Vniversitat de València
Facultat d' Economia

Unit 2. The Markowitz Portfolio Selection Model

Corporate Finance

Degree in International Business

Dr. Inmaculada Bel Oms

Agenda

1. Introduction
2. Markowitz and the portfolio selection
3. Efficient portfolios
4. Capital allocation and the separation property

REFERENCES

BLANCO, F.; FERRANDO, M. y MARTÍNEZ, M. F. (2015): *Teoría de la inversión*. Madrid. Pirámide. → CAPÍTULO 10

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BREALEY, R.A.; MYERS, S.C. y ALLEN, F. (2010). *Principios de Finanzas Corporativas*. McGraw Hill, México D.F. → CAPÍTULO 10

ROSS, WESTERFIELD and JAFFE. Corporate Finance. Tenth edition. McGraw-Hill (2013) → Chapter 11.3 (p.341); 11.5 (pp. 350-352) and 11.7 (pp. 356-359).

2.1. Introduction

- **THE MARKOWITZ MODEL** (1952-1959): development of a mathematical model.
  - The model studies the rational behavior of the investor → A portfolio that maximizes the investor's profit → ***Optimal Portfolio***.

- Given a set of financial assets, the model detects a range of possible combinations (portfolios) from which to choose the one which best suits the investor:
  - Portfolio that maximizes return (given a certain level of risk).
  - Portfolio that minimizes risk (given a certain level of return).

  - ***Objective:*** *to maximize the utility function of a rational and risk-averse individual who wishes to invest their entire budget in the N listed risky assets.*

2.2. Markowitz and portfolio selection

2.2.1 Hypothesis on the investor's behavior

- Assumptions:
  - Investor hypothesis:
    - Investors base their decisions on two parameters: mean (return) and variance (risk) → Decision context return – variance:

$$E\left[U\left(\tilde{R}_j\right)\right] = f\left[E\left(\tilde{R}_j\right), \sigma\left(\tilde{R}_j\right)^2\right] \qquad (2.1)$$

    - Investors are supposed to act rationally → to prefer portfolios with high returns and low risk (risk aversion). This is expressed in the utility function:

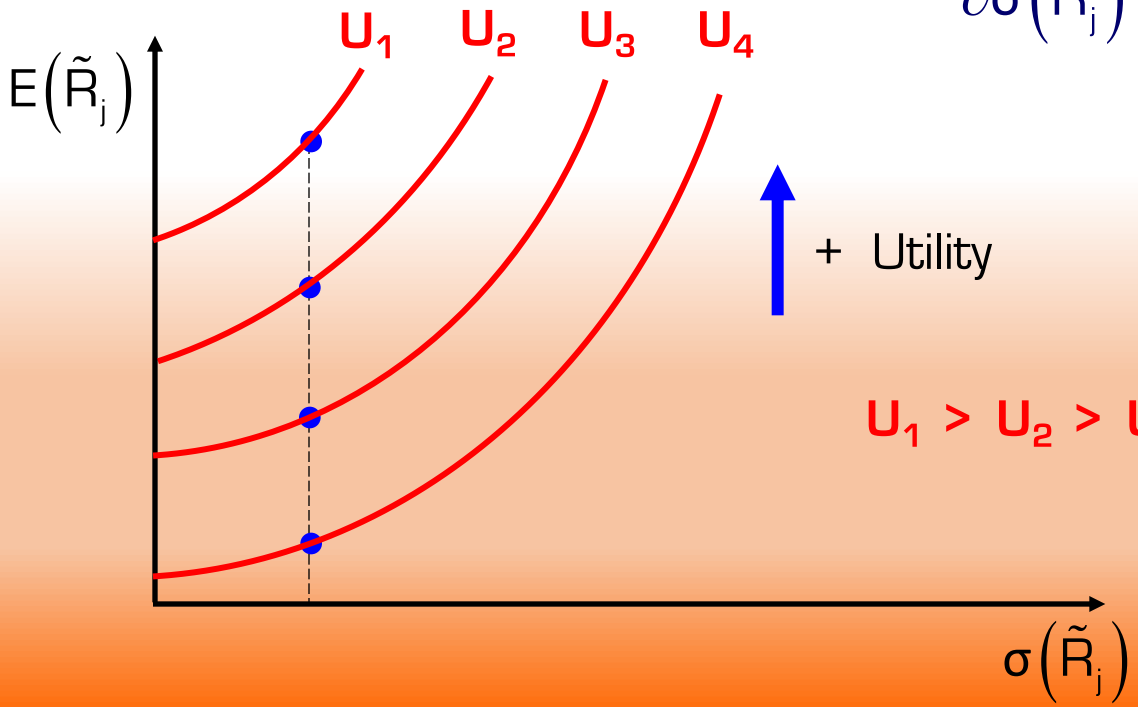
$$\frac{\partial U}{\partial \sigma\left(\tilde{R}_j\right)^2} \leq 0 \quad ; \quad \frac{\partial U}{\partial E\left(\tilde{R}_j\right)} \geq 0 \qquad (2.2)$$

