Cylindrical surface waveguide modes using a surface impedance dyadic method

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Indexing terms: Waveguides and waveguide components, Microwave passive components

Abstract: The fields and characteristic parameters of nonradiating modes of a dielectric-coated wire with an intervening airgap are derived using a surface impedance dyadic method. This method shows several advantages and provides a straightforward method for working out the characteristic equation, cutoff conditions, far-from-cutoff conditions and field coefficients. Important results about the hybrid nature of these modes are shown, and some of them are common to any cylindrical surface waveguide.

List of principal symbols

(ρ, ϕ, z)	= cylindrical co-ordinates
$\boldsymbol{u}_\rho, \boldsymbol{u}_\phi, \boldsymbol{u}_z$	= unit vectors
a, b, c	= radii of the waveguide
ε_0, μ_0	= vacuum permittivity and permeability, respectively
$\varepsilon_r, \mu_r, \tan\delta$	= relative permittivity, relative permeability and loss tangent of the dielectric medium, respectively
σ, δ	= conductivity and skin depth of the conductor, respectively
$\boldsymbol{E}, \boldsymbol{H}$	= electric and magnetic fields, respectively
a_i, c_i	= field coefficients
γ, α, β	= axial propagation factor and its real and imaginary parts, respectively
j	$= \sqrt{-1}$
J_n, Y_n, I_n, K_n	= Bessel functions and modified Bessel functions of first and second kind of order n
λ_0, k_0	= wavelength and wavenumber in a vacuum, respectively
λ_g	= wavelength in the waveguide
h, k	= radial propagation factors in air and in the dielectric medium, respectively
x, y	= normalised radial propagation factors
X	= surface impedance dyadic
Γ	= 1.78107
f_0	= normalised frequency
y_c, k_c, f_{0_c}	= cutoff values of y, k and f_0, respectively
y_f, k_f	= far-from-cutoff values of y and k, respectively
Z_0	= intrinsic impedance of vacuum
P, P_1, P_2, P_3	= total power flow and the contributions of each medium
W_1, W_2, W_3	= contributions to the stored energy of each medium
P_{ld}, P_{lc}	= power losses in the dielectric medium and in the conductor, respectively
R	$= jZ_0(H_z/E_z)$, when $\rho > c$

Paper 5219H (E12), first received 23rd July and in revised form 28th October 1986

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1 Introduction

The fields of a surface waveguide penetrate in the medium that surrounds the guide and this gives rise to a wide variety of possible applications. These applications include guided radar systems [1, 2], continuous access guided communications [3], communications in mines and tunnels [4] and sensor applications [5]. Investigations on microwave pulse propagation have shown that surface waveguides with a circular cross-section are particularly well behaved [6], which is outstanding because of the application in time domain dielectric spectroscopy.

The surface impedance dyadic method [7] is a generalised surface impedance method. The characteristic equation of any cylindrical surface waveguide can be expressed as a function of the elements of the dyadic, and it can be regarded as the condition that the surface impedance of the waveguide has to satisfy to support a guided mode. The internal structure of a particular cylindrical surface waveguide will determine the expression of the elements of the dyadic as a function of the propagation factors. This method gives rise to a set of results common to all cylindrical surface waveguides which are obtained in terms of the surface impedance dyadic elements. Previously, the general cutoff conditions were derived, and now we extend these results giving the far-from-cutoff conditions and the expression of the R parameter. This parameter provides direct information about the TM and TE contributions to the structure of the hybrid modes.

The nonradiating modes of a dielectric coated wire with an intervening airgap are investigated using the surface impedance dyadic method. Such a cylindrical structure was introduced by Rao and Hamid [8, 9], and it has been shown that the thickness of the airgap is a new parameter that can be used to control the penetration of the fields in the surrounding medium, at the same time that the propagation characteristics of the waveguide are improved [10]. The application of the surface impedance dyadic method shows several advantages, providing a straightforward method for working out the expression of the fields and the characteristic parameters of each mode.

2 Characteristic equation

Fig. 1 shows the geometry of the cylindrical surface waveguide and the parameters of each medium. The longitudinal and transverse components of the electric and

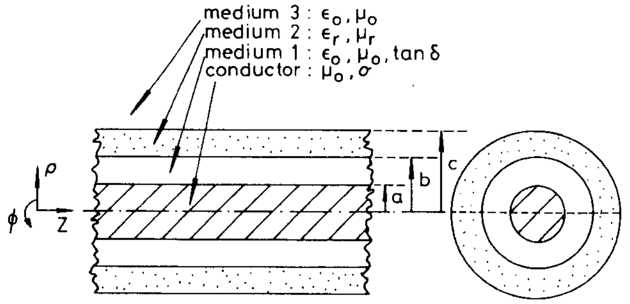

Fig. 1 *Geometry of the cylindrical surface waveguide*

magnetic fields of a guided mode can be expressed as follows:

medium 1: $a < \rho < b, t = h\rho$

$$\left.\begin{aligned}
E_{z1} &= a_1 I_n(t) + a_2 K_n(t)\\
E_{\rho 1}/j &= \frac{k_0 c}{x}\left[\frac{\gamma}{jk_0}(a_1 I'_n(t) + a_2 K'_n(t)) + \frac{n}{t}(c_1 I_n(t) + c_2 K_n(t))\right]\\
E_{\phi 1} &= -\frac{k_0 c}{x}\left[\frac{n\gamma}{jk_0 t}(a_1 I_n(t) + a_2 K_n(t)) + c_1 I'_n(t) + c_2 K'_n(t)\right]\\
jZ_0 H_{z1} &= c_1 I_n(t) + c_2 K_n(t)\\
Z_0 H_{\rho 1} &= \frac{k_0 c}{x}\left[\frac{n}{t}(a_1 I_n(t) + a_2 K_n(t)) + \frac{\gamma}{jk_0}(c_1 I'_n(t) + c_2 K'_n(t))\right]\\
Z_0 H_{\phi 1}/j &= \frac{k_0 c}{x}\left[a_1 I'_n(t) + a_2 K'_n(t) + \frac{n\gamma}{jk_0 t}(c_1 I'_n(t) + c_2 K'_n(t))\right]
\end{aligned}\right\} \quad (1)$$

medium 2: $b < \rho < c, s = k\rho$

$$\left.\begin{aligned}
E_{z2} &= a_3 J_n(s) + a_4 Y_n(s)\\
E_{\rho 2}/j &= -\frac{k_0 c}{y}\left[\frac{\gamma}{jk_0}(a_3 J'_n(s) + a_4 Y'_n(s)) + \frac{n\mu_r}{s}(c_3 J_n(s) + c_4 Y_n(s))\right]\\
E_{\phi 2} &= \frac{k_0 c}{y}\left[\frac{n\gamma}{jk_0 s}(a_3 J_n(s) + a_4 Y_n(s)) + \mu_r(c_3 J'_n(s) + c_4 Y'_n(s))\right]\\
jZ_0 H_{z2} &= c_3 J_n(s) + c_4 Y_n(s)\\
Z_0 H_{\rho 2} &= -\frac{k_0 c}{y}\left[\frac{n\varepsilon'_r}{s}(a_3 J_n(s) + a_4 Y_n(s)) + \frac{\gamma}{jk_0}(c_3 J'_n(s) + c_4 Y'_n(s))\right]\\
Z_0 H_{\phi 2}/j &= -\frac{k_0 c}{y}\left[\varepsilon'_r(a_3 J'_n(s) + a_4 Y'_n(s)) + \frac{n\gamma}{jk_0 s}(c_3 J_n(s) + c_4 Y_n(s))\right]
\end{aligned}\right\} \quad (2)$$

