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Recursive and bargaining values

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ABSTRACT

We introduce two families of values for TU-games: the *recursive* and *bargaining* values. Bargaining values are obtained as the equilibrium payoffs of the symmetric non-cooperative bargaining game proposed by Hart and Mas-Colell (1996). We show that bargaining values have a recursive structure in their definition, and we call this property *recursiveness.* All efficient, linear, and symmetric values that satisfy recursiveness are called recursive values. We generalize the notions of potential, and balanced contributions property, to characterize the family of recursive values. Finally, we show that if a time discount factor is considered in the bargaining model, every bargaining value has its corresponding discounted bargaining value.

1. Introduction

In this paper we present two families of values for cooperative games with transferable utility (TU-games): the *recursive values* and the *bargaining values*. They belong to the broad family of efficient, linear and symmetric values (ELS-values). The main feature that defines a recursive value, as its name suggests, is its recursive formulation: the payments within a coalition N with a number of n players are determined by some distribution of the marginal contributions made by each player to the coalition and by the payments when there are $n-1$ players. Each size of coalition n has its corresponding distribution coefficient α_n, $n = 1, 2, \ldots$. In general, the coefficients α_n are real numbers (even negative). The family of *bargaining values* arises when $\alpha_n \in [0, 1]$. In that case, this means that if coalition N forms, each player receives a fraction α_n of her contribution to the coalition, and the rest is shared equally among the remaining players.

In a cooperative setting, a value expresses a particular way in which players share the benefits of their cooperation. Following the *Nash program*, one can determine a particular value either through a set of properties that the value satisfies (the axiomatic approach) or through a non cooperative game which reflects a plausible negotiation process (the strategic approach). In the latter case cooperative agreement is obtained as the equilibrium payoffs of the non cooperative game. The two approaches are considered as complementary and hopeful of mutual reinforcement.[1] We use both approaches in the present paper. We see that the simple recursive design of these values play a key role in their axiomatic characterization and in their strategic support.

In the axiomatic approach, we find that *recursiveness* enables the *potential approach* to be extended to the whole family of recursive values. Hart and Mas-Colell (1989) introduce the notion of *potential* of a game and prove that there is a unique potential function P and that the gradient of the potential yields the Shapley (1953) value. Myerson (1980) introduces the property of *balanced contributions* and shows that the Shapley value is characterized by this property and by *efficiency*. Xu et al. (2016) extend this approach, adjusting the notions of potential and balanced contributions, to characterize the solidarity value. Following the same approach, we show that each recursive value also admits an *adjusted potential* P^{α}, that is, it can be obtained as the *adjusted gradient* of P^{α}. In the same way, we extend the property of quasi-balanced contributions from Xu et al. (2016), defining the α*-balanced contributions* property. Thus, it holds that a value satisfies efficiency and the α-balanced contributions property if, and only if, it coincides with the recursive value. In addition, we show that adding the properties of *positivity*[2] and *coalitional monotonicity*[3] suffices to characterize the family of all *bargaining values*.

[1] For a good survey on the Nash program readers are referred to Serrano (2005, 2008).

[2] In monotonic games the payoffs of the players must be non-negative.

[3] If the worth of a coalition increases, the payoffs of the members of such coalition should not decrease.

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The term bargaining values comes from the fact that they can be obtained as the subgame perfect equilibrium payoffs of a non-cooperative negotiation process. This strategic approach complements the axiomatic characterization given in the first part of this work. We follow a multilateral negotiating procedure presented in Hart and Mas-Colell (1996). There, it is proved that in games with transferable utility, when the probability of breakdown vanishes, the payoffs associated with the equilibria converge to the Shapley value. When utility is not transferable, in pure bargaining games, the equilibrium payoffs converge to the Nash (1953) bargaining solution, and in the general case, the equilibria payoffs converge to the consistent values introduced by Maschler and Owen (1989, 1992).

In the same paper (Hart and Mas-Colell, 1996, Section 6), the negotiation procedure is generalized, expanding the set of values supported by non-cooperative bargaining. Restricting the analysis to the *symmetric case* in which the rules of negotiation do not discriminate against players by name, the values obtained include the *equal split value*[4] (all players receive the same); the *egalitarian Shapley values* (which are convex combinations between the Shapley value and the equal split value) considered also in Joosten (1996) and van den Brink et al. (2013); the *solidarity value* considered in Sprumont (1990) and Nowak and Radzik (1994); and the family of *solidarity values* introduced by Casajus and Huettner (2014). In this paper we characterize all the values that can be obtained with this alternating random proposer protocol. We show that *the parameters that determine each bargaining value are precisely the probabilities of its associated non-cooperative bargaining game.*

We end this paper considering *discounted values*. These are values that also depend on a specific parameter, which is called the *discount factor* $d \in [0, 1]$. The parameter d (given exogenously) determines the discounting of the available worth going from one round of negotiation to the next after a rejection of the proposal. In Joosten (1996) the discounted Shapley values were introduced as a parametric family, where the Shapley value is obtained when $d = 1$, and the equal split value when $d = 0$. In van den Brink and Funaki (2015), a bargaining game that implements the family of discounted Shapley values is offered, by using the bidding mechanism given by Pérez-Castrillo and Wettstein (2001) that implements the Shapley value. In Calvo and Gutiérrez-López (2016), it is shown that the discounted Shapley values can be obtained by using the Hart and Mas-Colell bargaining model if the *time cost factor* $\delta \in [0, 1]$ is considered. Note that it is implicitly assumed that the value should be obtained from a negotiation process that takes place over time, and time spent is costly. The main result obtained there is that when the risk of breakdown, $(1 - \rho)$, and the time cost factor, δ, are considered *simultaneously*, the subgame perfect equilibria of the Hart and Mas-Colell bargaining yields a discounted Shapley value, where the discount is a function that depends on the time cost factor and the risk of breakdown. Namely,

$$d = \frac{\delta(1-\rho)}{1-\delta\rho}.$$

Kawamori (2016) presents a parallel result with a similar variant of the Hart and Mas-Colell model. In Calvo and Gutiérrez-López (2018) this approach has also been extended to the discounted solidarity values.

We prove that this result can be extended to the whole family of bargaining values: the general (symmetric) bargaining procedure of Hart and Mas-Colell yields a strategic support to every discounted bargaining value. Therefore, discounted bargaining values should not be considered as just another value within the family of bargaining values, but rather as their associated discounted values that appear as soon as the time cost factor is considered.

The paper is organized as follows. In Section 2, the recursive and the bargaining values are introduced. Section 3 is devoted to the axiomatic approach. Section 4 considers the strategic approach. Finally, Section 5 is devoted to the discounted values.

2. Recursive and bargaining values

First, we recall some preliminary notions.

Let $U = \{1, 2, \ldots\}$ be the (infinite) set of potential players. A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where $N \subset U$ is a non empty and finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and each non empty subset S of N is a *coalition*. The real number $v(S)$ is called the *worth* of coalition S, and is interpreted as the total payoff that the coalition S, if it forms, can obtain for its members. $\mathcal{G}^N$ denotes the set of all cooperative TU-games with player set N, and $\mathcal{G}$ the set of all cooperative TU-games. We write v when there is no place for confusion. For each $S \subseteq N$, we denote the restriction of (N, v) to S as (S, v). For the sake of simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \setminus i$ instead of $N \setminus \{i\}$, and $v(i)$ instead of $v(\{i\})$. The cardinality of set S is denoted by its lowercase letter s, when no confusion arise, i.e. $|S| = s$.

A *value* is a function ψ which assigns to every TU-game (N, v) and every player $i \in N$ a real number $\psi_i(N, v)$, which represents an assessment made by i of her gains from participating in the game. For $S \subseteq N$ and $x \in \mathbb{R}^N$ we denote $\sum_{i \in S} x_i$ by x_S.

Some well-known properties for values are the following.

Efficiency. A value ψ is efficient if $\sum_{i \in N} \psi_i(N, v) = v(N)$, for all games $v \in \mathcal{G}$.

Linearity. A value ψ is linear if $\psi(\alpha v + \beta w) = \alpha\psi(v) + \beta\psi(w)$ for all real numbers α and β, and games v and w. When the equality holds for $\alpha = \beta = 1$ the value is said to be *additive*.

Anonymity. Let π be a permutation of the player set N. For any $S \subseteq N$, define $\pi S = \{\pi(i) : i \in S\}$. The game $(N, \pi v)$ is defined by $\pi v(\pi S) = v(S)$ for all $S \subseteq N$. A value ψ satisfies anonymity if $\psi_i(N, v) = \psi_{\pi(i)}(N, \pi v)$ for all $i \in N$.

Equal treatment. Two players $i, j \in N$ are said to be interchangeable in v if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. A value ψ satisfies the equal treatment property if $\psi_i(N, v) = \psi_j(N, v)$ when i and j are interchangeable in the game (N, v).