2.2. Markowitz and portfolio selection

2.2.1 Hypothesis on the investor's behavior

- The investment considered is a one-period investment.
- Attitude towards risk will be reflected in the indifference curves (increasing and concave with respect to the ordinate axis):

$$\frac{\partial E\left(\tilde{R}_j\right)}{\partial \sigma\left(\tilde{R}_j\right)^2} \geq 0 \quad ; \quad \frac{\partial E\left(\tilde{R}_j\right)}{\partial \sigma^2\left(\tilde{R}_j\right)^2} \geq 0 \qquad (2.3)$$



+ Utility

$U_1 > U_2 > U_3 > U_4$

$\uparrow \sigma\left(\tilde{R}_j\right) \rightarrow \uparrow E\left(\tilde{R}_j\right)$

2.2. Markowitz and portfolio selection

2.2.1 Hypothesis on financial assets and markets

- Hypothesis on finanical assets and markets:

  - There are N risky assets in the market.
  - The returns of an asset/portfolio are a random variable whose probability function is Normal (mean and variance parameters).
  - Capital markets are perfect:
    - Investors do not influence price formation.
    - The information is accessible to all agents and at zero cost.
    - There are no taxes or transaction costs.
    - Short sales are not allowed.

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (I)

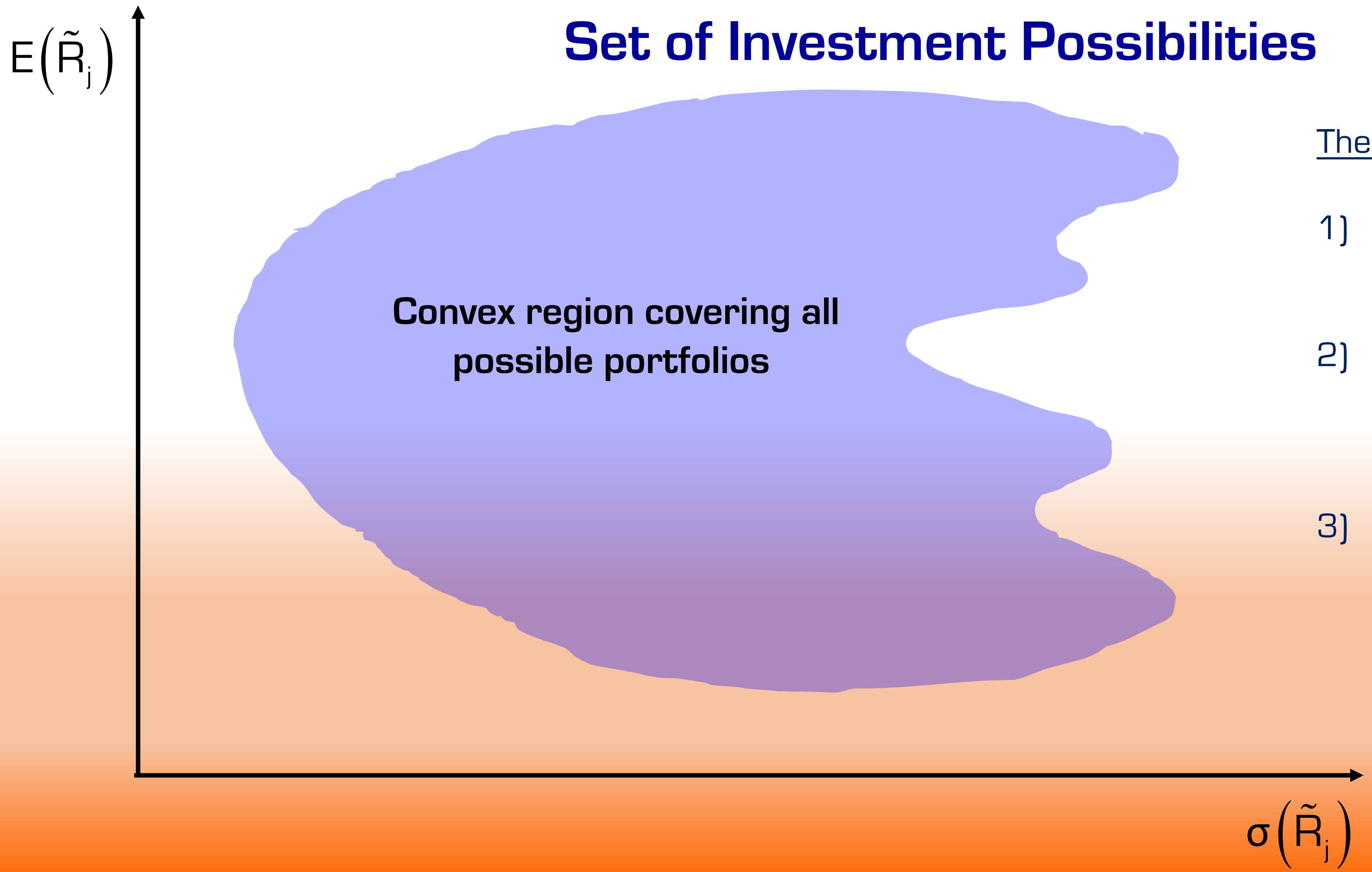
– **PHASE 1: DETERMINING THE SET OF INVESTMENT POSSIBILITIES**

- Analysis of the set of assets that are traded in the market. For all assets it will be necessary to estimate:
  - The expected return (mean ∀ assets).
  - The risk (variance ∀ assets).
  - The covariances between assets (in pairs) that can be formed in the portfolios.

– Graphically represent all possible assets/portfolios with their means and variances to obtain the **Set of Investment Possibilities or Viable Set (point cloud):**

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (II)



The viable set

1) The viable set is infinite but limited.