medium 3: $c < \rho, t = h\rho$

$$\left.\begin{aligned}
E_{z3} &= a_5 K_n(t)\\
E_{\rho 3}/j &= \frac{k_0 c}{x}\left[\frac{\gamma}{jk_0} a_5 K'_n(t) + \frac{n}{t} c_5 K_n(t)\right]\\
E_{\phi 3} &= -\frac{k_0 c}{x}\left[\frac{n\gamma}{jk_0 t} a_5 K_n(t) + c_5 K'_n(t)\right]\\
jZ_0 H_{z3} &= c_5 K_n(t)\\
Z_0 H_{\rho 3} &= \frac{k_0 c}{x}\left[\frac{n}{t} a_5 K_n(t) + \frac{\gamma}{jk_0} c_5 K'_n(t)\right]\\
Z_0 H_{\phi 3}/j &= \frac{k_0 c}{x}\left[a_5 K'_n(t) + \frac{n\gamma}{jk_0 t} c_5 K_n(t)\right]
\end{aligned}\right\} \quad (3)$$

where we have omitted the factor $\exp(j\omega t - \gamma z + jn\phi)$ in each component. The integer n fixes the angular dependence of the fields, and the radial propagation factors are defined by the expressions

$$\left.\begin{aligned}
h^2 &= -\gamma^2 - k_0^2\\
k^2 &= k_0^2 \varepsilon_r \mu_r + \gamma^2\\
\gamma &= \alpha + j\beta\\
x &= hc\\
y &= kc
\end{aligned}\right\} \quad (4)$$

At the moment we will neglect the losses in the waveguide that can be calculated (Appendix 8) using a perturbation technique. Therefore, we will substitute γ by $j\beta$. The surface impedance dyadic X relates the tangential components of the electric ($\boldsymbol{E}_T$) and magnetic ($\boldsymbol{H}_T$) fields at the external interface between a cylindrical surface waveguide and the air ($\rho = c$):

$$\boldsymbol{E}_T = jX(\boldsymbol{u}_\rho x \boldsymbol{H}_T)$$

$$\left.\begin{aligned}
E_\phi &= j(X_{12} H_\phi - X_{11} H_z)\\
E_z &= j(X_{22} H_\phi - X_{21} H_z)
\end{aligned}\right\} \quad (5)$$

where X_{ij} are the elements of the surface impedance dyadic. Eqns. 3 are common to any cylindrical surface waveguide and when substituted into eqns. 5 yield the following expression for the characteristic equation:

$$\bar{X}_{22}\Phi_n^2(x) + (1 + \bar{X}_{12}\bar{X}_{21} - \bar{X}_{11}\bar{X}_{22})\Phi_n - \left[\frac{n\beta}{k_0 x^2}(\bar{X}_{21} + \bar{X}_{12}) + \bar{X}_{11} + \left(\frac{n\beta}{k_0 x^2}\right)^2 \bar{X}_{22}\right] = 0 \quad (6)$$

where

$$\left.\begin{aligned}
\Phi_n(x) &= \frac{K'_n(x)}{xK_n(x)}\\
\bar{X}_{11} &= \frac{X_{11}}{k_0 c Z_0} \qquad \bar{X}_{12} = \frac{X_{12}}{Z_0}\\
\bar{X}_{22} &= \frac{k_0 c}{Z_0} X_{22} \qquad \bar{X}_{21} = \frac{X_{21}}{Z_0}
\end{aligned}\right\} \quad (7)$$

The characteristic equation (eqn. 6) is the condition that the elements of the surface impedance X of a given cylindrical surface waveguide have to satisfy to support a guided mode. This expression of the characteristic equation has the advantage of being written as a summation of terms proportional to Φ_n^0, Φ_n^1 and Φ_n^2, and such powers provide the basic properties of the equation at cutoff and far from cutoff.

When n is zero, if we assume that the elements X_{12} and X_{21} are also zero, the characteristic equation can be satisfied in two ways, corresponding to the symmetrical TM modes and TE modes:

$$\left.\begin{aligned} &\text{TM modes:} \quad \bar{X}_{22}\,\Phi_0(x) + 1 = 0 \\ &\text{TE modes:} \quad \Phi_0(x) - \bar{X}_{11} = 0 \end{aligned}\right\} \tag{8}$$

The internal structure of the cylindrical surface waveguide will determine the expressions of the element X_{ij} as a function of the radial propagation factors. The boundary conditions that the fields in eqns. 1–3 have to satisfy are

$$\left.\begin{aligned} &\rho = a:\ E_{z1} = 0 && E_{\phi 1} = 0 \\ &\rho = b:\ E_{z1} = E_{z2} && E_{\phi 1} = E_{\phi 2} \\ &\qquad\quad H_{z1} = H_{z2} && H_{\phi 1} = H_{\phi 2} \\ &\rho = c:\ E_{z2} = E_{z3} && E_{\phi 2} = E_{\phi 3} \\ &\qquad\quad H_{z2} = H_{z3} && H_{\phi 2} = H_{\phi 3} \end{aligned}\right\} \tag{9}$$

The conditions at $\rho = c$ are expressed through eqn. 5, which has to be satisfied simultaneously by the fields in eqns. 2 and 3. When the conditions at $\rho = a$ and $\rho = b$ are taken into account, and eqns. 2 are substituted into eqn. 5, the expressions of the elements $\bar{X}_{ij}$ can be derived:

$$\begin{aligned} \bar{X}_{22} &= \frac{y\xi(y)}{\varepsilon_r \xi'(y)} \\ \bar{X}_{12} = \bar{X}_{21} &= \frac{y}{\varepsilon_r \xi'(y)}\left[\frac{n\beta}{y^2 k_0}\,\xi(y) + \frac{\varepsilon_r \mu_r}{y}\,\Psi\right] \\ \bar{X}_{11} &= \frac{y}{\varepsilon_r \xi'(y)}\left[-\frac{\varepsilon_r \mu_r}{y^2}\,\zeta'(y) + \left(\frac{n\beta}{y^2 k_0}\right)^2 \xi(y) \right. \\ &\qquad \left. + 2\,\frac{\varepsilon_r \mu_r}{y}\,\frac{n\beta}{y^2 k_0}\,\Psi\right] \end{aligned} \tag{10}$$

