

Definition A.3.2. A *Young diagram* is a finite collection of boxes arranged in left-justified rows and such that a row is not shorter than the one below[4]. The number of boxes in a Young diagram is its *size*.

Young diagrams of size k are in bijectively correspondence with partitions of k, hence its use.

Example A.3.3.

- The partition $(5,4,1) \vdash 10$ is associated to the Young diagram .
- The partition $(5,2,2) \vdash 9$ is associated to the Young diagram .
- The partition $(2,2,1,1,1) \vdash 7$ is associated to the Young diagram .

[3] Or see the many resources of character tables one can find online.

[4] A *Ferrers diagram* is the same thing, but usually with circles.

Definition A.3.4. If we *fill* each box of a Young diagram of size k with the numbers from 1 to k, we get a *Young tableau*. In that case, we say that the Young diagram, or the partition associated to it, that was filled is the *shape* of the Young tableau.

If the filling is made in a way that every number is smaller than the left and below ones we are doing a *standard filling*, and the result is a *standard Young tableau*.

Young tableaux of size k are in correspondence with permutations in Σ_k, hence its utility.

Example A.3.5.

- The permutation $(67948)(1352) \in \Sigma_{10}$ is associated to the Young tableau $\begin{array}{|c|c|c|c|c|}\hline 6&7&9&4&8\\ \hline 1&3&5&2\\ \cline{1-4} 10\\ \cline{1-1}\end{array}$.
- The partition $(43185)(26)(79) \in \Sigma_9$ is associated to the Young tableau $\begin{array}{|c|c|c|c|c|}\hline 4&3&1&8&5\\ \hline 2&6\\ \cline{1-2} 7&9\\ \cline{1-2}\end{array}$.
- The partition $(81)(64) \in \Sigma_7$ is associated to the Young tableau $\begin{array}{|c|c|}\hline 8&1\\ \hline 6&4\\ \hline 2\\ \cline{1-1} 3\\ \cline{1-1} 5\\ \cline{1-1} 7\\ \cline{1-1}\end{array}$. But also with others, for example, with $\begin{array}{|c|c|}\hline 8&1\\ \hline 6&4\\ \hline 2\\ \cline{1-1} 5\\ \cline{1-1} 3\\ \cline{1-1} 7\\ \cline{1-1}\end{array}$.

There is an obvious action of Σ_k over the Young tableaux of size k, by permutation of the boxes of the tableaux.

Example A.3.6.

- $(164)(27)(98)\ \begin{array}{|c|c|c|c|c|}\hline 6&7&9&4&8\\ \hline 1&3&5&2\\ \cline{1-4} 10\\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&2&8&6&9\\ \hline 4&3&5&7\\ \cline{1-4} 10\\ \cline{1-1}\end{array}$.
- $(864)(12)\ \begin{array}{|c|c|c|c|c|}\hline 4&3&1&8&5\\ \hline 2&6\\ \cline{1-2} 7&9\\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|c|}\hline 6&3&2&4&5\\ \hline 1&8\\ \cline{1-2} 7&9\\ \cline{1-2}\end{array}$.
- $(2847)(631)\ \begin{array}{|c|c|}\hline 8&1\\ \hline 6&4\\ \hline 2\\ \cline{1-1} 3\\ \cline{1-1} 5\\ \cline{1-1} 7\\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2&1\\ \hline 6&8\\ \hline 7\\ \cline{1-1} 5\\ \cline{1-1} 3\\ \cline{1-1} 4\\ \cline{1-1}\end{array}$.

Two important subgroups of Σ_k appear with this action: the subgroups that fix the rows and the columns.

Definition A.3.7. The *row stabilizer* of a Young tableau T, $R(T)$, is the subgroup that fixes the rows of T. Similarly, the *column stabilizer*, $C(T)$, is the analogous definition for columns.

A *tabloid* is the class of equivalence of tableaux under the equivalence $T \sim T'$ if, and only if, $T = \sigma T'$ for $\sigma \in R(T)$. They are denoted as $[T]$.

Example A.3.8. $\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&6 \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|c|}\hline 1&3&4&2\\ \hline 5&6 \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|c|}\hline 2&1&3&4\\ \hline 6&5 \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|c|}\hline 4&2&3&1\\ \hline 5&6 \\ \cline{1-2}\end{array} \sim \cdots$

Finally, our main objects.

Definition A.3.9. Let T be a Young tableau, its associated *polytabloid* is

$$e_T \coloneqq \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\big[\sigma T\big].$$

Example A.3.10.