Equal treatment is a weaker property than anonymity. In the literature, symmetry is referred to anonymity in some papers and to equal treatment in others. However, as Malawski (2013) points out, in the presence of efficiency and linearity, anonymity and equal treatment are equivalent. For that reason, we use the term "*efficient, linear, symmetric value*" (ELS-value) to describe an efficient, linear value that satisfies anonymity or equal treatment. There are several alternative characterizations of the family of ELS-values on $\mathcal{G}$. In all cases, the payment received by each player sharing the grand coalition N depends on the values that can be distributed in all subcoalitions S of N. To the best of our knowledge, they are all given by Ruiz et al. (1998) and Driessen and Radzik (2003) (see also Radzik and Driessen, 2016), Hernández-Lamoneda et al. (2008), Chameni and Andjiga (2008), Chameni (2012), and Casajus (2012).

We use here the Chameni's characterization.

Let $\Delta^i(S, v) = v(S) - v(S \setminus i)$ be the *marginal contribution* of player $i \in S$ to coalition $S \subseteq N$.

Proposition 1 (*Chameni, 2012*)**.** *A value ξ^α on $\mathcal{G}$ is an ELS-value if, and only if, there exists a sequence of parameters $\left(\left(\alpha_s^n\right)_{s=1}^n\right)_{n=1,2\ldots}$,*

[4] Also called indistinctly in literature as *egalitarian value*.

with $\alpha_s^n \in \mathbb{R}$ for all n and s, and $\alpha_1^n = 1$, such that

$$\xi_i^{\alpha}(N,v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(s-1)!(n-s)!}{n!} \Delta^{\alpha_s^n|i}(S,v)\,, \quad (i \in N \subset U,\ v \in \mathcal{G})\,, \tag{1}$$

where

$$\Delta^{\alpha_s^n|i}(S,v) = \alpha_s^n \Delta^i(S,v) + \frac{1-\alpha_s^n}{s-1}\sum_{k\in S\setminus i}\Delta^k(S,v)\ . \tag{2}$$

One well-known example of ELS-value is introduced by Shapley (1953). The *Shapley value* of the game (N,v) is the payoff vector $Sh(N,v) \in \mathbb{R}^N$ defined for each $i \in N$ by

$$Sh_i(N,v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(s-1)!(n-s)!}{n!}\Delta^i(S,v)\,, \quad (i\in N\subset U)\,, \tag{3}$$

Alternatively, $Sh(N,v)$ can be obtained recursively[5] by

$$Sh_i(N,v) = \frac{1}{n}\Delta^i(N,v) + \frac{1}{n}\sum_{j\in N\setminus i} Sh_i(N\setminus j, v), \quad (i\in N)\,, \tag{4}$$

starting with

$$Sh_i(\{i\},v) = v(i) \ \ (i\in N)\ .$$

We mention two more examples of values that can also be obtained in a recursive way: the *solidarity value* and the *equal split value*.

Sprumont (1990, Section 5) introduces an example of a *population monotonic allocation scheme*, defined recursively by

$$Sl_i(N,v) = \frac{1}{n}\Delta^{av}(N,v) + \frac{1}{n}\sum_{j\in N\setminus i} Sl_i(N\setminus j, v), \quad (i\in N\subset U)\,, \tag{5}$$

starting with

$$Sl_i(\{i\},v) = v(i), \quad (i\in N)\ ,$$

where $\Delta^{av}(S,v)$ is the average of the marginal contributions of all players of coalition S, that is

$$\Delta^{av}(S,v) = \frac{1}{s}\sum_{k\in S}\Delta^k(S,v), \quad (S\subseteq N)\,.$$

The following formula

$$Sl_i(N,v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(n-s)!\,(s-1)!}{n!}\Delta^{av}(S,v)\,, \quad (i\in N\subset U)\ , \tag{6}$$

is introduced by Nowak and Radzik (1994) in order to define what they called the *Solidarity value* of the game (N,v). Calvo (2008) shows that definitions (5) and (6) are equivalent.

The *equal split value* E is defined by

$$E_i(N,v) = \frac{v(N)}{n}, \quad (i\in N\subset U)\ . \tag{7}$$

It can be checked that (7) can be obtained recursively by

$$E_i(N,v) = \frac{1}{n}\sum_{k\in N\setminus i}\frac{\Delta^k(N,v)}{(n-1)} + \frac{1}{n}\sum_{k\in N\setminus i} E_i(N\setminus k, v), \quad (i\in N\subset U)\ , \tag{8}$$

starting with

$$E_i(\{i\},v) = v(i), \quad (i\in N)\ .$$

[5] See Hart and Mas-Colell (1996).

The above recursive formulation of these values introduces the question about how many ELS-values can be obtained in this way. To answer this question we introduce formally the notion of *recursiveness*.

Definition 2. Let ξ^{α} be an ELS-value on $\mathcal{G}$ specified by $\left(\left(\alpha_s^n\right)_{s=1}^n\right)_{n=1,2,\ldots}$, with $\alpha_1^n = 1$ for all n. ξ^{α} is said to be α- *recursive* if, for all $v\in\mathcal{G}$, it holds that

$$\xi_i^{\alpha}(N,v) = \frac{1}{n}\Delta^{\alpha_n^n|i}(N,v) + \frac{1}{n}\sum_{k\in N\setminus i}\xi_i^{\alpha}(N\setminus k, v), \quad (i\in N\subset U)\ . \tag{9}$$

We show that recursive ELS-values appear when the parameters α_s^n are *independent* of n.

Theorem 3. *An ELS-value ξ^{α} is α-recursive if, and only if, $\alpha_s^n = \alpha_s^{n-1}$ for all $s\le n-1$.*

Proof. Let $(N,v)\in\mathcal{G}^N$. By (1), it holds that

$$\begin{aligned}\xi_i^{\alpha}(N,v) &= \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(n-s)!\,(s-1)!}{n!}\Delta^{\alpha_s^n|i}(S,v)\\ &= \frac{1}{n}\Delta^{\alpha_n^n|i}(N,v) + \frac{1}{n}\sum_{\substack{S\subsetneq N\\ i\in S}}\frac{(n-s)!\,(s-1)!}{(n-1)!}\Delta^{\alpha_s^n|i}(S,v)\,,\end{aligned}$$

and

$$\begin{aligned}\sum_{k\in N\setminus i}\xi_i^{\alpha}(N\setminus k,v) &= \sum_{k\in N\setminus i}\sum_{\substack{S\subseteq N\setminus k\\ i\in S}}\frac{(n-s-1)!\,(s-1)!}{(n-1)!}\Delta^{\alpha_s^{n-1}|i}(S,v)\\ &= \sum_{\substack{S\subsetneq N\\ i\in S}}\frac{(n-s)\,(n-s-1)!\,(s-1)!}{(n-1)!}\Delta^{\alpha_s^{n-1}|i}(S,v)\\ &= \sum_{\substack{S\subsetneq N\\ i\in S}}\frac{(n-s)!\,(s-1)!}{(n-1)!}\Delta^{\alpha_s^{n-1}|i}(S,v)\,.\end{aligned}$$

Then, ξ^{α} is recursive if, and only if, for all $(N,v)\in\mathcal{G}$, it holds that

$$\sum_{\substack{S\subsetneq N\\ i\in S}}\frac{(n-s)!\,(s-1)!}{(n-1)!}\left[\Delta^{\alpha_s^n|i}(S,v) - \Delta^{\alpha_s^{n-1}|i}(S,v)\right] = 0\ .$$

From (2), we have

$$\begin{aligned}&\Delta^{\alpha_s^{n-1}|i}(S,v) - \Delta^{\alpha_s^n|i}(S,v)\\ &= \left(\alpha_s^n - \alpha_s^{n-1}\right)\left(v(S\setminus i) - \frac{1}{s-1}\sum_{k\in S\setminus i}v(S\setminus k)\right)\ .\end{aligned}$$

Therefore, ξ^{α} is recursive if, and only if, for all $(N,v)\in\mathcal{G}$, it holds that

$$\sum_{\substack{S\subsetneq N\\ i\in S}}\frac{(n-s)!\,(s-1)!}{(n-1)!}\left(\alpha_s^n-\alpha_s^{n-1}\right)\left(v(S\setminus i) - \frac{1}{s-1}\sum_{k\in S\setminus i}v(S\setminus k)\right) = 0,\ i\in N. \tag{10}$$

Take $i,k\in N$, and let $v = u_{N\setminus i\setminus k}$, then expression (10) is reduced to:

$$\frac{1}{n-1}\left(\alpha_{n-1}^n - \alpha_{n-1}^{n-1}\right) = 0 \Rightarrow \alpha_{n-1}^n = \alpha_{n-1}^{n-1}.$$

By induction, assume that $\alpha_s^n = \alpha_s^{n-1}$, for all $s\ge r>2$. Take any $T\subseteq N$ with $|T| = r-1$ and $i\in T$, and let $v = u_{T\setminus i}$. Expression (10) and the induction hypothesis imply:

$$\frac{(n-r+1)!\,(r-2)!}{(n-1)!}\left(\alpha_{r-1}^n - \alpha_{r-1}^{n-1}\right) = 0 \Rightarrow \alpha_{r-1}^n = \alpha_{r-1}^{n-1}.$$

Thus, we prove that $\alpha_s^n = \alpha_s^{n-1}$ for all $s \leq n-1$ and all n. ■

Consequently, if an ELS-value is α-recursive, a sequence of real numbers $(\alpha_s)_{s=1,2,\ldots}$, with $\alpha_1 = 1$ suffices to determine it. We call all recursive ELS-values *recursive values*, and we denote them as φ^α.