where

$$\left.\begin{aligned} \xi(s) &= A(s)B(y) - \varepsilon_r \mu_r \left[\frac{2c\Delta_n}{\pi y^2 b}\right]^2 Y_n(k) Y_n(s) \\ \zeta(s) &= A(s)B'(y) - \varepsilon_r \mu_r \left[\frac{2c\Delta_n}{\pi y^2 b}\right]^2 Y'_n(k) Y_n(s) \\ \Psi &= \frac{2c\Delta_n}{\pi y^2 b}\,(A'(y)Y_n(k) - A(y)Y'_n(k)) \\ A(s) &= g_y\left(H_n V_n(s) + \frac{\varepsilon_r}{y}\,W_n(s)\right) - \Delta_n^2 Y_n(kb) V_n(s) \\ B(s) &= h_y\left(G_n V_n(s) + \frac{\mu_r}{y}\,W_n(s)\right) - \Delta_n^2 Y_n(kb) V_n(s) \\ g_y &= G_n Y_n(kb) + \frac{\mu_r}{y}\,Y'_n(kb) \\ h_y &= H_n Y_n(kb) + \frac{\varepsilon_r}{y}\,Y'_n(kb) \\ H_n &= \frac{P'_n(hb)}{xP_n(hb)} \\ G_n &= \frac{Q'_n(hb)}{xQ_n(hb)} \\ \Delta_n &= \frac{n\beta c}{k_0 b}\left[\frac{1}{x^2} + \frac{1}{y^2}\right] \\ V_n(s) &= J_n(s)Y_n(kb) - J_n(kb)Y_n(s) \\ W_n(s) &= J_n(s)Y'_n(kb) - J'_n(kb)Y_n(s) \\ P_n(t) &= I_n(t)K_n(ha) - I_n(ha)K_n(t) \\ Q_n(t) &= I_n(t)K'_n(ha) - I'_n(ha)K_n(t) \end{aligned}\right\} \tag{11}$$

Substituting eqn. 10 into eqn. 6, the general characteristic equation can be now rewritten for our particular cylindrical surface waveguide:

$$\begin{aligned} &\frac{y\xi(y)}{\varepsilon_r \xi'(y)}\,\Phi_n^2(x) + \left(1 + \frac{\mu_r \zeta(y)}{\varepsilon_r \xi'(y)}\right)\Phi_n(x) \\ &\quad - \frac{b^2 y}{c^2 \varepsilon_r \xi'(y)}\left[\Delta_n^2 \xi(y) + 2\varepsilon_r \mu_r \frac{c\Delta_n}{by}\,\Psi \right. \\ &\qquad\qquad \left. - \varepsilon_r \mu_r \left(\frac{c}{by}\right)^2 \zeta'(y)\right] = 0 \end{aligned} \tag{12}$$

When $n = 0$, the expressions of the elements $\bar{X}_{ij}$ (eqn. 10) are simplified:

$$\left.\begin{aligned} \bar{X}_{22} &= \frac{y}{\varepsilon_r}\,\frac{H_0 V_0(y) + \dfrac{\varepsilon_r}{y}\,W_0(y)}{H_0 V'_0(y) + \dfrac{\varepsilon_r}{y}\,W'_0(y)} \\ \bar{X}_{11} &= -\frac{\mu_r}{y}\,\frac{G_0 V'_0(y) + \dfrac{\mu_r}{y}\,W'_0(y)}{G_0 V_0(y) + \dfrac{\mu_r}{y}\,W_0(y)} \\ \bar{X}_{12} &= \bar{X}_{21} = 0 \end{aligned}\right\} \tag{13}$$

and substituting into eqn. 8 yields the characteristic equation of the symmetrical TM and TE modes.

Applying the condition $b \to a$ to the expressions of the elements $\bar{X}_{ij}$ (eqn. 10), the results corresponding to the Goubau line [7] are reobtained:

$$\begin{aligned} \bar{X}_{22} &= \frac{yV_n(y)}{\varepsilon_r V'_n(y)} \\ \bar{X}_{12} = \bar{X}_{21} &= \frac{n\beta}{k_0 y^2}\,\frac{yV_n(y)}{\varepsilon_r V'_n(y)} \\ \bar{X}_{11} &= \left(\frac{n\beta}{k_0 y^2}\right)^2 \frac{yV_n(y)}{\varepsilon_r V'_n(y)} - \frac{\mu_r W'_n(y)}{yW_n(y)} \end{aligned} \tag{14}$$

Fig. 2 shows λ_g/λ_0 as a function of the normalised frequency f_0 (eqn. 15), for the first solutions of the characteristic equation. A provisional nomenclature HM_{nm} (Figs. 2*b* and *c*) is used, where the hybrid modes are referred by means of two subscripts. The first subscript is the integer n that determines the angular dependence of the fields, and the second subscript is the number of the order when the solutions are arranged from large radial propagation factors h to small values. When $n = 0$, then the well established TM-TE nomenclature is used (Fig. 2*a*). To establish a proper nomenclature from the hybrid modes, a careful analysis of the fields and characteristic parameters of these modes is required [11].

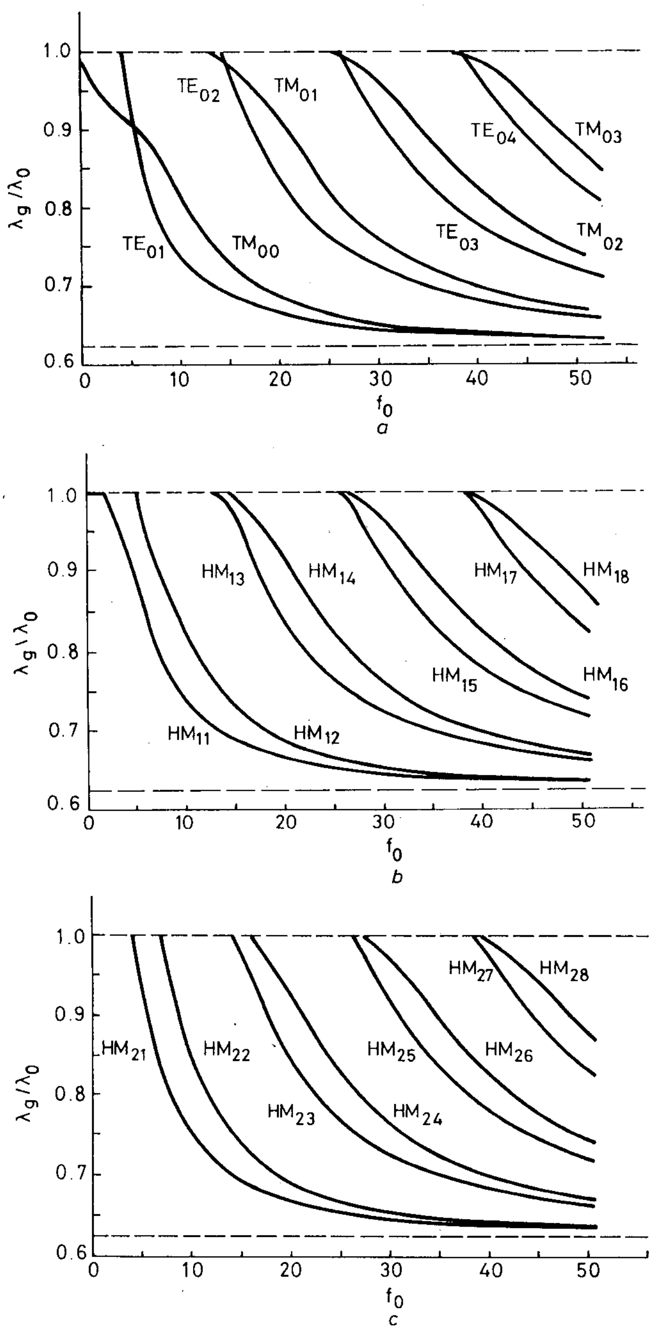

Fig. 2 *First solutions of the characteristic equation*
a $n = 0$
b $n = 1$
c $n = 2$