- $e_{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline\end{array}} = \Big[\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline\end{array}\Big]$

- As we have that $C\left(\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}\right) = \{(14),(25),(14)(25)\}$, then

$$\begin{aligned} e_{\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}} &= \operatorname{sgn}(\mathrm{id}_{\Sigma_5})\left[\mathrm{id}_{\Sigma_5}\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}\right] + \operatorname{sgn}\big((14)\big)\left[(14)\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}\right] \\ &\quad + \operatorname{sgn}\big((25)\big)\left[(25)\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}\right] + \operatorname{sgn}\big((14)(25)\big)\left[(14)(25)\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}\right] \\ &= \left[\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}\right] - \left[\begin{array}{|c|c|c|}\hline 4&2&3\\ \hline 1&5 \\ \cline{1-2}\end{array}\right] - \left[\begin{array}{|c|c|c|}\hline 1&5&3\\ \hline 4&2 \\ \cline{1-2}\end{array}\right] + \left[\begin{array}{|c|c|c|}\hline 4&5&3\\ \hline 1&2 \\ \cline{1-2}\end{array}\right]. \end{aligned}$$

These polytabloids are what we shall use to find all the irreducible representations of Σ_k. To achieve this, it is obvious that we need an action of the group on the polytabloids, which is $\sigma e_T = e_{\sigma T}$. This can be taken as a definition or can be deduced from an action over the tabloids. Finally, it is not surprising that the $\mathbb{C}\Sigma_k$-modules we are looking for are spanned by polytabloids that come form a certain partition of k.

Definition A.3.11. The *Specht module* associated with the partition $\lambda \vdash k$ is the $\mathbb{C}\Sigma_k$-module spanned by all the polytabloids e_T that have T of shape λ. We shall denote them by S^λ.

Take for example $\lambda = (k)$, the only tabloid of that shape is $\Big[\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}\ \cdots\ \begin{array}{|c|}\hline k\\ \hline\end{array}\Big]$ and it coincides with its associated polytabloid. From here, it is easy to see that the action of Σ_k is trivial and that $S^{(k)}$ is the trivial representation of Σ_k.

A not-much more sophisticated example appears when we take $\lambda = (1,1,\ldots,1) \vdash k$, where every tabloid has one Young tableau because $R(T) = \mathrm{id}_{\Sigma_k}$. Here, $C(T) = \Sigma_k$ for every Young tableau T, so we only have one polytabloid modulo the sign. Furthermore,

$$\begin{aligned}
\kappa e_T &= \kappa \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\big[\sigma T\big] \\
&= \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\big[\kappa\sigma T\big] \\
&= \sum_{\sigma \in C(T)} \operatorname{sgn}(\kappa^{-1}\kappa\sigma)\big[\kappa\sigma T\big] \\
&= \sum_{\sigma \in C(T)} \operatorname{sgn}(\kappa^{-1})\operatorname{sgn}(\kappa\sigma)\big[\kappa\sigma T\big] \\
&= \operatorname{sgn}(\kappa^{-1}) \sum_{\kappa\sigma \in C(T)} \operatorname{sgn}(\kappa\sigma)\big[\kappa\sigma T\big] \\
&= \operatorname{sgn}(\kappa^{-1}) e_T = \operatorname{sgn}(\kappa) e_T
\end{aligned}$$

for any $\kappa \in \Sigma_k$. This proves that $S^{(1,\dots,1)}$ is the alternating representation of Σ_k.

It is not a coincidence that these modules are irreducible (cf. Corollary A.2.9).

Theorem A.3.12 (see [Sag01, Theorems 2.4.4 and 2.4.6])**.** *The Specht modules are irreducible, hence, they are all the irreducible representations of the symmetric groups.*

Notation A.3.13. On behalf of this theorem, a Specht module S^λ can also be called the *λ-representation of* Σ_k, provided that $\lambda \vdash k$.

Regarding the degree of these representations, we give, without proofs, a basis and the dimension of them.

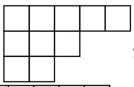
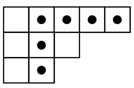
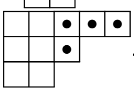
Theorem A.3.14 (see [Sag01, Theorem 2.5.2])**.** *The set of polytabloids*

$$\{e_T : T \text{ is a standard Young tableau of shape } \lambda\}$$

is a basis of the Specht module S^λ*.*

The dimension of a Specht module S^λ is determined, not unexpectedly, uniquely by λ because it is the number of standard Young tableaux of that shape, but the computation is not completely straightforward[5].

Definition A.3.15. The *hook-length* of the entry indexed by (i,j), $h_{i,j}$, in a Young tableau is the number of boxes below at the same column and at the right in the same row plus one, the entry (i,j) itself.

For example, if the Young diagram is , then $h_{1,2} = 6$ corresponding to  and $h_{1,3} = 4$ corresponding to .

With this, we have the *hook-length formula*, that gives the dimension of S^λ, $\lambda \vdash k$.