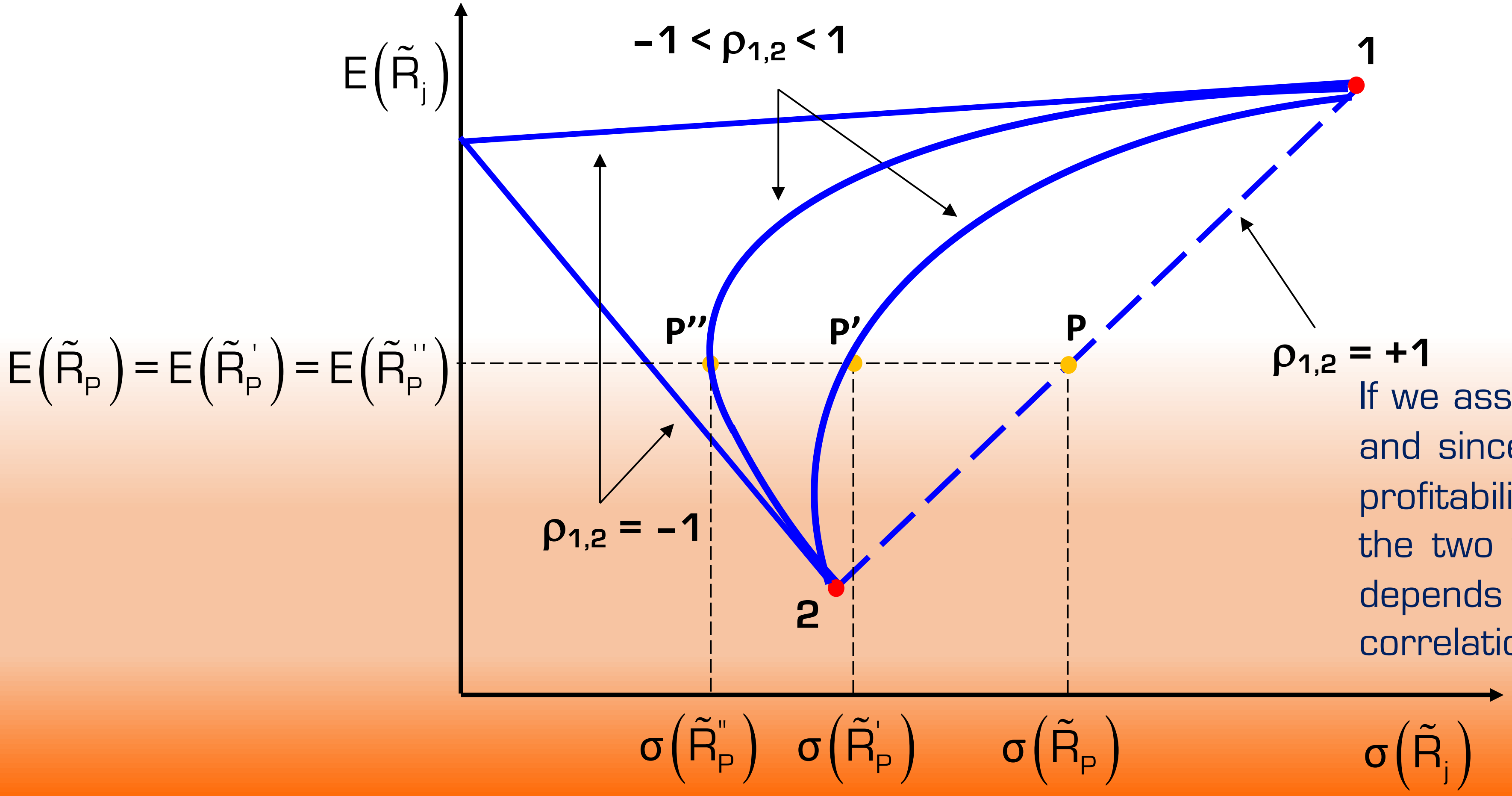
2) It represents all the investment possibilities this market represents (titles and portfolios).

3) There are no investment opportunities outside the viable set.

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (III)

- **Assumption:** there are only two assets and their correlation can range from $-1 < \rho_{1,2} < +1$. How does this affect the shape of the Set of Investment Possibilities?



If we assume there are only two titles, and since in practice $-1 < \rho_{A,B} < 1$, the profitability-risk combinations between the two titles will be on a curve that depends on the value of this correlation coefficient.

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (IV)