3 Cutoff conditions and fundamental modes

The cutoff frequencies of surface waveguide modes are the values of frequency at which the external fields exhibit no transverse attenuation, and therefore the true guiding properties disappear. The transverse attenuation of the external fields of a cylindrical surface waveguide is controlled by the radial propagation factor h, through the Bessel function $K_n(h\rho)$ which tends asymptotically to $\exp(-h\rho)$ for large values of ρ. The condition $h \to 0$ will determine the cutoff frequencies. The limit when $h \to 0$ of the characteristic equation (eqn. 6) yields the cutoff conditions in terms of the elements of the surface impedance dyadic.

The normalised radial propagation factors x and y are related by the equation

$$x^2 + y^2 = f_0^2 \qquad f_0 = k_0 c\sqrt{(\varepsilon_r\mu_r - 1)} \tag{15}$$

This relationship is derived from eqn. 4 and shows that the normalised cutoff frequencies f_{0_c} are equal to the cutoff values of the radial propagation factor in the dielectric medium y_c. Taking into account the expansions ($n > 0$),

$$\left.\begin{aligned} \Phi_n(x) &= -\frac{1}{x^2}(n + x^2\phi_1 + x^4\phi_2 + \cdots) \\ \frac{n\beta}{k_0} &= n\left(1 + x^2\frac{\varepsilon_r\mu_r - 1}{2y^2} + x^4\frac{1 - \varepsilon_r\mu_r}{2y^4} \right. \\ &\qquad \left. \times\left(1 + \frac{\varepsilon_r\mu_r - 1}{4}\right) + \cdots\right) \end{aligned}\right\} \tag{16}$$

and substituting eqn. 16 into eqn. 4, an approximate expression of the characteristic equation near cutoff can be derived ($n > 0$):

$$\begin{aligned} &\frac{1}{x^2}\left[n\left(\frac{1 - \varepsilon_r\mu_r}{y^2} + 2\phi_1\right)\bar{X}_{22} \right. \\ &\qquad \left. - (1 + \bar{X}_{12}\bar{X}_{21} - \bar{X}_{11}\bar{X}_{22} + \bar{X}_{12} + \bar{X}_{21})\right] \\ &\qquad + n\frac{1 - \varepsilon_r\mu_r}{2y^2}(\bar{X}_{12} + \bar{X}_{21}) + n^2\frac{\varepsilon_r\mu_r - 1}{y^4}\bar{X}_{22} \\ &\qquad + \bar{X}_{22}(\phi_1^2 + 2\phi_0\phi_2) \\ &\qquad - \phi_1(1 + \bar{X}_{12}\bar{X}_{21} - \bar{X}_{11}\bar{X}_{22}) - \bar{X}_{11} = 0 \end{aligned} \tag{17}$$

This equation has the structure $F_1/x^2 + F_2 = 0$, whose solutions $x^2 = -F_2/F_1$, when $x \to 0$, are the roots of the function F_2, provided that both F_2 and F_1 have been written with common denominators and that the numerator of F_1 has no infinity of order greater than $1/x^2$. Under these assumptions, the characteristic equation can be approximated near cutoff by the expression ($n > 0$)

$$\begin{aligned} &n\left(\frac{1 - \varepsilon_r\mu_r}{y^2} + 2\phi_1\right)\bar{X}_{22} \\ &\qquad - (1 + \bar{X}_{12}\bar{X}_{21} - \bar{X}_{11}\bar{X}_{22} + \bar{X}_{12} + \bar{X}_{21}) = 0 \end{aligned} \tag{18}$$

Taking now into account the expressions for ϕ_1:

$$\left.\begin{aligned} n = 1&: \ \phi_1 = \ln\frac{2}{\Gamma x} \\ n > 1&: \ \phi_1 = \frac{1}{2(n-1)} \end{aligned}\right\} \tag{19}$$

and the approximation of $\Phi_n(x)$ when $n = 0$:

$$\lim_{x \to 0}\Phi_0(x) \to \frac{1}{x^2\ln\dfrac{1}{\Gamma x}} \tag{20}$$

eqns. 8 (for $n = 0$) and 18 (for $n > 0$) determine the cutoff conditions in terms of the elements $\bar{X}_{ij}$:

$$\left.\begin{aligned} &n = 0,\ \text{TM modes:}\ \bar{X}_{22} = 0 \\ &n = 0,\ \text{TE modes:}\ \frac{1}{\bar{X}_{11}} = 0 \\ &n = 1:\ \bar{X}_{22} = 0 \\ &n > 1:\ n\left(\frac{1 - \varepsilon_r\mu_r}{y^2} + \frac{1}{n-1}\right)\bar{X}_{22} \\ &\qquad - (1 + \bar{X}_{12}\bar{X}_{21} - \bar{X}_{11}\bar{X}_{22} \\ &\qquad + \bar{X}_{12} + \bar{X}_{21}) = 0 \end{aligned}\right\} \tag{21}$$

The elements $\bar{X}_{ij}$ (eqn. 10) have an explicit dependence on x, and therefore the limits when $x \to 0$ of these expressions have to be calculated to work out the particular cutoff conditions that we have given only in terms of $\bar{X}_{ij}$. Such limits are determined by

$$\left.\begin{aligned}
\lim_{x\to 0} x^4 \xi &= R V_n(y)\Bigg(K V_n(y) + (\varepsilon_r g_0 + \mu_r h_0)\frac{W_n(y)}{y}\Bigg)\\
\lim_{x\to 0} x^4 \xi' &= R\Bigg(K V_n(y) V_n'(y) + \frac{\varepsilon_r}{y} g_0 V_n(y) W_n'(y) + \frac{\mu_r}{y} h_0 V_n'(y) W_n(y)\Bigg)\\
\lim_{x\to 0} x^4 \zeta &= R\Bigg(K V_n(y) V_n'(y) + \frac{\varepsilon_r}{y} g_0 V_n'(y) W_n(y) + \frac{\mu_r}{y} h_0 V_n(y) W_n'(y)\Bigg)\\
\lim_{x\to 0} x^4 \zeta' &= R V_n'(y)\Bigg(K V_n'(y) + (\varepsilon_r g_0 + \mu_r h_0)\frac{W_n'(y)}{y}\Bigg)\\
\lim_{x^4\to 0} x^4 \Psi &= -R\frac{n}{y}\left(\frac{2c}{\pi y b}\right)^2
\end{aligned}\right\} \quad (22)$$