[5] An interesting anecdote is related to this formula, see [Sag01, p. 125].

Theorem A.3.16 (Hook-length formula, see [Sag01, Theorem 3.10.2])**.** *With the notation above,*

$$\dim S^{\lambda} = \frac{k!}{\prod_{i,j} h_{i,j}}.$$

For example, $S^{(k)}$ has dimension $k!/k!$, and so does $S^{(1,\ldots,1)}$. Another example is $S^{(2,1\ldots,1)}$, that has dimension $k!/k(k-2)! = k-1$.

A.4. Substructures: restrictions and projections

At the beginning of this section, we talked about the *substructures* a representation of G on V has, which is a natural question to ask, and said that we could consider subgroups or invariant subspaces. We talked about the second one and gave a reason to call a G-invariant subspace a subrepresentation, instead of something related to a subgroup: because a representation is the same than a $\mathbb{C}G$-module and, with this definition of *subrepresentation*, a subrepresentation coincides with a submodule. Now, we are going to complete this topic.

Definition A.4.1. If we have a representation of G on V and we consider a subgroup H, we can consider the *restricted* action. In this case, the representation of H on the same space is called the *restricted representation* of G and, usually, they are denoted by $\mathrm{Res}^G_H V$. If the context is clear, they are also denoted simply by $\mathrm{Res}V$ or $V{\downarrow_H}$.

Similarly, if we have a representation of a subgroup H on $W \subseteq V$ then we say that a representation of G on V is *induced* by W if

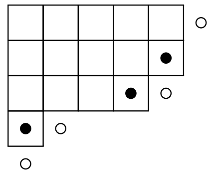
$$V = \bigoplus_{\overline{g} \in G/H} \overline{g} \cdot W,$$

which is well defined because $gh \cdot W = g \cdot W$, and, usually, they are denoted by $\mathrm{Ind}^G_H W$. If the context is clear, they are also denoted simply by $\mathrm{Ind}W$ or $W{\uparrow^G}$.

If we focus on the group Σ_k, there is a nice relation between the induction and restriction of (irreducible) representations, although there are similar relations in other contexts. These kind of relations are called *branching rules* or *branching theorems*, and give a restriction of an irreducible representation in terms of irreducible representations of the subgroup (and vice versa with the induced representation). In the case of the permutation group, we will see that it is quite visual.

Definition A.4.2. If we have a Young diagram associated to a partition $\lambda \vdash k$, an *inner box* is a box of the diagram that, when we delete it, it leaves a new Young diagram associated to a partition of $k-1$. An *outer box* is a new box in a position that, when we add it, gives a new Young diagram associated to a partition of $k+1$. The set of these partitions associated to the Young diagrams that appear when we delete (resp. add) an inner (resp. outer) box is denoted by $\downarrow(\lambda)$ (resp. $\uparrow(\lambda)$).

Example A.4.3. If we have $\lambda = (5,5,4,1) \vdash 15$, its associated Young diagram is



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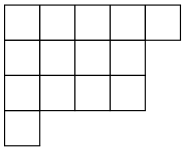
where the dots point the inner boxes and the circles the outer boxes. Hence, when we delete an inner box we get one of these:



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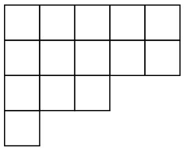
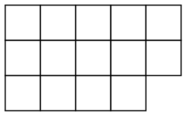
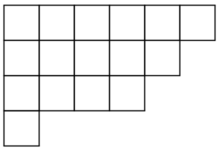


, or



,

and the partitions in $\downarrow(5,5,4,1)$ are, respectively, $(5,4,4,1), (5,5,3,1)$, and $(5,5,4)$. If we add an outer box we get one of these:



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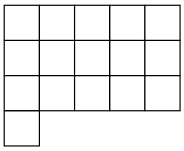

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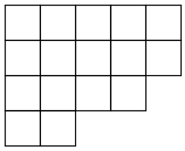
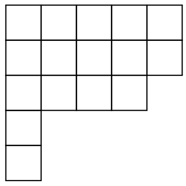


, or



.

and the partitions in $\uparrow(5,5,4,1)$ are, respectively, $(6,5,4,1), (5,5,5,1), (5,5,4,2)$ and $(5,5,4,1,1)$.

Theorem A.4.4 (Branching Rules, see [Sag01, Theorem 2.8.3])**.** *If we have an irreducible representation of Σ_k, say S^λ with $\lambda \vdash k$, then*

(i) $S^\lambda\downarrow_{\Sigma_{k-1}} \cong \bigoplus_{\gamma \in \downarrow(\lambda)} S^\gamma$, *and*

(ii) $S^\lambda\uparrow^{\Sigma_{k+1}} \cong \bigoplus_{\gamma \in \uparrow(\lambda)} S^\gamma$.