– **PHASE 2: DETERMINING THE EFFICIENT FRONTIER**

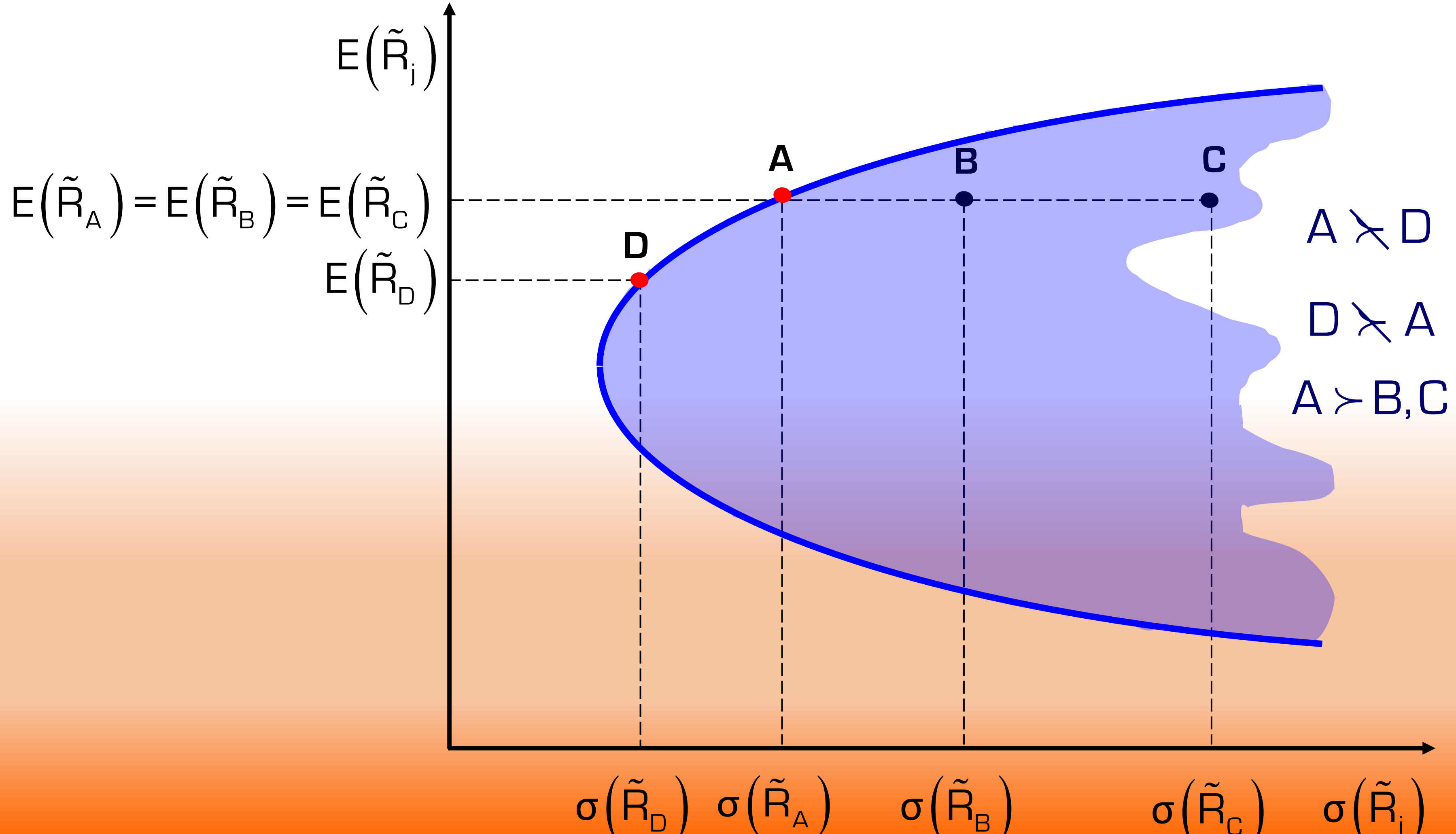
- **Efficient Portfolio**
  - OPTION 1: for an expected level of return, risk is minimized (there is no other portfolio with a lower risk).
  - OPTION 2: for an expected level of risk, profitability is maximized (there is no other portfolio with greater profitability).

  - If these two postulates are not met → **Inefficient Portfolio**

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (V)