where

$$\left.\begin{aligned}
R &= Y_n(kb)\left[K Y_n(kb) + (\varepsilon_r g_0 + \mu_r h_0)\frac{Y_n'(kb)}{y}\right]\\
K &= h_0 g_1 + g_0 h_1 - \left(\frac{nc}{b}\right)^2 \frac{\varepsilon_r \mu_r + 1}{y^2}\\
H_n &= \frac{1}{x^2}(h_0 + x^2 h_1 + \cdots) \quad n \geqslant 0\\
G_n &= \frac{1}{x^2}(g_0 + x^2 g_1 + \cdots) \quad n \geqslant 1\\
G_0 &= g_0 + x^2 g_1 + \cdots \quad n = 0\\
n = 0:\ g_0 &= \frac{a}{2c} m_1,\ h_0 = \frac{c}{b \ln \frac{b}{a}}\\
g_1 &= \frac{a^2 b}{8c^3}\left[2 - 2\left(\frac{a}{b}\right)^2 \ln\frac{a}{b} - \frac{1}{2}\left(\frac{b}{a}\right)^2 - \frac{3}{2}\left(\frac{a}{b}\right)^2\right]\\
h_1 &= \frac{b}{4c \ln \frac{b}{a}}\left[2\ln\frac{b}{a} - 2 + \frac{1 - \left(\frac{a}{b}\right)^2}{\ln\frac{b}{a}}\right]\\
n \geqslant 1:\ g_0 &= \frac{nc}{b}\frac{m_n}{s_n},\ h_0 = \frac{nc}{b}\frac{s_n}{m_n}\\
n = 1:\ g_1 &= \frac{b}{c s_1^2}\left[\left(\frac{a}{b}\right)^2 \ln\frac{b}{a} + \frac{1}{4}\left(\frac{b}{a}\right)^2 - \frac{5}{4}\left(\frac{a}{b}\right)^2 + 1\right]\\
h_1 &= \frac{a}{c m_1^2}\left[\frac{a}{b}\ln\frac{b}{a} - \frac{b}{a} + \frac{1}{4}\left(\left(\frac{b}{a}\right)^3 + 3\frac{a}{b}\right)\right]\\
n \geqslant 2:\ g_1 &= \frac{b}{2 c s_n^2}\left[2 + \frac{1}{n+1}\left(\frac{b}{a}\right)^{2n} - \frac{1}{n-1}\left(\frac{a}{b}\right)^{2n} - 2\frac{n^2 - 2}{n^2 - 1}\left(\frac{a}{b}\right)^2\right]\\
h_1 &= \frac{n s_n}{4 m_n}\left[\frac{b}{nc}\left[\frac{n+2}{n+1}\left(\frac{b}{a}\right)^n - \frac{n-2}{n-1}\left(\frac{a}{b}\right)^n\right] - \frac{a^2}{bc}\left[\frac{1}{n-1}\left(\frac{b}{a}\right)^n - \frac{1}{n+1}\left(\frac{a}{b}\right)^n\right]\right]\Bigg/ S_n\\
&\quad + \left[\frac{\frac{a}{c} s_1 m_n - n\frac{a}{c} m_1 s_n}{(n+1)(n-1) m_n}\right]\\
s_n &= \left(\frac{b}{a}\right)^n + \left(\frac{a}{b}\right)^n,\ m_n = \left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n
\end{aligned}\right\} \quad (23)$$

Therefore, the cutoff conditions of the cylindrical surface waveguide of Fig. 1 are

$$\left.\begin{aligned}
&n = 0,\ \text{TM modes:}\ V_0(y) = 0\\
&n = 0,\ \text{TE modes:}\ \frac{1}{2}\left(\frac{b}{a} - \frac{a}{a}\right)\frac{a}{c} V_0(y) + \frac{\mu_r}{y} W_0(y) = 0\\
&n = 1:\ (a)\ V_1(y) = 0\\
&\qquad (b)\ K V_1(y) + (\varepsilon_r g_0 + \mu_r h_0)\frac{W_1(y)}{y} = 0\\
&n > 1:\ \left[\left(n\frac{\varepsilon_r \mu_r + 1}{y^2} - \frac{1}{n-1}\right)V_n(y) + (\varepsilon_r + \mu_r)\frac{V_n'(y)}{y}\right]\left[K V_n(y) + (\varepsilon_r g_0 + \mu_r h_0)\frac{W_n(y)}{y}\right]\\
&\qquad + \frac{c}{b}\left(\frac{2}{\pi y^2}\right)^2 (\varepsilon_r \sqrt{(g_0)} - \mu_r \sqrt{(h_0)})^2 = 0
\end{aligned}\right\} \quad (24)$$

We observe that the cutoff condition (eqn. 21) ($n = 1$) splits into two equations, like the conditions for $n = 0$, but this is prevented by an adding term when $n > 1$. The cutoff conditions for $n = 0$ and $n = 1$ have been worked out [9], and they agree with the results derived here, apart from some differences in condition 24 ($n = 1$, b). These differences can be analysed applying the limit when $x \to 0$ to the particular expression of the characteristic equation [9], which was derived using a boundary value technique instead of the surface impedance dyadic method. The conclusion of this analysis [12] is that expressions 24 seem correct, because they are reobtained with these alternative calculations.

If we apply the limit when $b \to a$ to the cutoff conditions 24, we obtain the cutoff conditions of the Goubau line:

$$n = 0, \text{ TM modes: } V_0(y) = 0$$
$$n = 0, \text{ TE modes: } W_0(y) = 0$$
$$n = 1: (a)\ V_1(y) = 0$$
$$(b)\ W_1(y) = 0 \tag{25}$$
$$n > 1: \left[\left(n\frac{\varepsilon_r \mu_r + 1}{y^2} - \frac{1}{n-1}\right)V_n(y) + (\varepsilon_r + \mu_r)\frac{V_n'(y)}{y}\right]W_n(y) + \mu_r \frac{c}{yb}\left(\frac{2}{\pi y}\right)^2 = 0$$

which agree with the results given by Fikioris and Roumeliotis [13].