This theorem gives a way of going from a representation of Σ_k to a representation of Σ_{k-1} seen as a subgroup of Σ_k, for example by fixing the first entry, and vice versa.

As the situation with module and submodule is considerably simpler because we just deal with linear spaces, there should be an analogous way of passing from a module to a submodule: we are looking for a *projection* of V into a subrepresentation W.

Theorem A.4.5 (see [FH91, Section 2.4])**.** *If we have a representation V of G that is decomposed as in Corollary A.2.10 into the sum of irreducible representations $\bigoplus V_i^{\oplus a_i}$, then*

$$\frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_{V_i}(g)} g : V \to V$$

is the projection of V onto the factor $V_i^{\oplus a_i}$.

Appendix B

Spectral sequences

> It has been suggested that the name "spectral" was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.
>
> Ravi Vakil, *Spectral sequences: friend or foe?*

In this section, we give a very basic introduction of spectral sequences.

As general references, we recommend [Mit69, DR17] as an introduction to the topic, [McC01] as a basic reference, the extra and fifth chapter of Hatcher's famous book *Algebraic topology* (see [Hat02, Chapter 5]) to have an introduction to some famous spectral sequences, and [Cho06] to lie to yourself and think that you understand the topic. In particular, we highlight [DR17] and [McC01], which we follow, for their nice examples and explanations.

B.1. Informal introduction

If one wants to study an algebraic object with some graduation, such as a graded $\mathbb{K}$-vector space or a graded $\mathbb{K}$-algebra, but without knowing exactly what is the object, one could use a *spectral sequence*. For example, if we want to know the (co)homology of some topological space we may use some well-known spectral sequences.

Broadly speaking, a *spectral sequence* plays the role of a series converging to something that we want to know, but with more algebraic structure (see Table B.1). What we mean by this is that a series is a sequence of partial sums, S_n, where each term of the sequence is related with the previous one by means of a sum, $S_{n+1} = S_n + a_{n+1}$, but a *spectral sequence* is an ordered countable set of *grids*, $E_r^{*,*}$, with some algebraic structure (the so-called *pages*) where a grid is related with the next one by means of the structure of the former one in a *homological fashion*, $E^{*,*}_{r+1} \cong H(E^{*,*}_r, d_r)$.

Furthermore, when a series $\sum_0^\infty a_n$ is such that $a_{n_0+i} = 0$ for some n_0 and every $i \geq 0$, then the partial sums S_n are constant if $n \geq n_0$. The same thing could happen between the *pages* $E^{*,*}_r$ if the *differentials* d_{r_0+i} are zero for some r_0 and every $i \geq 0$, what is called *collapsing of the spectral sequence*. Moreover, when the partial sums are constant or when the spectral sequence collapses, finding the *limit* ($\sum_0^\infty a_n$ and $E^{*,*}_\infty$, respectively) is very easy. However, both concepts have a general notion of *limit*.

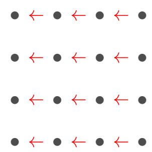
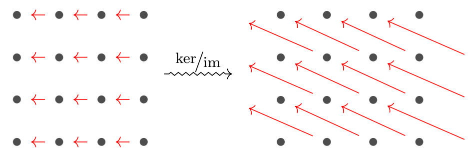
Series	**Spectral sequences**
S_n	$E^{*,*}_r =$ 
$S_{n+1} = S_n + a_{n+1}$	 $E^{*,*}_{r+1} \cong H(E^{*,*}_r, d_r)$
$S_{n_0+i} = S_{n_0}$ if $a_{n_0+i} = 0$ $\forall i \geq 0$	$E^{*,*}_{r_0+i} \cong E^{*,*}_{r_0}$ if $d_{r_0+i} \equiv 0$ $\forall i \geq 0$
$\sum_0^\infty a_n = \sum_0^{n_0} a_n$	$E^{*,*}_\infty \cong E^{*,*}_{r_0}$
$\sum_0^\infty a_n = \lim_n S_n$	$E^{*,*}_\infty \overset{E_0^{*,*}}{\rightsquigarrow} H^*$

Table B.1: Comparison between the elements and basic occurrences of a series and a spectral sequence (form top to bottom): elements, relation between elements, collapsing, limit when collapses, and limit.