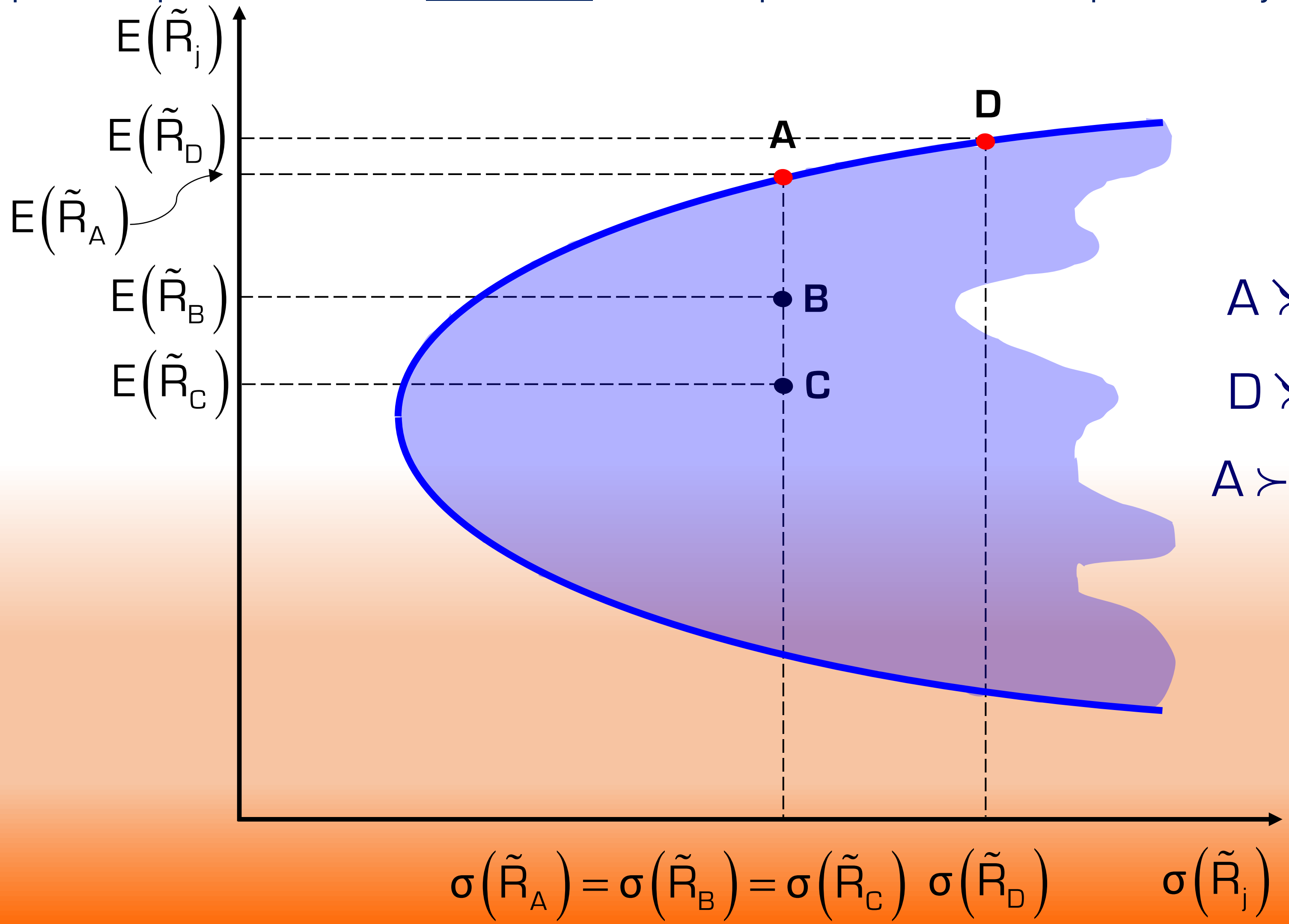
- Graphical representation of OPTION 1: for an expected level of return, the risk is minimized.



2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining Optimal Portfolio (VI)

- Graphical representation of <u>OPTION 2</u>: for an expected level of risk, profitability is maximized.