Careful consideration has to be given to the possible existence of modes without a cutoff frequency. These modes are solutions of the characteristic equation that exist even for very low frequencies, and only when $f_0 \to 0$ then $h \to 0$. To work out the low-frequency approximation of the characteristic equation, we have to realise that, from eqn. 15, both radial propagation factors x and y will tend to zero at the same time that $f_0 \to 0$. If $n = 0$, then $\bar{X}_{11}$ and $\bar{X}_{22}$ may be approximated by

$$\left.\begin{aligned} \lim_{\substack{x\to 0\\ y\to 0}} \bar{X}_{22} &= \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{\varepsilon_r}\left[\ln\frac{b}{c} + \varepsilon_r \frac{x^2}{y^2}\ln\frac{b}{a}\right] \\ \lim_{\substack{x\to 0\\ y\to 0}} \bar{X}_{11} &= \frac{b}{2c}\left[\left(\frac{b}{a} - \frac{a}{b}\right)\frac{a}{c} + \mu_r\left(\frac{c}{b} - \frac{b}{c}\right)\right] \end{aligned}\right\} \tag{26}$$

Substituting eqns. 20 and 26 into eqn. 8, the low-frequency approximations of the characteristic equation corresponding to the TM and TE modes are obtained:

$$\left.\begin{aligned} &\text{TM modes: } x^2 \ln\frac{2b}{\Gamma x a} = \frac{y^2}{\varepsilon_r}\ln\frac{c}{b} \\ &\text{TE modes: } x^2 \ln\frac{2}{\Gamma x} = -\frac{b}{2c} \times \frac{1}{\left(\frac{b}{a} - \frac{a}{b}\right)\frac{a}{c} + \mu_r\left(\frac{c}{b} - \frac{b}{c}\right)} \end{aligned}\right\} \tag{27}$$

Therefore, there is a TM mode with no cutoff frequency, but no TE mode because eqns. 27 (TE) have no solution when $x \to 0$.

If $n > 0$, the low-frequency approximations of the elements $\bar{X}_{ij}$ can be derived taking into account the limits

$$\left.\begin{aligned}
\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4\xi}{R}\right) &= -\frac{v_0}{y^2}\left[\left(\frac{nc}{b}\right)^2(\varepsilon_r\mu_r + 1)v_0 + (\varepsilon_r g_0 + \mu_r h_0)\omega_0\right] = -\frac{\xi_0}{y^2} \\
\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4\xi'}{R}\right) &= -\frac{1}{y^3}\left[\left(\frac{nc}{b}\right)^2(\varepsilon_r\mu_r + 1)v_0 v_0' + \varepsilon_r g_0 v_0 \omega_0' + \mu_r h_0 v_0' \omega_0\right] = -\frac{\xi_0'}{y^3} \\
\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4\zeta}{R}\right) &= -\frac{1}{y^3}\left[\left(\frac{nc}{b}\right)^2(\varepsilon_r\mu_r + 1)v_0 v_0' + \varepsilon_r g_0 v_0' \omega_0 + \mu_r h_0 v_0 \omega_0'\right] = -\frac{\zeta_0}{y^3} \\
\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4\zeta'}{R}\right) &= -\frac{v_0'}{y^4}\left[\left(\frac{nc}{b}\right)^2(\varepsilon_r\mu_r + 1)v_0' + (\varepsilon_r g_0 + \mu_r h_0)\omega_0'\right] = -\frac{\zeta_0'}{y^4} \\
\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4\psi}{R}\right) &= -\frac{n}{y^3}\left(\frac{2c}{\pi b}\right)^2 = -\frac{\psi_0}{y^3} \\
v_0 &= -\lim_{y\to 0} V_n(y) = \frac{1}{\pi n}\left[\left(\frac{c}{b}\right)^n - \left(\frac{b}{c}\right)^n\right] \\
v_0' &= -y \lim_{y\to 0} V_n'(y) = \frac{1}{\pi}\left[\left(\frac{c}{b}\right)^n + \left(\frac{b}{c}\right)^n\right] \\
\omega_0 &= y \lim_{y\to 0} W_n(y) = \frac{c}{\pi b}\left[\left(\frac{c}{b}\right)^n + \left(\frac{b}{c}\right)^n\right] \\
\omega_0' &= y^2 \lim_{y\to 0} W_n'(y) = \frac{nc}{\pi b}\left[\left(\frac{c}{b}\right)^n - \left(\frac{b}{c}\right)^n\right]
\end{aligned}\right\} \tag{28}$$

Substituting eqn. 28 into eqn. 18:

$$\phi_1 = \frac{1}{2y^2} \times \left[\frac{\varepsilon_r \xi_0' + \mu_r \zeta_0 + 2n\xi_0 + 2\varepsilon_r \mu_r \psi_0}{\xi_0} + n(\varepsilon_r \mu_r - 1)\right] \tag{29}$$

The square brackets of eqn. 29 define a positive factor C. Taking into account the expressions for ϕ_1 (eqns. 19), the low-frequency approximation of the characteristic equation ($n > 0$) is given by

$$\left.\begin{aligned} n = 1:\ &\ln\frac{2}{\Gamma x} = \frac{C}{2y^2} \\ n \geqslant 2:\ &\frac{1}{n-1} = \frac{C}{y^2} \end{aligned}\right\} \tag{30}$$

As a consequence, there is another mode without a cutoff frequency which belongs to the group of symmetry $n = 1$, but there are no other modes without a cutoff frequency with $n > 1$, because the second equation cannot be satisfied when $y \to 0$. This waveguide exhibits two fundamental modes without cutoff frequencies and, consequently, if we are interested in single mode propagation, we will have to take a lot of care on the launching device to avoid the excitation of the undesirable mode.

4 Far-from-cutoff conditions

As the frequency tends to infinity, the transversel attenuation of the fields in the external medium increases and the radial propagation factor h also tends to infinity. The limit when $h \to \infty$ of the characteristic equation yields the far-from-cutoff conditions. The radial propagation factor k has to satisfy these conditions for large frequencies. The solutions of these equations k_f, and the cutoff values k_c, define the characteristic interval (k_c, k_f) of each mode. This pair of values k_c, k_f are the lower and upper limits of the radial propagation factor k for a given mode.

Taking into account the limits:

$$\left.\begin{aligned} \lim_{x\to\infty} \Phi_n(x) &= \lim_{x\to\infty} -\frac{1}{x} \\ \lim_{x\to\infty} \frac{\beta}{k_0} &= \sqrt{(\varepsilon_r \mu_r)} \end{aligned}\right\} \quad (31)$$

it is straightforward to derive from the characteristic equations 6 and 8 the far-from-cutoff conditions in terms of the elements $\bar{X}_{ij}$:

$$\begin{aligned} &n = 0, \text{ TM modes:} && \frac{1}{\bar{X}_{22}} = 0 \\ &n = 0, \text{ TE modes:} && \bar{X}_{11} = 0 \\ &n > 0: && \bar{X}_{11} = 0 \end{aligned} \quad (32)$$

The particular expression of these conditions, which correspond to the surface waveguide of Fig. 1, can be calculated by means of the limits of the elements $\bar{X}_{ij}$ (eqns. 10 and 13) when $x \to \infty$:

$$\begin{aligned} n = 0:\ & \lim_{x\to\infty} \bar{X}_{11} = -\frac{\mu_r W_0'(y)}{y W_0(y)} \\ n = 0:\ & \lim_{x\to\infty} \bar{X}_{22} = \frac{y W_0(y)}{\mu_r W_0'(y)} \\ n > 0:\ & \lim_{x\to\infty} \bar{X}_{11} = \frac{\mu_r}{y}\left[\frac{n}{y}\left(W_n(y) - \frac{nc}{by} V_n(y)\right) - W_n'(y) + \frac{nc}{by} V_n'(y)\right] \\ & \times \frac{\left[\frac{n}{y}\left(W_n(y) + \frac{nc}{by} V_n(y)\right) + W_n'(y) + \frac{nc}{by} V_n'(y)\right]}{W_n(y) W_n'(y) - \left(\frac{nc}{by}\right)^2 V_n(y) V_n'(y)} \end{aligned} \quad (33)$$

Substituting eqn. 33 into eqn. 32 yields the far-from-cutoff conditions:

$$n = 0, \text{ TM modes and TE modes: } W_0'(y) = 0 \quad (34)$$

$$n > 0 \begin{cases} \frac{n}{y}\left(W_n(y) + \frac{nc}{by} V_n(y)\right) + W_n'(y) + \frac{nc}{by} V_n'(y) = 0 & (34a) \\ \frac{n}{y}\left(W_n(y) - \frac{nc}{by} V_n(y)\right) - W_n(y) + \frac{nc}{by} V_n'(y) = 0 & (34b) \end{cases}$$

which are common to the symmetric TM and TE modes, and split into two equations when $n > 0$.