Definition 4. Let $(\alpha_s)_{s=1,2\ldots}$ be a sequence of real numbers with $\alpha_1 = 1$. The recursive value φ^α on $\mathcal{G}$ is defined by

$$\varphi_i^\alpha(N, v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(n-s)!\,(s-1)!}{n!} \Delta^{\alpha_s|i}(S, v), \quad (i \in N \subset U) \ . \quad (11)$$

By Theorem 3, the above definition is equivalent to

Definition 5. Let $(\alpha_s)_{s=1,2\ldots}$ be a sequence of real numbers with $\alpha_1 = 1$. The recursive value φ^α on $\mathcal{G}$ is defined by

$$\varphi_i^\alpha(N, v) = \frac{1}{n}\Delta^{\alpha_n|i}(N, v) + \frac{1}{n}\sum_{k\in N\setminus i} \varphi_i^\alpha(N\setminus k, v), \quad (i \in N \subset U) \ , \quad (12)$$

starting with

$$\varphi_i^\alpha(\{i\}, v) = v(i), \quad (i \in N) \ .$$

Below we show a selection (not exhaustive) of recursive values:

(a) The *Shapley value, Sh*, corresponds to the case when $\alpha_s = 1$, for all s.

(b) The *equal split value, E*, corresponds to the case when $\alpha_s = 0$, for all $s > 1$.

(c) The *solidarity value, Sl*, corresponds to the case when $\alpha_s = 1/s$, for all $s > 1$.

(d) The *egalitarian Shapley values* (Hart and Mas-Colell, 1996; Joosten, 1996; van den Brink et al., 2013) are the class of all convex combinations between the Shapley value and the equal split value, $Sh^\theta = \theta Sh + (1-\theta)E$, for a fixed $\theta \in [0, 1]$. They correspond to the case when $\alpha_s = \theta$. Note that the solidarity value is not included in the family of egalitarian Shapley values.

(e) Casajus and Huettner (2014) introduce a family of values called *generalized solidarity values* So^ξ, for $\xi \in [0, 1]$. The two extreme points of that family are the equal split value, $So^1 = E$ (when $\xi = 1$), and the Shapley value, $So^0 = Sh$ (when $\xi = 0$). In the middle (when $\xi = 1/2$) is the solidarity value, $So^{1/2} = Sl$. It turns out that So^ξ corresponds to recursive value $\varphi^{\alpha(\xi)}$ where

$$\alpha_s(\xi) = \frac{1-\xi}{(s-2)\xi + 1}, \quad (s > 1) \ .$$

This is a straightforward consequence of Casajus and Huettner (2014, Corollary 4), and formula (11).

(f) The *egalitarian solidarity values* (Gutiérrez-López, 2020) are the class of all convex combinations between the solidarity value and the equal split value, $Sl^\theta = \theta Sl + (1-\theta)E$, for a fixed $\theta \in [0, 1]$. They correspond to the case when $\alpha_s = \theta/s$, for all $s > 1$.

Next, we show some examples of ELS-values which are not recursive values.

(g) Malawski (2013) introduces the family of *procedural values*. He specifies a procedure for sharing the gains of cooperation following a random order approach. We briefly recall its definition. Let $\Pi(N)$ be the set of all permutations of the set N. Every $\pi \in \Pi(N)$ induces an order on N. We denote by $P^{\pi(i)}$ the set of all predecessors which come before i in the order π (including i); that is, $P^{\pi(i)} = \{j \in N : \pi(j) \leq \pi(i)\}$. The marginal contribution of player i in the order π is defined by $m_{i,\pi}(N, v) = \left[v\left(P^{\pi(i)}\right) - v\left(P^{\pi(i)}\setminus i\right)\right]$. The procedure is as follows:

1. The players enter in random order π, with all orders being equally probable.
2. There is a fixed set of parameters $\left(\alpha_s^n\right)_{s=1}^n$, where $\alpha_1^n = 1$ and $\alpha_s^n \in [0, 1]$ for all $s \geq 2$, which specifies how to share the marginal contribution of player i when she comes up in the order $\pi(i) = s$.
3. Each player i in the order π receives a proportion $\alpha_{\pi(i)}^n$ of her marginal contribution, $\alpha_{\pi(i)}^n m_{i,\pi}(N, v)$, and the rest is shared equally among the remaining predecessors, that is, each player j $(\pi(j) < \pi(i))$ receives $\left(1-\alpha_{\pi(i)}^n\right) / (\pi(i)-1)\, m_{i,\pi}(N, v)$.
4. The *procedural value* of a player is the expected value of the income earned in each order, with all orders being equally likely.

Since in each order π, player i receives $\alpha_{\pi(i)}^n m_{i,\pi}(N, v)$, plus $\left(1-\alpha_{\pi(j)}^n\right) / (\pi(j)-1)\, m_{j,\pi}(N, v)$ from each of her successors ($j : \pi(j) > \pi(i)$), the procedural value is given by (see Malawski, 2013; formula (2))

$$a_i^N = \sum_{\pi\in\Pi} \frac{1}{n!}\left(\alpha_{\pi(i)}^n m_{i,\pi}(N, v) + \sum_{\substack{j\in N\\ \pi(j)>\pi(i)}} \frac{1-\alpha_{\pi(j)}^n}{\pi(j)-1} m_{j,\pi}(N, v)\right) \ . \quad (13)$$

However, under an easy manipulation of this formula, we can also express any procedural value in terms of the Chameni's coefficients. Therefore, a procedural value turns out to be an ELS-value ξ^α with $\alpha_1^n = 1$ and $\alpha_s^n \in [0, 1]$ for all $2 \leq s \leq n$. Actually, as Malawski points out (Section 4 in Malawski, 2013), it is possible to obtain any ELS-value with this random order approach if we allow that $\alpha_s^n \in \mathbb{R}$ for all $2 \leq s \leq n$.

Finally, we show two ELS-values with $\alpha_s^n \notin [0, 1]$.

(h) The *center of imputation set value*[6] (CIS) (see Driessen and Funaki, 1991), defined by

$$CIS_i(N, v) = v(i) + \frac{1}{n}\left[v(N) - \sum_{k\in N} v(k)\right], \quad (i \in N \subset U) \ .$$

This value corresponds to the case $\alpha_2^n = n-1$ and $a_s^n = 0$ for all $2 < s \leq n$. From this fact, it follows that the *consensus value*, $Cs = \frac{1}{2}Sh + \frac{1}{2}CIS$, introduced by Ju et al. (2007), or any other convex combination of Sh and CIS values, are not recursive values.

(i) The *equal allocation of non separable cost value* (ENSC) (see Moulin, 1985), is the *dual value* of CIS, i.e.

$$ENSC(N, v) = CIS(N, v^*),$$

where v^* is the *dual game* of v defined as $v^*(S) = v(N) - v(N\setminus S)$ for all $S \subseteq N$. This value corresponds to the case $\alpha_n^n = n-1$ and $a_s^n = 0$ for all $1 < s < n$. Therefore, it also follows that convex combinations of CIS, $ENSC$ and E values, considered in van den Brink and Funaki (2009), are not recursive values.

We end this section introducing the bargaining values. A remarkable feature of procedural values is that, for every coalition S, $\alpha_s^n \in [0, 1]$. This fact allows a nice interpretation for these coefficients: if coalition S forms, each player $i \in S$ receives a fraction α_s^n of her marginal contribution $\Delta^i(S, v)$, with the rest $\left(1-\alpha_s^n\right)\Delta^i(S, v)$ being shared equally among the remaining players in the coalition. Thus, player i receives a share α_s^n of her own marginal contribution, plus a share $\left(1-\alpha_s^n\right)/(s-1)$ of the marginal contribution of each of the other players in the coalition. We call *bargaining values* all recursive values that satisfy $\alpha_s \in [0, 1]$, and will be denoted by ϕ^α.