$A \not\succ D$

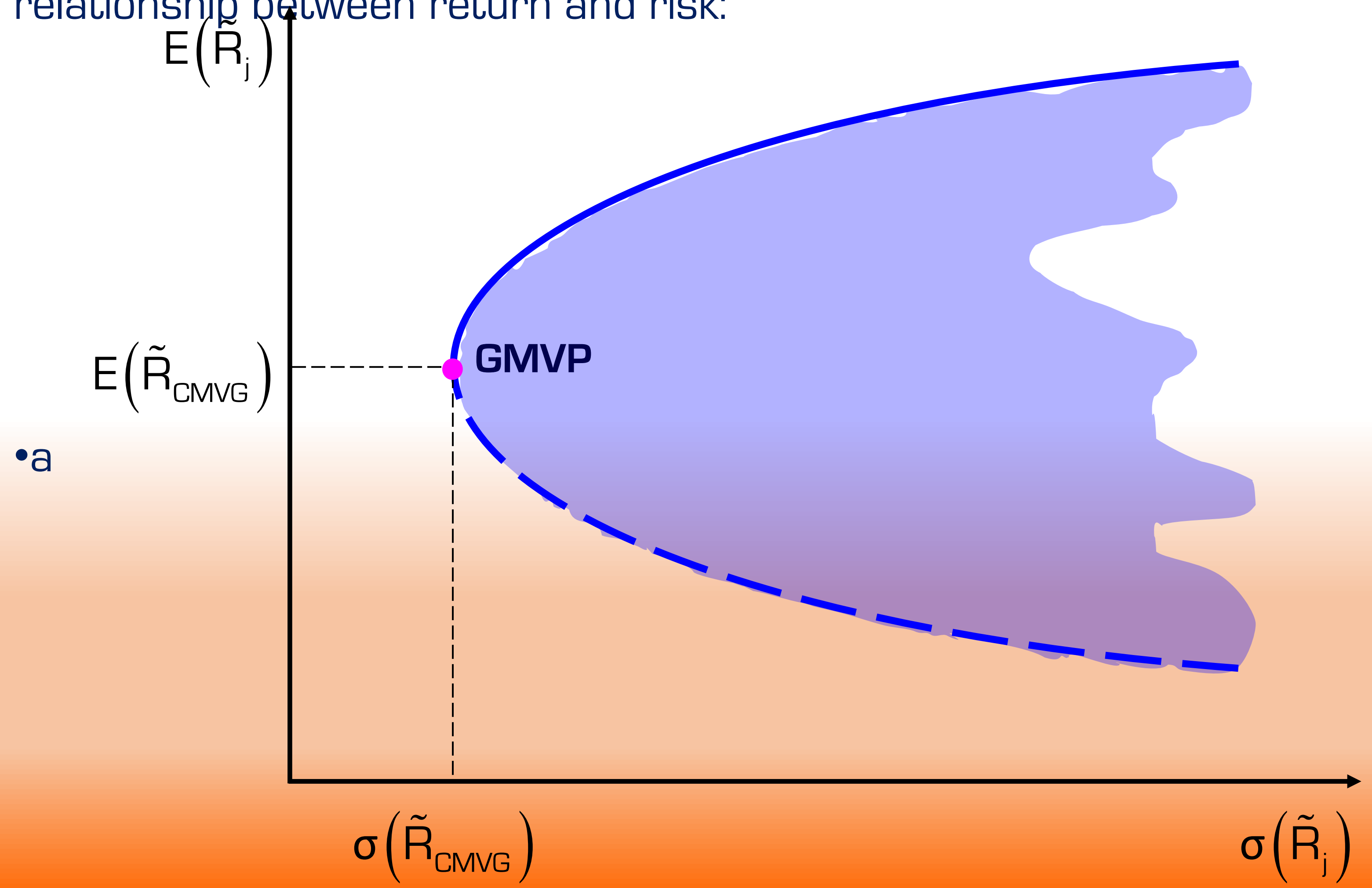
$D \not\succ A$

$A \succ B,C$

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (VII)