5 Field coefficients and parameter *R*

Adler [14] showed that the nonradiating modes of a surface waveguide are, except for some particular solutions, hybrid modes, and both axial components of the electric and magnetic fields are nonzero. It is important to introduce a parameter to evaluate the relative contributions of TM and TE to the field structure of the hybrid modes. We have defined the R parameter as the normalised relation between the axial components of the magnetic and electric fields in the external medium:

$$R \equiv jZ_0\left[\frac{H_z}{E_z}\right]_{\rho > c} = \frac{c_5}{a_5} \quad (35)$$

Substituting eqn. 3 into eqn. 5 yields the following expression for R in terms of the elements of the surface impedance dyadic:

$$R = -\frac{1 + \bar{X}_{22}\Phi_n(x)}{\frac{n\beta}{k_0 x^2}\bar{X}_{22} + \bar{X}_{21}} = -\frac{\bar{X}_{12}\Phi_n(x)\frac{n\beta}{k_0 x^2}}{\frac{n\beta}{k_0 x^2}\bar{X}_{12} + \bar{X}_{11} - \Phi_n(x)} \quad (36)$$

Taking into account the limits when $x \to 0$ (eqn. 16) and when $x \to \infty$ (eqn. 31), and assuming that the elements $\bar{X}_{ij}$ remain finite, we can derive the asymptotic values of R at cutoff and far from cutoff:

$$\lim_{x\to 0} R = 1 \quad (37a)$$

$$\lim_{x\to\infty} R = -\frac{1}{\bar{X}_{21}} \quad (37b)$$

These results are interesting because they show that the hybrid nature of such modes is persistent even at cutoff and far from cutoff. This is a radically different property with respect to hybrid modes of inhomogeneous closed waveguides, which always exhibit a TM or TE structure at cutoff [14].

The R parameter determines the relative values of the coefficients c_5 and a_5. One of these two coefficients can be chosen as a free parameter to control the amount of power flow. Once the characteristic equation is solved, and the R parameter is evaluated by means of eqn. 36, the other field coefficients can be calculated from a_5 and c_5:

$$\left.\begin{aligned} a_3 &= \frac{DK_n(hc)}{\xi}\left[B(y)a_5 + \frac{2\mu_r c}{\pi y^2 b}\Delta_n Y_n(kc)c_5\right] \\ c_3 &= \frac{DK_n(hc)}{\xi}\left[\frac{2\varepsilon_r c}{\pi y^2 b}\Delta_n Y_n(kc)a_5 + A(y)c_5\right] \\ a_4 &= \frac{1}{D}\left[(\Delta_n^2 J_n(kb)Y_n(kb) - g_y h_j)a_3 - \frac{2\mu_r c}{\pi y^2 b}\Delta_n c_3\right] \\ c_4 &= \frac{1}{D}\left[-\frac{2\mu_r c}{\pi y^2 b}\Delta_n a_3 + (\Delta_n^2 J_n(kb)Y_n(kb) - h_y g_j)c_3\right] \\ a_1 &= \frac{K_n(ha)}{P_n(hb)}[J_n(kb)a_3 + Y_n(kb)a_4] \\ c_1 &= \frac{K_n'(ha)}{Q_n(hb)}[J_n(kb)c_3 + Y_n(kb)c_4] \\ a_2 &= -\frac{I_n(ha)}{K_n(ha)}a_1 \end{aligned}\right\} \quad (38)$$

$$c_2 = -\frac{I_n'(ha)}{K_n'(ha)} c_1$$

where

$$\left.\begin{aligned} D &= g_y h_y - \Delta_n^2 Y_n(kb)^2 \\ h_j &= H_n J_n(kb) + \frac{\varepsilon_r}{y} J_n'(kb) \\ g_j &= G_n J_n(kb) + \frac{\mu_r}{y} J_n'(kb) \end{aligned}\right\} \quad (39)$$

and where the other symbols have already been defined (eqn. 11). If $n = 0$, then the expressions of the field coefficients a_i and c_i (eqn. 38) can be simplified.

These analytical expressions of the field coefficients are useful to provide a straightforward evaluation of the fields themselves, as well as the power flow and the stored energy. In Appendix 8 we have calculated the power flow P, the relative contributions of each medium P_1, P_2 and P_3, the time-averaged stored energy W, the relative contributions of each medium W_1, W_2 and W_3, the power losses in the dielectric medium P_{ld} and over the surface of the conductor P_{lc}, and the attenuation factor α. All these characteristic parameters are expressed in terms of the field coefficients.

It is interesting to particularise the asymptotic value of R (eqn. 37b) for the surface waveguide under investigation. The limit of R when $x \to \infty$ changes its sign as a function of which the far-from-cutoff condition is satisfied (eqn. 34a or 34b):

$$n > 0: \lim_{x \to \infty} R = \sqrt{\frac{\varepsilon_r}{\mu_r}} \quad (40a)$$

$$n > 0: \lim_{x \to \infty} R = -\sqrt{\frac{\varepsilon_r}{\mu_r}} \quad (40b)$$

This result makes it possible to distinguish two different sets of hybrid modes as a function of their field structure. There is a set of modes that exhibit a change of sign of the R parameter as a function of frequency, so they must have a zero or an infinity for some particular frequency.

6 Conclusions

The surface impedance dyadic method has shown several advantages in working out the fields and characteristic parameters of cylindrical surface waveguide modes. Among these advantages are the expressions of the characteristic equation, cutoff conditions, far from cutoff conditions, the R parameter and its asymptotic values at cutoff and far from cutoff, all of which are given in terms of the elements of the surface impedance dyadic. This method has been applied to a dielectric-coated wire with an intervening airgap and, as a consequence, it is possible to calculate the characteristic parameters of any non-radiating mode supported by this waveguide. A careful analysis of the fields and the R parameter is required to discuss the TM-TE structure of the hybrid modes [11].