Definition 6. Let $(\alpha_s)_{s=1,2\ldots}$ be a sequence of parameters with $\alpha_1 = 1$ and $\alpha_s \in [0, 1]$ for all s. The bargaining value ϕ^α on $\mathcal{G}$ is defined by

$$\phi_i^\alpha(N, v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(n-s)!\,(s-1)!}{n!} \Delta^{\alpha_s|i}(S, v), \quad (i \in N \subset U) \ .$$

[6] This value is also called the *equal surplus division value*.

Obviously, the family of bargaining values is nothing more than the family of *recursive procedural values*. Apart from its intuitive interpretation in terms of redistribution of marginal contributions, this family of bargaining values has a special interest, since we will prove latter that the payoffs given by each bargaining value ϕ^α are precisely the equilibrium payoffs associated to a non-cooperative bargaining game, determined by the sequence $(\alpha_s)_{s=1,2...}$ that now will also have an interpretation in terms of probabilities.

3. Axiomatic approach

A significant consequence of recursiveness is that each recursive value admits a simple *potential* representation and can be characterized by a generalization of the *balanced contributions* property.

Hart and Mas-Colell (1989) introduce the notion of a potential function P for a cooperative game. The potential P is a real function $P : \mathcal{G} \to \mathbb{R}$, with $P(\emptyset, v) = 0$, such that $\sum_{i\in N} \Delta^i P(N, v) = v(N)$, where $\Delta^i P(N, v) = P(N, v) - P(N\backslash i, v)$, for all $i \in N$ and all $(N, v) \in \mathcal{G}$. An explicit representation of the potential in terms of the game is given by:

$$P(N, v) = \sum_{S\subseteq N} \frac{(n-s)!\,(s-1)!}{n!} v(S) .$$

Given a value ψ on $\mathcal{G}$, the contribution that a player i makes to the payoff of another player j is the difference between what j gets in the game with and without i. We denote this contribution by $\Delta^i\psi_j(N, v) = \psi_j(N, v) - \psi_j(N\backslash i, v)$. The balanced contributions property was introduced in Myerson (1980): a value ψ on $\mathcal{G}$ satisfies the *balanced contributions property* if

$$\Delta^j\psi_i(N, v) = \Delta^i\psi_j(N, v), \quad (i, j \in N \subset U,\ v \in \mathcal{G}) .$$

Balanced contributions is a principle of *reciprocity*: what you contribute to other is the same as what you get from the other. Two noteworthy results are that the Shapley value can be obtained as the gradient of the potential, i.e. $\Delta^i P(N, v) = Sh_i(N, v)$, for every game $v \in \mathcal{G}$ and player $i \in N$ (Hart and Mas-Colell, 1989, Theorem A); and that a value ψ on $\mathcal{G}$ satisfies efficiency and the balanced contributions property if, and only if, $\psi = Sh$ (Myerson, 1980, Theorem 1).

Xu et al. (2016) extend this approach, revising the notions of potential and balanced contributions, to characterize the solidarity value Sl. Let $P^* : \mathcal{G} \to \mathbb{R}$ be a function with $P^*(\emptyset, v) = 0$. P^* is called an *A-potential function* if it satisfies

$$\sum_{i\in N}\left[\Delta^i P^*(N, v) + \frac{1}{n} v(N\backslash i)\right] = v(N), \quad (i \in N \subset U,\ v \in \mathcal{G}) .$$

As Xu et al. (2016) point out: "Comparing with the potential function P by Hart and Mas-Colell (1989), an adjustment compensation $\frac{1}{n}v(N\backslash i)$ is added to the marginal contribution $\Delta^i P^*(N, v)$ for each player $i \in N$, to satisfy the efficiency normalization condition of the A-potential function P^*" (page 88); and "The adjustment implies precisely the factor of egalitarianism for the Solidarity value". (page 87).

Proposition 7 (Xu et al., 2016; *Theorem 3.2*). *There exists a unique A-potential function P^* on $\mathcal{G}$. Moreover, it holds that*

$$\Delta^i P^*(N, v) + \frac{1}{n} v(N\backslash i) = Sl_i(N, v), \quad (i \in N \subset U;\ v \in \mathcal{G}) .$$

Second, they introduce the notion of *quasi-balanced contributions*. A value ψ on $\mathcal{G}$ satisfies the quasi-balanced contributions property if

$$\Delta^j\psi_i(N, v) - \frac{1}{n} v(N\backslash i) = \Delta^i\psi_j(N, v) - \frac{1}{n} v(N\backslash j) .$$

Proposition 8 (Xu et al., 2016; *Theorem 4.2*). *A value ψ defined on $\mathcal{G}$ satisfies efficiency and the quasi-balanced contributions property if, and only if, $\psi = Sl$.*

We extend this adjustment approach for each recursive value.

First we show that if we add the compensation $\frac{1-\alpha_n}{n-1} v(N\backslash i)$ for each player $i \in N$, in the definition of the potential, we obtain that each recursive value is the gradient of its corresponding adjusted potential.

Definition 9. Let $(\alpha_s)_{s=1,2,...}$ be a sequence of real numbers with $\alpha_1 = 1$. A function $P^\alpha : \mathcal{G} \to \mathbb{R}$, with $P^\alpha(\emptyset, v) = 0$, is called an *adjusted α-potential function* if it satisfies

$$\sum_{i\in N}\left[\Delta^i P^\alpha(N, v) + \frac{1-\alpha_n}{n-1} v(N\backslash i)\right] = v(N), \quad (N \subset U,\ v \in \mathcal{G}) , \tag{14}$$

From Definition 9, it follows that

$$nP^\alpha(N, v) - \sum_{i\in N} P^\alpha(N\backslash i, v) = v(N) - \frac{1-\alpha_n}{n-1}\sum_{i\in N} v(N\backslash i) .$$

Therefore, there is an equivalent recursive formulation for P^α:

$$P^\alpha(N, v) = \frac{1}{n}\left[\sum_{i\in N} P^\alpha(N\backslash i, v) + v(N) - \frac{1-\alpha_n}{n-1}\sum_{i\in N} v(N\backslash i)\right], \quad (N \subset U,\ v \in \mathcal{G}) , \tag{15}$$

with $P^\alpha(\emptyset, v) = 0$.

We show that P^α can also be expressed directly in terms of the game v and the coefficients $(a_s)_{s=1,2...}$:

Proposition 10. *For every game $v \in \mathcal{G}$,*

$$P^\alpha(N, v) = \frac{1}{n} v(N) + \sum_{S\subsetneq N} \frac{(n-s)!\,(s-1)!}{n!}\alpha_{s+1} v(S), \quad (N \subset U) .$$

Proof. We prove by induction on the cardinality of the player set. The one player case, $P^\alpha(i, v) = v(i)$, is obvious. Assume that for all $i \in N$ it holds that

$$P^\alpha(S, v) = \frac{1}{s} v(S) + \sum_{T\subsetneq S} \frac{(s-t)!\,(t-1)!}{s!}\alpha_{t+1} v(T), \quad (S \subsetneq N) .$$

From (15) and the induction hypothesis,

$$\begin{aligned}
P^\alpha(N, v) &= \frac{1}{n}\left[\sum_{i\in N} P^\alpha(N\backslash i, v) + v(N) - \frac{1-\alpha_n}{n-1}\sum_{i\in N} v(N\backslash i)\right] \\
&= \frac{1}{n}\left[\sum_{i\in N}\frac{1}{n-1} v(N\backslash i)\right. \\
&\quad + \sum_{i\in N}\sum_{S\subsetneq N\backslash i}\frac{(n-1-s)!\,(s-1)!}{(n-1)!}\alpha_{s+1} v(S) \\
&\quad \left. - \frac{1-\alpha_n}{n-1}\sum_{i\in N} v(N\backslash i)\right] + \frac{1}{n} v(N) \\
&= \frac{1}{n} v(N) + \frac{1}{n}\left[\sum_{i\in N}\frac{\alpha_n}{n-1} v(N\backslash i)\right. \\
&\quad \left. + \sum_{i\in N}\sum_{S\subsetneq N\backslash i}\frac{(n-1-s)!\,(s-1)!}{(n-1)!}\alpha_{s+1} v(S)\right] \\
&= \frac{1}{n} v(N) + \frac{1}{n}\left[\sum_{i\in N}\sum_{S\subseteq N\backslash i}\frac{(n-1-s)!\,(s-1)!}{(n-1)!}\alpha_{s+1} v(S)\right]
\end{aligned}$$

$$= \frac{1}{n}v(N) + \frac{1}{n}\left[\sum_{S\subsetneq N} \frac{(n-s)\,(n-1-s)!\,(s-1)!}{(n-1)!}\alpha_{s+1}v(S)\right]$$
$$= \frac{1}{n}v(N) + \sum_{S\subsetneq N} \frac{(n-s)!\,(s-1)!}{n!}\alpha_{s+1}v(S)\ . \quad \blacksquare$$

Now we show that each recursive value φ^α can be obtained as the *adjusted gradient* of P^α.