Assume that we want to study vector spaces over some field. In this case, each *page* of a *spectral sequence* is a bigraded vector space $E^{*,*}_r$. Evidently, a bigraded vector space can be arranged in a grid or integral lattice, hence the concept of a *sequence of grids*. As we have said, the relation between a *page* and the next one, say $E^{*,*}_r$ and $E^{*,*}_{r+1}$, comes from the extra structure we give to the bigraded vector space, the *differential structure*. This way, each *page* $E^{*,*}_r$ has an endomorphism d_r such that $d_r \circ d_r = 0$ and of bidegree $(r, 1-r)$, i.e., it has the induced maps

$$d_r : E^{p,q}_r \to E^{p+r,q+1-r}_r.$$

Finally, the relation is by means of homology: $E^{*,*}_{r+1}$ is the homology of $(E^{*,*}_r, d_r)$ or, in other words,

$$E^{p,q}_{r+1} \cong \ker d_r : E^{p,q}_r \to E^{p+r,q+1-r}_r \big/ \operatorname{im} d_r : E^{p-r,q-1+r}_r \to E^{p,q}_r.$$

B.2. Basics on spectral sequences

The minimal object we are going to deal with are *differential bigraded modules.*

Definition B.2.1. A *bigraded module over a ring* R is an R-module

$$E^{*,*} = \bigoplus_{p,q\in\mathbb{Z}} E^{p,q},$$

where $\{E^{p,q}\}_{p,q\in\mathbb{Z}}$ is a family of R-modules. We say that the elements x of $E^{p,q}$ have *bidegree* (p,q) and it is denoted $bideg(x)$. If an element x has bidegree (p,q), we say that it has *total degree* $p+q$, and it is denoted $deg(x)$.

A *morphism* ϕ *between bigraded* R*-modules* is a morphism of R-modules such that, for some (i,j), we have induced morphisms

$$\phi| : E^{p,q} \to E^{p+i,q+i}$$

for every (p,q). In that case, we say that ϕ has *bidegree* (i,j).

Actually, the pieces of a *spectral sequence* are not bigraded R-modules, we need a *differential structure* (see Figure B.1).

Definition B.2.2. A *differential bigraded* R*-module* is a bigraded R-module with an endomorphism $d : E^{*,*} \to E^{*,*}$ of bidegree $(s, 1-s)$ or $(-s, s-1)$, for some integer s, and such that $d \circ d \equiv 0$.

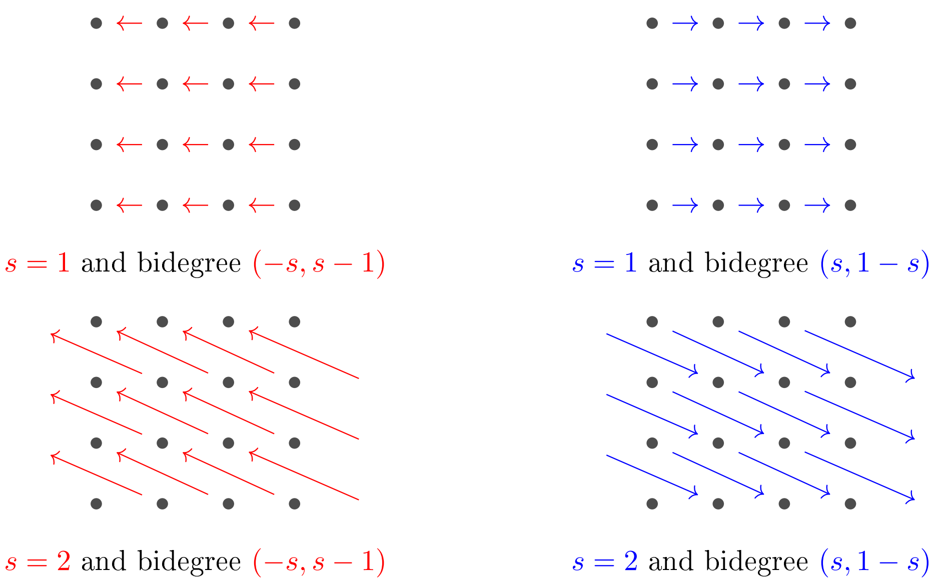


Figure B.1: Sketch of the possible differentials in a bigraded R-module.

The differential structure allows us to take the homology of the differential bigraded R-module (see Figure B.2), i.e.,

$$H^{p,q}(E^{*,*}, d) = {\ker d{:}E^{p,q} \to E^{p+s,q+1-s}}\big/{\operatorname{im} d{:}E^{p-s,q-1+s} \to E^{p,q}}.$$

Observe that we consider this as a bigraded R-module once more. If it has a differential structure as well and we can iterate this process, we have essentially how a *spectral sequence* works.

Definition B.2.3. A *spectral sequence* is a family of differential bigraded R-modules, $\{(E_r^{*,*}, d_r)\}_{r\in\mathbb{Z}}$, such that the differentials d_r are all of bidegree $(-r, r-1)$ or all of bidegree $(r, 1-r)$ and $E_{r+1}^{p,q}$ is isomorphic to $H^{p,q}(E_r^{*,*}, d_r)$, for all $p, q, r \in \mathbb{Z}$.