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8 Appendix: Power flow, stored energy and attenuation

The power flow P can be obtained by integrating the Poynting vector over a transverse cross-section of the surface waveguide. Making use of the expressions for the fields eqns. 1–3 it is possible to calculate the contribution P_1, P_2 and P_3 of each medium (Fig. 1) to the total flux of the Poynting vector. Such contributions are:

$$\left.\begin{aligned} P_1 &= \pi \frac{k_0^2 c^4}{Z_0 x^4} \left[\frac{\beta}{k_0} [L + M]_{\rho=a}^{\rho=b} + n\left(1 + \frac{\beta^2}{k_0^2}\right)[L_n M_n]_{\rho=a}^{\rho=b} \right] \\ P_2 &= \pi \frac{k_0^2 c^4}{Z_0 y^4} \left[\frac{\beta}{k_0} [\varepsilon_r E + \mu_r F]_{\rho=b}^{\rho=c} + n\left(\varepsilon_r \mu_r + \frac{\beta^2}{k_0^2}\right)[E_n F_n]_{\rho=b}^{\rho=c} \right] \\ P_3 &= \pi \frac{k_0^2 c^4}{Z_0 x^4} \left[\frac{\beta}{k_0} (a_5^2 + c_5^2)[T]_{\rho=c}^{\rho=\infty} + n\left(1 + \frac{\beta^2}{k_0^2}\right) a_5 c_5 [K_n(h\rho)^2]_{\rho=c}^{\rho=\infty} \right] \end{aligned}\right\} \quad (41)$$

where

$$\begin{aligned} L &= \int dt \left(\frac{n^2}{t} L_n^2 + tL_n'^2\right) = \left(n - \frac{t^2}{2}\right)L_n^2 + (1+n)tL_n L_{n+1}^* + \frac{t^2}{2} L_{n+1}^{*2} \\ M &= \int dt \left(\frac{n^2}{t} M_n^2 + tM_n'^2\right) = \left(n - \frac{t^2}{2}\right)M_n^2 + (1+n)tM_n M_{n+1}^* + \frac{t^2}{2} M_{n+1}^{*2} \end{aligned}$$

$$
\left.
\begin{aligned}
E &= \int ds \left(\frac{n^2}{s} E_n^2 + sE_n'^2\right)\left(n + \frac{s^2}{2}\right)E_n^2 \\
&\quad - (1+n)sE_n E_{n+1} + \frac{s^2}{2} E_{n+1}^2 \\
F &= \int ds \left(\frac{n^2}{s} F_n^2 + sF_n'^2\right) = \left(n + \frac{s^2}{2}\right)F_n^2 \\
&\quad - (1+n)sF_n F_{n+1} + \frac{s^2}{2} F_{n+1}^2 \\
T &= \int dt \left(\frac{n^2}{t} K_n(t)^2 + tK_n'(t)^2\right) \\
&= \left(n - \frac{t^2}{2}\right)K_n(t)^2 \\
&\quad - (1+n)tK_n(t)K_{n+1}(t) + \frac{t^2}{2} K_{n+1}(t)^2 \\
L_n(t) &= a_1 I_n(t) + a_2 K_n(t) \\
L_n(t)^* &= a_1 I_n(t) - a_2 K_n(t) \\
M_n(t) &= c_1 I_n(t) + c_2 K_n(t) \\
M_n(t)^* &= c_1 I_n(t) - c_2 K_n(t) \\
F_n(s) &= a_3 J_n(s) + a_4 Y_n(s) \\
F_n(s) &= c_3 J_n(s) + c_4 Y_n(s) \\
t &= h\rho \\
s &= k\rho
\end{aligned}
\right\} \quad (42)
$$

The time-averaged energies stored by the electric and magnetic fields of a surface waveguide mode are equal [14]. Therefore, the total time-averaged stored energy can be evaluated as twice the electric stored energy:

$$
\left.
\begin{aligned}
W_1 &= \pi\varepsilon_0\left(\frac{c}{x}\right)^2\left[N + \left(\frac{k_0 c}{x}\right)^2\right. \\
&\quad \left.\times\left[\left(\frac{\beta}{k_0}\right)^2 L + M + \frac{2n\beta}{k_0} L_n M_n\right]\right]_{\rho=a}^{\rho=b} \\
W_2 &= \pi\varepsilon_0\varepsilon_r\left(\frac{c}{y}\right)^2\left[S + \left(\frac{k_0 c}{y}\right)^2\right. \\
&\quad \times\left[\left(\frac{\beta}{k_0}\right)^2 E + \mu_r F\right. \\
&\quad \left.\left. + \frac{2n\beta}{k_0} \mu_r E_n F_n\right]\right]_{\rho=b}^{\rho=c} \\
W_3 &= \pi\varepsilon_0\left(\frac{c}{x}\right)^2\left[a_5^2 U + \left(\frac{k_0 c}{y}\right)^2\right. \\
&\quad \times\left[\left(\left(\frac{\beta}{k_0} a_5\right)^2 + c_5^2\right)T\right. \\
&\quad \left.\left. + \frac{2n\beta}{k_0} a_5 c_5 K_n(h\rho)\right]\right]_{\rho=c}^{\rho=\infty}
\end{aligned}
\right\} \quad (43)
$$

where

$$
\left.
\begin{aligned}
N &= \int dt\, tL_n^2 = \frac{t^2}{2} L_n^2 - ntL_n L_{n+1}^* - \frac{t^2}{2} L_{n+1}^{*2} \\
S &= \int ds\, sE_n^2 = \frac{s^2}{2} E_n^2 - nsE_n E_{n+1} + \frac{s^2}{2} E_{n+1}^2 \\
U &= \int dt\, tK_n(h\rho)^2 = \frac{t^2}{2} K_n(h\rho)^2 \\
&\quad + ntK_n(h\rho)K_{n+1}(h\rho) - \frac{t^2}{2} K_{n+1}(h\rho)^2
\end{aligned}
\right\} \quad (44)
$$

and where the other symbols have been defined in eqn. 42.

We will assume that the waveguide exhibits small losses, due to the loss tangent of the dielectric medium and the finite conductivity of the conductor. Using a perturbation technique [15], the attenuation factor $\alpha = \text{Real}(\gamma)$ can be evaluated integrating the losses in the dielectric medium P_{ld} and over the surface of the conductor P_{lc}:

$$
\begin{aligned}
\alpha &= \frac{P_{ld} + P_{lc}}{2P} \\
P_{ld} &= W_2 \frac{k_0}{\sqrt{(\varepsilon_0 \mu_0)}} \tan\delta \\
P_{lc} &= \tfrac{1}{2}\pi\, \delta k_0 Z_0\, a(H_z H_z^* + H_\phi H_\phi^*)_{\rho=a} \\
\delta &= \sqrt{\frac{1}{\sigma k_0 Z_0}}
\end{aligned}
\quad (45)
$$

where we have assumed that the permeability of the conductor is μ_0, and the fields of the lossless waveguide are used to evaluate P_{ld} and P_{lc}, as a 1st-order approximation. For a low-loss dielectric medium such as polystyrene ($\tan\delta = 0.0035$) and a good conductor such as copper, we can expect that the main contribution to the attenuation factor is P_{ld}, which is proportional to the amount of energy stored in the dielectric medium.