Theorem 11. *Let $(\alpha_s)_{s=1,2,\ldots}$ be a sequence of real numbers with $\alpha_1 = 1$. Let φ^α be its associated recursive value. Then, it holds that*

$$\varphi_i^\alpha(N, v) = \Delta^i P^\alpha(N, v) + \frac{1-\alpha_n}{n-1}v(N\backslash i),\ \ (v \in \mathcal{G}, i \in N \subset U)\ .$$

Proof. We prove by induction on the cardinality of the player set. In the one player case, from (15), $P^\alpha(i, v) = v(i)$, so $\Delta^i P^\alpha(i, v) = v(i) = \varphi_i^\alpha(i, v)$.

Assume that for all $S \subsetneq N$, and $i \in S$, it holds that

$$\varphi_i^\alpha(S, v) = \Delta^i P^\alpha(S, v) + \frac{1-\alpha_s}{s-1}v(S\backslash i)\ .$$

By (15) and the induction hypothesis,

$$\Delta^i P^\alpha(N, v) + \frac{1-\alpha_n}{n-1}v(N\backslash i) = P^\alpha(N, v) - P^\alpha(N\backslash i, v)$$
$$+\frac{1-\alpha_n}{n-1}v(N\backslash i)$$
$$= \frac{1}{n}\left[\sum_{k\in N} P^\alpha(N\backslash k, v) + v(N) - \frac{1-\alpha_n}{n-1}\sum_{k\in N} v(N\backslash k)\right]$$
$$-P^\alpha(N\backslash i, v) + \frac{1-\alpha_n}{n-1}v(N\backslash i)$$
$$= \frac{1}{n}\left[v(N) - \frac{1-\alpha_n}{n-1}\sum_{k\in N} v(N\backslash k)\right] + \frac{1-\alpha_n}{n-1}v(N\backslash i)$$
$$+\frac{1}{n}\sum_{k\in N\backslash i}\left[P^\alpha(N\backslash k, v) - P^\alpha(N\backslash i, v)\right]$$
$$= \frac{1}{n}\left[v(N) - \frac{1-\alpha_n}{n-1}\sum_{k\in N} v(N\backslash k)\right] + \frac{1-\alpha_n}{n-1}v(N\backslash i)$$
$$+\frac{1}{n}\sum_{k\in N\backslash i}\left[\Delta^i P^\alpha(N\backslash k, v) - \Delta^k P^\alpha(N\backslash i, v)\right]$$
$$= \frac{1}{n}\left[v(N) - \frac{1-\alpha_n}{n-1}\sum_{k\in N} v(N\backslash k)\right] + \frac{1-\alpha_n}{n-1}v(N\backslash i)$$
$$+\frac{1}{n}\sum_{k\in N\backslash i}\left[\varphi_i^\alpha(N\backslash k, v) - \frac{1-\alpha_{n-1}}{n-2}v(N\backslash k\backslash i) - \varphi_k^\alpha(N\backslash i, v)\right.$$
$$\left.+\frac{1-\alpha_{n-1}}{n-2}v(N\backslash i\backslash k)\right]$$
$$= \frac{1}{n}\left[v(N) - \frac{1-\alpha_n}{n-1}\sum_{k\in N} v(N\backslash k)\right] + \frac{1-\alpha_n}{n-1}v(N\backslash i)$$
$$+\frac{1}{n}\sum_{k\in N\backslash i}\varphi_i^\alpha(N\backslash k, v) - \frac{1}{n}v(N\backslash i)$$
$$= \frac{1}{n}\left[v(N) - \alpha_n v(N\backslash i) - \frac{1-\alpha_n}{n-1}\sum_{k\in N\backslash i} v(N\backslash k)\right]$$
$$+\frac{1}{n}\sum_{k\in N\backslash i}\varphi_i^\alpha(N\backslash k, v)$$
$$= \frac{1}{n}\Delta^{\alpha|i}(N, v) + \frac{1}{n}\sum_{k\in N\backslash i}\varphi_i^\alpha(N\backslash k, v)$$
$$= \varphi_i^\alpha(N, v)\ . \quad \blacksquare$$

We also generalize the notion of quasi-balanced contributions. To do this, we use a slightly different interpretation of the axiom.

Recall that when $\alpha_n \in [0, 1]$ we can make an intuitive interpretation of this coefficient: if coalition N forms, each player i receives a fraction α_n of her marginal contribution $\Delta^i(N, v)$, with the rest $(1-\alpha_n)\,\Delta^i(N, v)$ being shared equally among the remaining $n-1$ players in the coalition. Correspondingly, every player $j \in N\backslash i$ contributes to the payoff of player i with the fraction $(1-\alpha_n)/(n-1)\Delta^j(N, v)$ of her own marginal contribution. Thus, two players $i, j \in N$ are said to be in *balance* when the mutual contributions to their payoffs follow the same relationship that the fraction $(1-\alpha_n)/(n-1)$ of their marginal contributions to the worth of the coalition.

Definition 12. Let $(\alpha_s)_{s=1,2,\ldots}$ be a sequence of real numbers with $\alpha_1 = 1$. A solution ψ on $\mathcal{G}$ satisfies the α-*balanced contributions* property if it holds that

$$\Delta^j\psi_i(N, v) - \Delta^i\psi_j(N, v) = \frac{1-\alpha_n}{n-1}\left[\Delta^j(N, v) - \Delta^i(N, v)\right]\ , \quad (16)$$

for all $i, j \in N \subset U$, and $v \in \mathcal{G}$.

In other words, the mutual impact on their payoffs follows the same relationship as their productivities. The fraction $(1-\alpha_n)/(n-1)$ measures the amount of her own productivity that each player shares with the others. When $\alpha_n = 0$, each player shares all of it, being the maximum degree of solidarity between them. Indeed, the recursive value φ^α corresponding to $\alpha_s = 0$, for all $s \geq 2$, is just the equal split value. The opposite is when $\alpha_n = 1$. In this case, each player reserves all her productivity for herself. Accordingly, $\alpha_s = 1$, for all s, corresponds with the Shapley value.

Obviously, for the general case of recursive values, it could happen that $\alpha_n \notin [0, 1]$. In that case, the sign of $(1-\alpha_n)/(n-1)$ only says whether the relationship between their productivity has a positive (direct) or negative (inverse) impact on their payment balance.

Proposition 13. *Let $(\alpha_s)_{s=1,2,\ldots}$ be a sequence of real numbers with $\alpha_1 = 1$. The associated recursive value φ^α satisfies the α-balanced contributions property.*

Proof. Let a game $(N, v) \in \mathcal{G}$, and $i, j \in N$ with $i \neq j$. By definition of P^α it is immediate that

$$\Delta^i P^\alpha(N, v) - \Delta^i P^\alpha(N\backslash j, v) = \Delta^j P^\alpha(N, v) - \Delta^j P^\alpha(N\backslash i, v)\ .$$

By Theorem 11, we have that

$$\Delta^j\psi_i(N, v) - \Delta^i\psi_j(N, v) = \left[\varphi_i^\alpha(N, v) - \varphi_i^\alpha(N\backslash j, v)\right]$$
$$-\left[\varphi_j^\alpha(N, v) - \varphi_j^\alpha(N\backslash i, v)\right] =$$
$$\left[\left(\Delta^i P^\alpha(N, v) + \frac{1-\alpha_n}{n-1}v(N\backslash i)\right)\right.$$
$$\left.-\left(\Delta^i P^\alpha(N\backslash j, v) + \frac{1-\alpha_{n-1}}{n-2}v(N\backslash i\backslash j)\right)\right]$$
$$-\left[\left(\Delta^j P^\alpha(N, v) + \frac{1-\alpha_n}{n-1}v(N\backslash j)\right)\right.$$
$$\left.-\left(\Delta^j P^\alpha(N\backslash i, v) + \frac{1-\alpha_{n-1}}{n-2}v(N\backslash i\backslash j)\right)\right] =$$
$$\Delta^i P^\alpha(N, v) - \Delta^i P^\alpha(N\backslash j, v) - \Delta^j P^\alpha(N, v) + \Delta^j P^\alpha(N\backslash i, v)$$

$$+\frac{1-\alpha_n}{n-1}\left[v(N\backslash i)-v(N\backslash j)\right]$$
$$=\frac{1-\alpha_n}{n-1}\left[\Delta^j(N,v)-\Delta^i(N,v)\right]. \quad \blacksquare$$