If all the differentials are of bidegree $(-r, r-1)$, it is a *spectral sequence of homological type*, and it is a *spectral sequence of cohomological type* if the differentials are of bidegree $(r, 1-r)$.

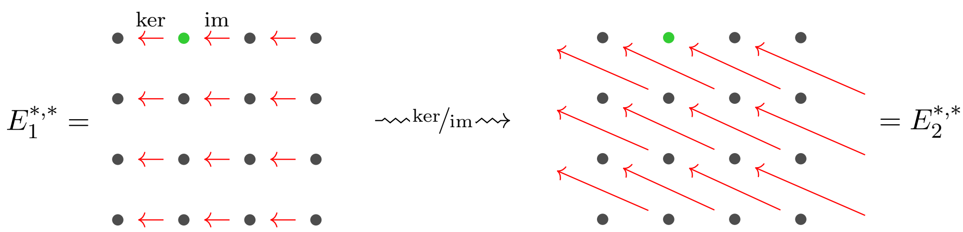


Figure B.2: Sketch of the process to go from a page to the next one.

There are a lot of things to say about Definition B.2.3. For instance, $E_r^{*,*}$ is called the *r-page of the spectral sequence*. Also, notice that, in the definition of spectral sequence, the r-page determines the next page by means of its differentials, but there is no information about the differentials of the $(r+1)$-page. Usually, there is more structure to determine the differentials, but in general they are not relevant.

Furthermore, many spectral sequences start at the first or second page, so the $E_r^{*,*}$ are not considered if r is less than 1 or 2. This doesn't change the definition of spectral sequence significantly.

A good example of how all the parts of a spectral sequence articulate between one another is [McC01, Example 1.E], which we reproduce here (with some extra comments) for the sake of completion. Also, for a nice *real* example, [DR17, Example 3.6] is recommended.

Example B.2.4 (see [McC01, Example 1.E])**.** Suppose $E_2^{*,*}$ is an algebra given by

$$E_2^{*,*} \cong \frac{\mathbb{Q}[x, y, z]}{(x^2 = y^4 = z^2 = 0)},$$

and assume we know that

$$\begin{aligned} \text{bideg}(x) &= (7,1), \\ \text{bideg}(y) &= (3,0), \\ \text{bideg}(z) &= (0,2), \end{aligned} \qquad \begin{aligned} d_2(x) &= y^3, \text{ and} \\ d_3(z) &= y. \end{aligned}$$

Usually, we are not given any information about the differentials, especially about the differentials after the first page of the spectral sequence ($E_2^{*,*}$ in this case). The information about d_2 and d_3 is given so the example works smoothly. Furthermore, we have talked about the structure of spectral sequences when each page is a differential bigraded R-module, not a differential bigraded *algebra*. This is not relevant, because we only need the structure of algebra (or $\mathbb{K}$-vector space, R-module, etc.) to fit with the bidegree and differential structure[1]. In the case of a *differential bigraded algebra*, the definition is completely analogous to Definition B.2.2 with the additional conditions on the inner product, i.e., we need to say how the inner product behaves with the differential (*Leibniz rule*) and how the bidegree works with the inner product (see [McC01, Definition 1.6]):

$$\begin{aligned} d(e \cdot e') &= d(e) \cdot e' + (-1)^{\deg e} e \cdot d(e') \text{ and} \\ \text{bideg}(e \cdot e') &= \text{bideg}(e) + \text{bideg}(e') \end{aligned}$$

for any $e \in E^{p,q}$ and $e' \in E^{p',q'}$.

The generators of the different algebras $E_2^{p,q}$ are represented in the following diagram taking into account the bidegrees of x, y and z and the relations of $E_2^{*,*}$.

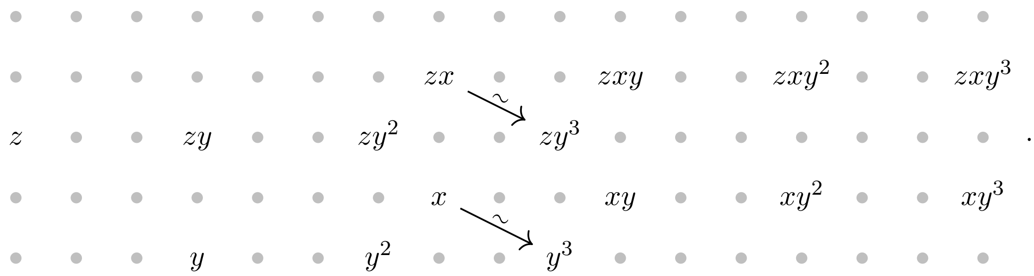
$$\begin{array}{ccccccccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & zx & \bullet & \bullet & zxy & \bullet & \bullet & zxy^2 & \bullet & \bullet & zxy^3 \\
z & \bullet & \bullet & zy & \bullet & \bullet & zy^2 & \bullet & \bullet & zy^3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & x & \bullet & \bullet & xy & \bullet & \bullet & xy^2 & \bullet & \bullet & xy^3 \\
\bullet & \bullet & \bullet & y & \bullet & \bullet & y^2 & \bullet & \bullet & y^3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}.$$