Theorem 14. *Let $(\alpha_s)_{s=1,2,\ldots}$ be a sequence of real numbers with $\alpha_1 = 1$. A value ψ defined on $\mathcal{G}$ satisfies efficiency and the α-balanced contributions property if, and only if, ψ is the recursive value φ^α.*

Proof. We know that φ^α is efficient by construction. Then, by Proposition 13, it only remains to show uniqueness. Let ψ be an efficient value on $\mathcal{G}$ satisfying α-balanced contributions. For any $(i, v) \in \mathcal{G}$, efficiency implies that $\psi_i(i, v) = v(i) = \varphi_i^\alpha(i, v)$. Now, assume that $\psi(T, v) = \varphi^\alpha(T, v)$ for any game (T, v) with $|T| < n$. Let $(N, v) \in \mathcal{G}$, $i, j \in N$ with $i \neq j$. Since ψ and φ^α satisfy α-balanced contributions, subtracting expression (16) for both values, we obtain

$$\psi_i(N,v)-\psi_i(N\backslash j,v)-\varphi_i^\alpha(N,v)+\varphi_i^\alpha(N\backslash j,v)$$
$$=\psi_j(N,v)-\psi_j(N\backslash i,v)-\varphi_j^\alpha(N,v)+\varphi_j^\alpha(N\backslash i,v),$$

and applying the induction hypothesis, $\psi_i(N\backslash j, v) = \varphi_i^\alpha(N\backslash j, v)$ and $\psi_j(N\backslash i, v) = \varphi_j^\alpha(N\backslash i, v)$, we have

$$\psi_i(N,v)-\varphi_i^\alpha(N,v)=\psi_j(N,v)-\varphi_j^\alpha(N,v),\ (i,j\in N),$$

which, jointly with efficiency, implies $\psi_i(N, v) = \varphi_i^\alpha(N, v)$, for all $i \in N$. $\blacksquare$

To end this section, recall that the family of bargaining values ϕ^α is a strict subfamily of the recursive values φ^α, since the parameters $(\alpha_s)_{s=1,2,\ldots}$ satisfy the additional restriction of $\alpha_s \in [0, 1]$ for all s. They are also a strict subfamily of the procedural values, since the parameters α_s are independent of n. By combining the properties that characterize the recursive values with those that characterize the procedural values, we characterize the family of bargaining values.

We recall the Malawski characterization of the procedural values. A TU-game is said to be *monotonic* if $v(T) \leq v(S)$ whenever $T \subseteq S$. We say that a value ψ satisfies *positivity*[7] on $\mathcal{G}$ when $\psi_i(N, v) \geq 0$ for all monotonic games $v \in \mathcal{G}$ and all $i \in N \subset U$. We say that a value ψ satisfies *coalitional monotonicity* on $\mathcal{G}$ when $\psi_i(N, v) \geq \psi_i(N, w)$ for all $i \in S \subseteq N$ if $v, w \in \mathcal{G}$ are such that $v(S) > w(S)$ for $S \subseteq N$, and $v(T) = w(T)$ otherwise.

Proposition 15 (*Malawski, 2013; Theorem 2*)**.** *A value ψ on G^N satisfies efficiency, linearity, symmetry, positivity, and coalitional monotonicity if, and only if, ψ is an ELS-value ξ^α with $\alpha_s^n \in [0, 1]$ for all $s \leq n$.*

In view of Proposition 15 and Theorems 14 and 3, two immediate corollaries emerge.

Corollary 16. *A value ψ on $\mathcal{G}$ satisfies efficiency, linearity, symmetry, positivity, coalitional monotonicity, and α-recursiveness if, and only if, ψ is a bargaining value ϕ^α.*

Corollary 17. *A value ψ on $\mathcal{G}$ satisfies efficiency, α-balanced contributions, positivity, and coalitional monotonicity if, and only if, ψ is a bargaining value ϕ^α.*

[7] In the literature, this axiom has also been called monotonicity, and weak monotonicity.

4. Strategic approach

This section sets out the *alternating random proposer bargaining model* and determines the family of symmetric values obtained from it. This non-cooperative approach has a long tradition in the literature. The idea is that after a process of negotiation between them, agents accept a value which is as simple and realistic as possible. This tradition begins with the Nash (1953) *demands game* and continues with the alternating offers model proposed by Ståhl (1972) and Rubinstein (1982). Binmore (see Binmore and Dasgupta, 1987, ch. 4) shows that if the per-round time discount factor is close to one, then the outcome of the unique subgame perfect equilibrium is close to the Nash (1953) bargaining solution. Binmore et al. (1986) (see also Roth, 1989) subsequently replace the time preferences (assuming that players are indifferent as to the timing of an agreement) with the risk of breakdown after the rejection of each proposal. In this, they prove that when the probability of breakdown converges to zero the equilibrium converges to the Nash bargaining solution. Hart and Mas-Colell (1996) extend the risk-of-breakdown model to the general setting of n-person cooperative games without transferable utility (NTU-games). They show that the limits of equilibrium payoffs are the *Shapley value* in TU-games, the *Nash bargaining solution* in pure bargaining problems, and the *consistent value* in NTU-games. In the same paper (Section 6) the bargaining rules are modified, also allowing the responders to leave the game after a breakdown. This is the setting that we consider here.

It is worth mentioning that this is an *approximate* implementation program, as the value is obtained *at the limit.* This is opposite to an *exact* implementation, in which the equilibrium payoffs coincide exactly with the value. An exact implementation of the Nash bargaining solution is given in Howard (1992), among others. Pérez-Castrillo and Wettstein (2001) provide a *bidding mechanism* for the Shapley value where players first bid for the right to be the proposer. Variations of the bidding mechanism are proposed in Ju and Wettstein (2009) to implement the Shapley, the consensus and the equal surplus values; in van den Brink and Funaki (2015) to implement the discounted Shapley values; and in van den Brink et al. (2013) to implement the egalitarian Shapley values.

In this paper we follow the alternating random proposer bargaining approach. Hart and Mas-Colell (1996, Section 6) consider a general multilateral bargaining procedure that players follow to find cooperative agreements. This is a sequential, noncooperative game where the proposer is chosen at random at each step and players drop out of the game randomly after proposals are rejected. In the general case, the proposer is not necessarily the only player to drop out after a proposal rejection: a respondent can also do so. We present here the general bargaining rules of the Hart and Mas-Colell model only for the *symmetric case in TU-games*:

> Let $(N, v) \in G^N$ be a TU-game. In each *round* there is a set $S \subseteq N$ of *active* players and a *proposer* $i \in S$. In the first round the active set is $S = N$. The proposer is chosen at random from S, with all players in S being equally likely to be selected. The proposer makes a feasible offer $a^{S,i} \in \mathbb{R}^S$, i.e. $\sum_{j\in N} a_j^{S,i} \leq v(S)$. If all members of S accept the offer (they are asked in a prespecified order) then the game ends with these payoffs. If the offer is rejected by even one member of S, the game moves on to the next round where, the set of active players is again S with probability $0 \leq \rho < 1$, and with probability $1-\rho$, a *breakdown* will occur: In this case, a player k is chosen at random from S to drop out, with probability α_s if $k = i$, and $(1 - \alpha_s) / (s - 1)$ if $k \in S\backslash i$.[8] Thus, player k receives a payoff of zero and the set of active players becomes $S\backslash k$.

[8] Obviously, $\alpha_s \in [0, 1]$ and $\alpha_1 = 1$.

The result in Hart and Mas-Colell (1996) for this symmetric case is the following:

Proposition 18 (*Hart and Mas-Colell, 1996, Propositions 1 and 9*). *Let $v \in \mathcal{G}$ be a monotonic TU-game. If $\rho < 1$, then there is a unique subgame perfect (SP) equilibrium. The proposals corresponding to an SP equilibrium are always accepted and are characterized by:*

(E.1) $a_i^{S,i}(\rho) = v(S) - \sum_{j \in S \setminus i} a_j^{S,i}(\rho)$ *for each* $i \in S \subseteq N$*; and*

(E.2) $a_j^{S,i}(\rho) = \rho a_j^S(\rho) + (1-\rho)\left[\alpha_s a_j^{S\setminus i}(\rho) + \sum_{k \in S \setminus i \setminus j} \frac{1-\alpha_s}{s-1} a_j^{S\setminus k}(\rho)\right]$ *for each* $i, j \in S$ *with* $i \neq j$*, and each* $S \subseteq N$*;*

where $a^S(\rho) = \frac{1}{s}\sum_{j \in S} a^{S,j}(\rho)$*. These proposals are unique and nonnegative. Moreover, these expected payoffs* $\left(a^S(\rho)\right)_{S \subseteq N}$ *are independent of* ρ *and satisfy*

$$a_i^S = \frac{1}{s}\Delta^{\alpha_s|i}(S, v) + \frac{1}{s}\sum_{k \in S \setminus i} a_i^{S\setminus k}, \quad (i \in S) \ , \tag{17}$$

where

$$\Delta^{\alpha_s|i}(S, v) = \alpha_s \Delta^i(S, v) + \frac{1-\alpha_s}{s-1}\sum_{k \in S \setminus i} \Delta^k(S, v) \ ,$$

starting with $a^i = v(i)$*, for all* $i \in N$*.*

Different specifications of parameters α_s, $s \geq 1$, yield different values.

Applying Theorem 3, we obtain the following equivalence:

Proposition 19. *Formula (17) is equivalent to*

$$a_i^N = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!\,(s-1)!}{n!}\Delta^{\alpha_s|i}(S, v), \quad (i \in N) \ . \tag{18}$$

Thus, the payoff allocations $\left(a^S\right)_{S \subseteq N}$ satisfying (18) is just the *bargaining value* ϕ^α.

Hence, the *symmetric bargaining procedure* of Hart and Mas-Colell *yields a strategic support to every bargaining value* ϕ^α on the class of monotonic games. It is noteworthy that the *parameters* $(\alpha_s)_{s=1,2...}$ *which determine each bargaining value* ϕ^α are exactly the probabilities of the proposer being dropped after a breakdown in the bargaining game.

5. Discounted values

A *discounted value* is a particular value that also depends on a specific parameter, which is called the *discount factor* $d \in [0, 1]$. As examples, the discounted Shapley values Sh^d were introduced in Joosten (1996), and considered in Driessen and Radzik (2003), and Radzik and Driessen (2016). This is a parametric family of values where the two extreme values are $Sh^1(v) = Sh(v)$ and $Sh^0(v) = E(v)$. Moreover, it holds that $Sh^d(v) = Sh(v_d)$, where the *discounted game* v_d is defined by $v_d(S) = d^{n-s}v(S)$, for all $S \subseteq N$.

In Calvo and Gutiérrez-López (2016), it is shown that the discounted Shapley values can be obtained in the Hart and Mas-Colell bargaining model if the *time cost factor* $\delta \in [0, 1]$ is considered. That is, players have preferences over the time at which the agreement is reached. In particular, the preferences over the amount x obtained at period $t \in \{0, 1, 2, \ldots\}$ can be represented by $\delta^t x$. This is a very natural assumption, as the time cost is a factor that should be taken into account because negotiations take place over time.[9] Each round requires time, and assuming that time is costly, should be natural and that players prefer to reach agreements at the beginning of bargaining.

The main result obtained in Calvo and Gutiérrez-López (2016) is that when the risk of breakdown, $(1-\rho)$, and the time cost factor, δ, are considered *simultaneously*, the subgame perfect equilibria of the Hart and Mas-Colell bargaining yields as *average payoffs* a discounted Shapley value Sh^d, where

$$d = \frac{\delta(1-\rho)}{1-\delta\rho} \ .$$

In Calvo and Gutiérrez-López (2018) this approach has also been extended to the discounted solidarity values.

In this section we show that these results can be reproduced for all bargaining values. Thus, the discounted bargaining values $\phi^{\alpha,d}$ should be considered as the discounted values that appear as soon as the time cost is considered in the bargaining process.

Definition 20. A value $\xi^{\alpha,d}$ on $\mathcal{G}$ is a discounted ELS-value if, and only if, there exists a sequence of parameters $\left(\left(\alpha_s^n\right)_{s=1}^n\right)_{n=1,2...}$, with $\alpha_1^n = 1$ and $\alpha_s^n \in \mathbb{R}$ for all $n, s \in \mathbb{N}$, and a discount factor $d \in [0, 1]$ such that

$$\xi_i^{\alpha,d}(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} d^{n-s}\Delta_d^{\alpha_s^n|i}(S, v), \quad (i \in N, \ v \in \mathcal{G}), \tag{19}$$

where

$$\Delta_d^{\alpha_s^n|i}(S, v) = \alpha_s^n \Delta_d^i(S, v) + \frac{1-\alpha_s^n}{s-1}\sum_{k \in S \setminus i} \Delta_d^k(S, v), \quad (S \subseteq N), \tag{20}$$

and

$$\Delta_d^i(S, v) = v(S) - dv(S \setminus i), \quad (i \in S) \ . \tag{21}$$

Proposition 21. *Let* $\delta \in [0, 1]$ *be a time cost factor and* (N, v) *be a monotonic TU-game. If* $0 \leq \rho < 1$*, then there is a unique SP equilibrium for each specification of the parameters* (ρ, δ)*. The proposals corresponding to an SP equilibrium are always accepted, and they are characterized by:*

(E.1) $a_i^{S,i} = v(S) - \sum_{j \in S \setminus i} a_j^{S,i}$ *for each* $i \in S \subseteq N$*; and*

(E.2) $a_j^{S,i} = \rho\delta a_j^S + (1-\rho)\left[\alpha_s \delta a_j^{S\setminus i} + \sum_{k \in S \setminus i \setminus j} \frac{1-\alpha_s}{s-1}\delta a_j^{S\setminus k}\right]$ *for each* $i, j \in S$ *with* $i \neq j$*, and each* $S \subseteq N$*;*

where $a^S = \frac{1}{s}\sum_{j \in S} a^{S,j}$*. Moreover, these are unique, nonnegative proposals whose payoffs* $\left(a^S\right)_{S \subseteq N}$ *satisfy*

$$a_i^S = d\left[\frac{1}{s}\Delta^{\alpha_s|i}(S, v) + \frac{1}{s}\sum_{k \in S \setminus i} a_i^{S\setminus k}\right] + (1-d)\frac{v(S)}{s} \ , \tag{22}$$

for all $i \in S \subseteq N$*, where*

$$d = \frac{\delta(1-\rho)}{1-\delta\rho} \ .$$

Proof. The proof follows the same lines used in Proposition 9 in Hart and Mas-Colell (1996). It is only necessary to replace the term "expected payoffs" by "discounted expected payoffs" and the same arguments apply here too. Hence, we omit it. In order to obtain (22), note that the proposer $i \in S$ offers each other player $j \in S \setminus i$ her discounted expected payoff in case of rejection, and

9 Given a rate of interest $r > 0$, the discount factor is equal to $\delta = 1/(1+r)$.

the proposer takes all the surplus. Therefore,

$$
\begin{aligned}
a_i^{S,i} &= v(S) - \sum_{j\in S\backslash i}\left(\rho\delta a_j^S + \delta(1-\rho)\right.\\
&\quad \times \left.\left[\alpha_s a_j^{S\backslash i} + \frac{1-\alpha_s}{s-1}\sum_{k\in S\backslash i\backslash j} a_j^{S\backslash k}\right]\right)\\
&= \delta\rho a_i^S + (1-\delta\rho)\, v(S) - \delta(1-\rho)\,\alpha_s v(S\backslash i) - \delta(1-\rho)\\
&\quad \times \frac{1-\alpha_s}{s-1}\sum_{j\in S\backslash i}\sum_{k\in S\backslash i\backslash j} a_j^{S\backslash k}.
\end{aligned}
$$

Note that

$$
\sum_{j\in S\backslash i}\sum_{k\in S\backslash i\backslash j} a_j^{S\backslash k} + \sum_{j\in S\backslash i} a_i^{S\backslash j} = \sum_{k\in S\backslash i} v(S\backslash k),
$$

and

$$
\begin{aligned}
(1-\delta\rho)\, v(S) &= (1-\delta)\, v(S) + \delta(1-\rho)\, v(S)\\
&= (1-\delta)\, v(S) + \delta(1-\rho)\\
&\quad \times \left[\alpha_s v(S) + \sum_{k\in S\backslash i}\frac{1-\alpha_s}{s-1} v(S)\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
a_i^{S,i} &= \delta\rho a_i^S + (1-\delta)\, v(S)\\
&\quad + \delta(1-\rho)\left[\alpha_s v(S) + \sum_{k\in S\backslash i}\frac{1-\alpha_s}{s-1} v(S)\right]\\
&\quad - \delta(1-\rho)\,\alpha_s v(S\backslash i) - \delta(1-\rho)\\
&\quad \times \frac{1-\alpha_s}{s-1}\left[\sum_{k\in S\backslash i} v(S\backslash k) - \sum_{j\in S\backslash i} a_i^{S\backslash j}\right]\\
&= \delta\rho a_i^S + (1-\delta)\, v(S) + \delta(1-\rho)\,\Delta^{\alpha_s|i} v(S)\\
&\quad + \delta(1-\rho)\,\frac{1-\alpha_s}{s-1}\sum_{j\in S\backslash i} a_i^{S\backslash j}
\end{aligned}
$$