Now, we have to complete this diagram with the differentials. To see if the bidegree of d_2 is $(2,-1)$ or $(-2,1)$ we use that $d_2(x) = y^3$, so the bidegree is $(2,-1)$. This shows that almost every differential in $E_2^{*,*}$ is trivial, because they have a trivial $E_2^{p,q}$ in their source or target. The exceptions are $d_2 : E_2^{7,1} \to E_2^{9,0}$ and, possibly, $d_2 : E_2^{7,3} \to E_2^{9,2}$, but Leibniz rule sows that the second one is an isomorphism as well:

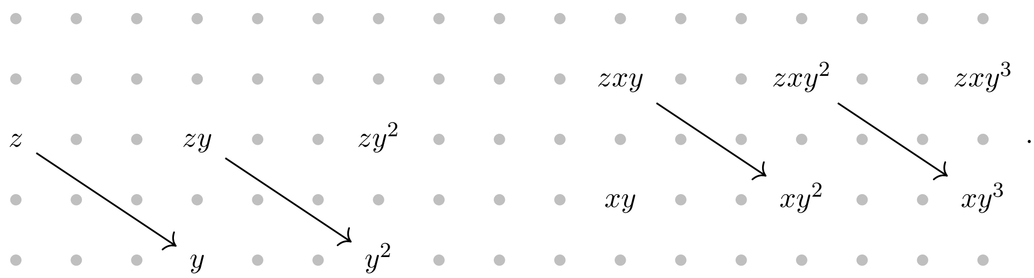
$$d_2(zx) = d_2(z)x + (-1)^2 z d_2(x) = 0 + zy^3.$$

[1] Actually, the structure of algebra does not appear after this example, consequently, the reader can forget about it if it is convenient.

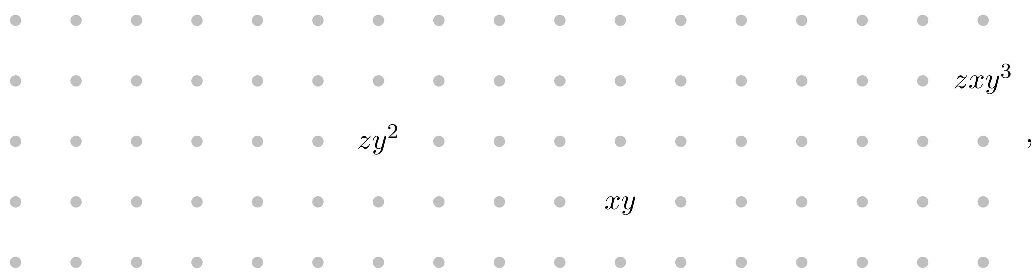
So, in conclusion, there are two non-trivial differentials and they are isomorphisms:



For the third page, we have to take the homology of these differentials, i.e., $E_3^{p,q} \cong H^{p,q}(E_2^{*,*}, d_2)$. Observe that, if we have an isomorphism between two entries of a page $E_r^{*,*}$, then those entries are trivial in $E_{r+1}^{*,*}$, because either $\ker d_r$ is trivial or $\operatorname{im} d_r$ is the total. For a similar reason, if a differential is zero between two entries, those entries *survive* in the next page. Hence, we have the following in $E_3^{*,*}$ (with the possible non-trivial differentials):



Observe that we know that the bidegree of d_3 has to be $(3, -2)$ because d_2 has bidegree $(2, -1)$. Also, with the Leibniz rule and knowing that $d_3(z) = y$, one can show that all the differentials we see in the previous diagram are isomorphisms. Therefore, in $E_4^{*,*}$, we have

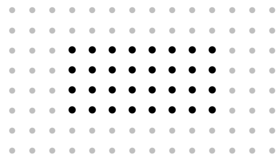


where all the differentials are zero. This implies that $E_5^{*,*} \cong E_4^{*,*}$, and the following pages will be also isomorphic to $E_4^{*,*}$ for the same reason.

As we were saying in the introduction, the spectral sequences are useful to study an algebraic object without knowing it exactly. We can do this if we have a spectral sequence *converging* to that object we want to study, therefore we need a notion of convergence. In Example B.2.4, we have seen a phenomenon simpler than convergence.

Definition B.2.5. A spectral sequence $\{(E_r^{*,*}, d_r)\}_r$ *collapses at the M-th term*, or *page*, if the differentials d_r are zero for $r \geq M$. In that case, the *page at infinity* is $E_\infty^{*,*} := E_M^{*,*}$.