Therefore, the expected payoff of player i is

$$
\begin{aligned}
a_i^S &= \frac{1}{s}\left(\delta\rho a_i^S + (1-\delta)\, v(S) + \delta(1-\rho)\,\Delta^{\alpha_s|i}(S,v)\right.\\
&\quad \left. + \delta(1-\rho)\,\frac{1-\alpha_s}{s-1}\sum_{j\in S\backslash i} a_i^{S\backslash j}\right)\\
&\quad + \frac{1}{s}\sum_{j\in S\backslash i}\left(\rho\delta a_i^S + \delta(1-\rho)\left[\alpha_s a_i^{S\backslash j} + \frac{1-\alpha_s}{s-1}\sum_{k\in S\backslash i\backslash j} a_i^{S\backslash k}\right]\right),
\end{aligned}
$$

and then,

$$
\begin{aligned}
& s(1-\delta\rho)\, a_i^S = (1-\delta)\, v(S) + \delta(1-\rho)\,\Delta^{\alpha_s|i}(S,v)\\
&\quad + \delta(1-\rho)\,\frac{1-\alpha_s}{s-1}\sum_{j\in S\backslash i} a_i^{S\backslash j}\\
&\quad + \delta(1-\rho)\left[\sum_{j\in S\backslash i}\alpha_s a_i^{S\backslash j} + \frac{1-\alpha_s}{s-1}\sum_{j\in S\backslash i}\sum_{k\in S\backslash i\backslash j} a_i^{S\backslash k}\right]
\end{aligned}
$$

Note that

$$
\sum_{j\in S\backslash i} a_i^{S\backslash j} + \sum_{j\in S\backslash i}\sum_{k\in S\backslash i\backslash j} a_i^{S\backslash k} = \sum_{j\in S\backslash i}(s-1)\, a_i^{S\backslash j},
$$

Then

$$
\begin{aligned}
& s(1-\delta\rho)\, a_i^S = (1-\delta)\, v(S) + \delta(1-\rho)\,\Delta^{\alpha_s|i}(S,v)\\
&\quad + \delta(1-\rho)\sum_{j\in S\backslash i} a_i^{S\backslash j},
\end{aligned}
$$

and dividing by $(1-\delta\rho)$, finally gives

$$
\begin{aligned}
a_i^S &= \frac{\delta(1-\rho)}{(1-\delta\rho)}\left[\frac{1}{s}\Delta^{\alpha_s|i}(S,v) + \frac{1}{s}\sum_{j\in S\backslash i} a_i^{S\backslash j}\right] + \frac{1-\rho}{1-\delta\rho}\frac{v(S)}{s}\\
&= d\left[\frac{1}{s}\Delta^{\alpha_s|i}(S,v) + \frac{1}{s}\sum_{k\in S\backslash i} a_i^{S\backslash k}\right] + (1-d)\,\frac{v(S)}{s},
\end{aligned}
$$

where

$$
d = \frac{\delta(1-\rho)}{1-\delta\rho}.
$$

which is Eq. (22). ■

It is clear that when $\delta = 1$ (22) reduces to (17), and when $\delta = 0$ it holds that $a_i^S = v(S)/s = E_i(S,v)$.

Moreover, if $\alpha_s = 1$, for all $s \geq 1$,

$$
a_i^S = \frac{v(S)}{s} + \frac{d}{s}\left[\sum_{k\in S\backslash i} a_i^{S\backslash k} - v(S\backslash i)\right],
$$

which defines the payoff configuration of the *discounted Shapley value*, which was introduced by Joosten (1996).

If $\alpha_s = 1/s$, for all $s > 1$, then

$$
a_i^S = \frac{v(S)}{s} + \frac{d}{s}\left[\sum_{k\in S\backslash i} a_i^{S\backslash k} - \frac{1}{s}\sum_{k\in S} v(S\backslash k)\right],
$$

which defines the payoff configuration of the *discounted solidarity value*, considered in Calvo and Gutiérrez-López (2018).

Notice that the payoff configuration $\left(a^S\right)_{S\subseteq N}$ defined recursively by (22) depends on game v, and parameters $(\alpha_s)_{s=1,2,\ldots}$ and d. Thus, below we write $a^N(d,v)$ as the payoff vector satisfying (22) and $a^N(v)$ as the payoff vector satisfying (17). By construction, the payoff vectors a^S are linear and efficient. This implies the following proposition.

Proposition 22. *For every game $v \in \mathcal{G}$, and parameters $(\alpha_s)_{s=1,2,\ldots}$ and d, it holds that $a^N(d,v) = a^N(v_d^n)$, where $v_d^n(S) = d^{n-s}v(S)$, for all $S \subseteq N$.*

Proof. The one player case is straightforward. Assume that the proposition holds for $n-1$ players. By induction, $a_i^{N\backslash k}(d,v) = a_i^{N\backslash k}\left(v_d^{n-1}\right)$. By definition of v_d^{n-1}, it holds that $v_d^n(S) = d v_d^{n-1}(S)$ for all $S \subseteq N\backslash k$. By linearity, it follows that

$$
a_i^{N\backslash k}\left(v_d^n\right) = d a_i^{N\backslash k}\left(v_d^{n-1}\right).
$$

Therefore,

$$
\begin{aligned}
a_i^N(d,v) &= \frac{v(N)}{n} + \frac{d}{n}\left[\sum_{k\in N\backslash i} a_i^{N\backslash k}(d,v) - \sum_{k\in N}\alpha_{k|i}(N) v(N\backslash k)\right]\\
&= \frac{v(N)}{n} + \frac{1}{n}\left[\sum_{k\in N\backslash i} d a_i^{N\backslash k}(d,v) - \sum_{k\in N}\alpha_{k|i}(N) d v(N\backslash k)\right]\\
&= \frac{v(N)}{n} + \frac{1}{n}\left[\sum_{k\in N\backslash i} d a_i^{N\backslash k}\left(v_d^{n-1}\right) - \sum_{k\in N}\alpha_{k|i}(N) d v(N\backslash k)\right]\\
&= \frac{v_d^n(N)}{n} + \frac{1}{n}\left[\sum_{k\in N\backslash i} a_i^{N\backslash k}\left(v_d^n\right) - \sum_{k\in N}\alpha_{k|i}(N) v_d^n(N\backslash k)\right]\\
&= a_i^N(v_d^n). \quad ■
\end{aligned}
$$

The meaning of the above proposition is that, if players discount payoffs for possible delays in agreements, it may be convenient to replace game v by the discounted game v_d. The reason is quite intuitive: Players are bargaining over agreements that can be delayed in time. Initially, they wish to share the worth $v(N)$, taking into account what each player would obtain eventually if a player i drops out of the game and players bargain over $v(N\backslash i)$ in the next period, sharing the expected discounted worth $dv(N\backslash i)$. Again, what they expect to obtain from $v(N\backslash i)$ is conditional on what they would obtain if a further player j drops out of the bargaining in a subsequent period, and they bargain over $v\left(N\backslash\{i,j\}\right)$, with an expected discounted worth of $d^2 v\left(N\backslash\{i,j\}\right)$, and so on. Therefore, any coalition $S \subseteq N$ of size $s < n$, needs $n-s$ periods of time, dropping players one by one, to be reached. Hence, the expected discounted worth to be shared is $d^{n-s}v(S)$.

Theorem 23. *Let $\left(a^N\right)_{N\subset U}$ be a sequence of payoff vectors satisfying (22). Then, it holds that $a^N = \phi^{\alpha,d}(N,v)$, for all $N \subset U$, where $\phi^{\alpha,d}$ is the discounted bargaining value associated to the sequence $(\alpha_s)_{s=1,2...}$, with $\alpha_s \in [0,1]$, and discount factor $d = \delta\left(1-\rho\right)/\left(1-\delta\rho\right)$.*

Proof. This is an immediate consequence of Proposition 22 and the fact that $\phi^{\alpha,d}(N,v) = \phi^{\alpha}(N,v_d^n)$. ∎

Hence, *the general (symmetric) bargaining procedure of Hart and Mas-Colell (1996, Section 6) yields a strategic support to every discounted bargaining value $\phi^{\alpha,d}$ on the class of monotonic games,* where $\alpha_s \in [0,1]$, for all $s > 1$ with $\alpha_1 = 1$, and $d = \delta\left(1-\rho\right)/\left(1-\delta\rho\right)$.

In summary, we have introduced a parameter $d \in [0,1]$ into the model which can be interpreted as the *cost of delay agreements* in bargaining. That parameter depends on the *time cost* δ and the *breakdown probability* $(1-\rho)$ *simultaneously*. As this cost of delay vanishes ($d \to 1$), the payoffs converge to the bargaining value ϕ^{α}, and when the cost of delay increases ($d \to 0$), the payoffs converge to the equal split value E.

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