Example B.2.6 (see [McC01, Example 1.B])**.** If $E_2^{p,q} = 0$ for $|p| > a$ or $|q| > b$, for some $a, b \in \mathbb{N}$, then the spectral sequence collapses.



Definition B.2.7 (see [McC01, Definition 2.4])**.** A spectral sequence $\{(E_r^{*,*}, d_r)\}_{r \geq 2}$ is said to *converge* to the graded R-module H^* if there is a filtration F^*, i.e.,

$$H^* \supseteq \cdots \supseteq F^n H^* \supseteq F^{n+1} \supseteq \cdots \supseteq \{0\},$$

such that $E_0^{*,*} \cong E_\infty^{*,*}$, where $E_0^{*,*}$ depends on H^* and F^* and

$$E_0^{p,q}(H^*, F^*) := \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}.$$

In this case, it is denoted as $E_2^{*,*} \Longrightarrow H^*$.

As we know, $E_\infty^{*,*}$ is very easy to determine if a spectral sequence collapses. There is a way of defining $E_\infty^{*,*}$ when a spectral sequence does not necessarily collapse, the interested reader could see this in [McC01, p. 30].

Recovering H^* from a spectral sequence is not straightforward. Indeed, the convergence of a spectral sequence is not unique in general, so it is not true that H^* is determined from $E_\infty^{*,*}$ (see [McC01, Example 1.J]). There are some *extension problems* that could occur. However, when we deal with vector spaces and the spectral sequence collapses, the situation is very simple and recovering H^* is trivial (see [McC01, Section 1.1]):

$$H^n = \bigoplus_{p+q=n} E_\infty^{p,q}.$$

In general, although the convergence is not unique, having a spectral sequence converging to something you want to compute is very good (see Appendix B.3 for example). So, as McCleary says in [McC01], we would like to have theorems of the following form:

Generic theorem of spectral sequences. *There is a spectral sequence (of R-modules, algebras, etc.) such that*

$$E_r^{*,*} \cong \quad \textit{something computable}$$

and converges to $H^* \cong$ something desirable.

B.3. A step further

There are a lot of things that can be said about spectral sequences, but, here, we will only talk about two more things: the *Poincaré series* and the *Euler characteristic* of a graded vector space.

Definition B.3.1. Let H^* be a graded vector space over some field $\mathbb{K}$ such that H^n is finite-dimensional for every n. Then, the *Poincaré series* for H^* is the formal power series

$$P(H^*, t) \coloneqq \sum_{n=0}^{\infty} (\dim_{\mathbb{K}} H^n)\, t^n.$$

In this case, the Euler characteristic for H^* is $\chi(H^*) \coloneqq P(H^*, -1)$, if this expression makes sense.

Example B.3.2. If H^* is the complex homology of some manifold M, $H^*(M, \mathbb{C})$, then $\chi(H^*)$ is the classical Euler-Poincaré characteristic of M.

Note that, if we have a bigraded vector space, $E^{*,*}$, it is a graded vector space considering the total degree (recall Definition B.2.1). That way, if $E^{*,*}$ is a bigraded vector space such that each $E^{p,q}$ is finite-dimensional for every p and q, we have a Poincaré series,

$$P(E^{*,*}, t) \coloneqq \sum_{n=0}^{\infty} \dim_{\mathbb{K}} \left(\bigoplus_{p+q=n} E^{p,q} \right) t^n,$$

and the induced Euler characteristic, $\chi(E^{*,*}) \coloneqq P(E^{*,*}, -1)$, if this makes sense.

We can provide a useful partial order relation on the Poincaré series: we will say that $P(A^*, t) \geq P(B^*, t)$ if the formal power series $P(A^*, t) - P(B^*, t)$ does not have any negative coefficient. For convenience, if $P(A^*, t) \geq P(B^*, t)$ and the two series are different, we will write $P(A^*, t) > P(B^*, t)$.

Proposition B.3.3 (see [McC01, Example 1.F]). *Suppose that the spectral sequence* $\left\{ \left(E_r^{*,*}, d_r \right) \right\}_{r \geq r_0}$ *collapses at the N-th page and converges to H^*. Furthermore, assume that $E_{r_0}^{p,q}$ is finite-dimensional for every $p, q \in \mathbb{Z}$. Then, H^n is also finite-dimensional for every n and*

$$P(E_{r_0}^{*,*}, t) \geq \cdots \geq P(E_{N-1}^{*,*}, t) > P(E_N^{*,*}, t) = P(E_\infty^{*,*}, t) = P(H^*, t).$$

Finally, whenever they make sense, $\chi(E_r^{*,*}) = \chi(H^*)$ for every r.

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