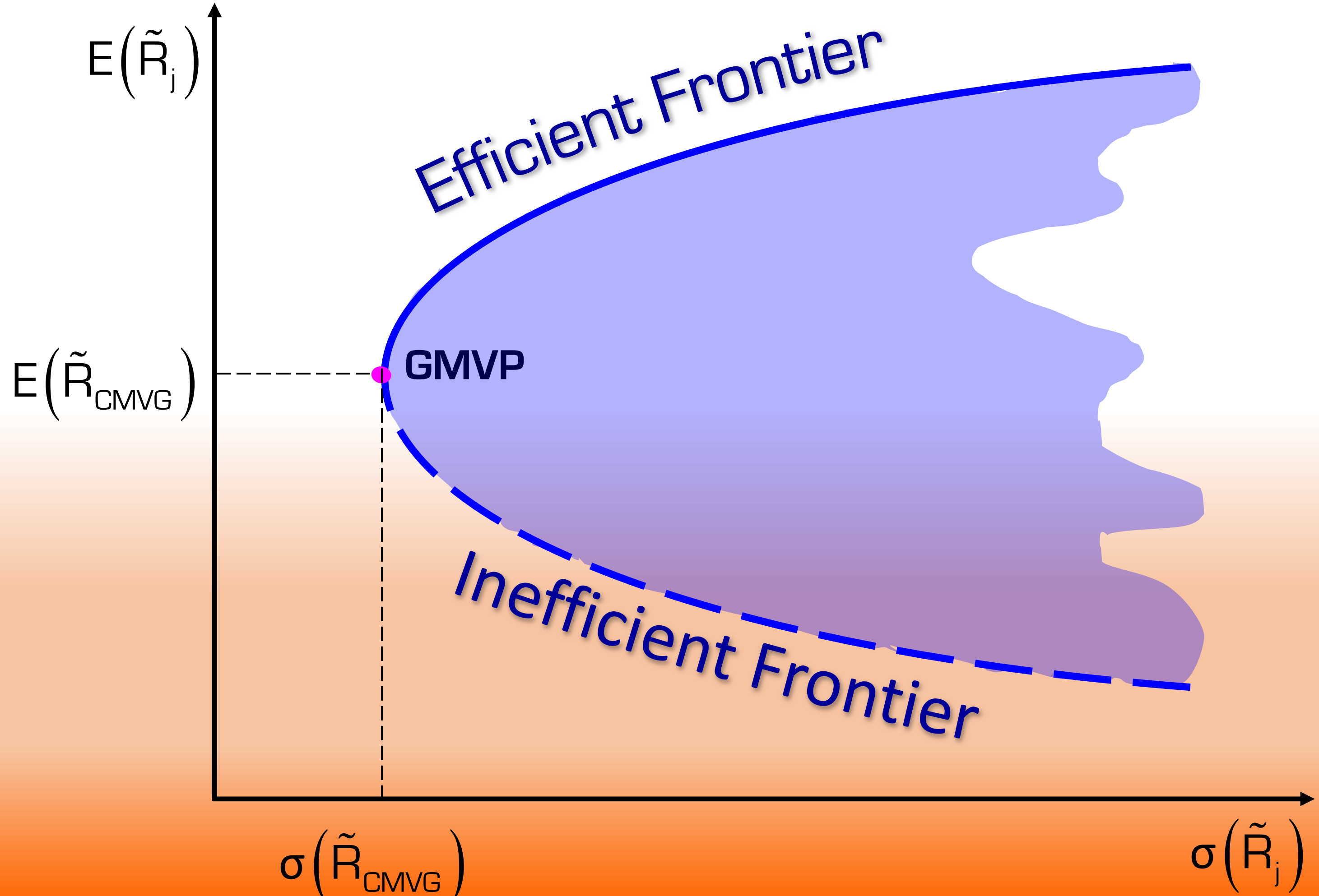
- **Minimum Global Variance Portfolio (MVGP):** since this one is efficient, it offers the lowest relationship between return and risk:



2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (VIII)

- The GMVP determines the EFFICIENT FRONTIER goes from the Minimum Global Variance Portfolio to the right (solid line).



2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (IX)

- Analytical determination of the Efficient Frontier; mathematical optimization through two alternative programs:
  - ❑ Maximize return for the level of risk we wish to assume:

$$\text{Máx. } E\left(\tilde{R}_P\right) = \sum_{i=1}^{N} \omega\, E\left(\tilde{R}_i\right) \qquad (2.4)$$

$$\text{s.a. } \sigma^2\left(\tilde{R}_P\right) = \sum_{i=1}^{N} \omega_i^2 \cdot \sigma^2\left(\tilde{R}_i\right) + \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \cdot \omega_j \cdot \sigma\left(\tilde{R}_i, \tilde{R}_j\right)$$

← The level of risk or variability to be supported is set: $\sigma^2\left(\tilde{R}_P\right)^*$

$$\sum_{i=1}^{N} \omega_i = 1$$

← Budget restrictions exist: the amount must match the initial investment budget.

$$\omega_i \geq 0$$

← Restrictions that indicate the non-negativity of the variables of the problem.

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (X)

- Minimize risk for the desired level of return:

$$\text{Mín. } \sigma^2\left(\tilde{R}_P\right) = \sum_{i=1}^{N} \omega_i^2 \cdot \sigma^2\left(\tilde{R}_i\right) + \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \cdot \omega_j \cdot \sigma\left(\tilde{R}_i, \tilde{R}_j\right) \tag{2.5}$$

$$\text{s.a. } E\left(\tilde{R}_P\right) = \sum_{i=1}^{N} \omega_i E\left(\tilde{R}_i\right)$$

← the level of return to be obtained is set: $E\left(\tilde{R}_P\right)^*$

$$\sum_{i=1}^{N} \omega_i = 1$$

← Budget restrictions exist: the amount must match the initial investment budget.

$$\omega_i \geq 0$$

← Restrictions that indicate the non-negativity of the variables of the problem.

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (XI)

- **Example 2.1** (extracted from Berk and DeMarzo, 2014): below are the expected returns and volatilities of the following portfolio combinations made up of Intel and Coca-Cola assets.

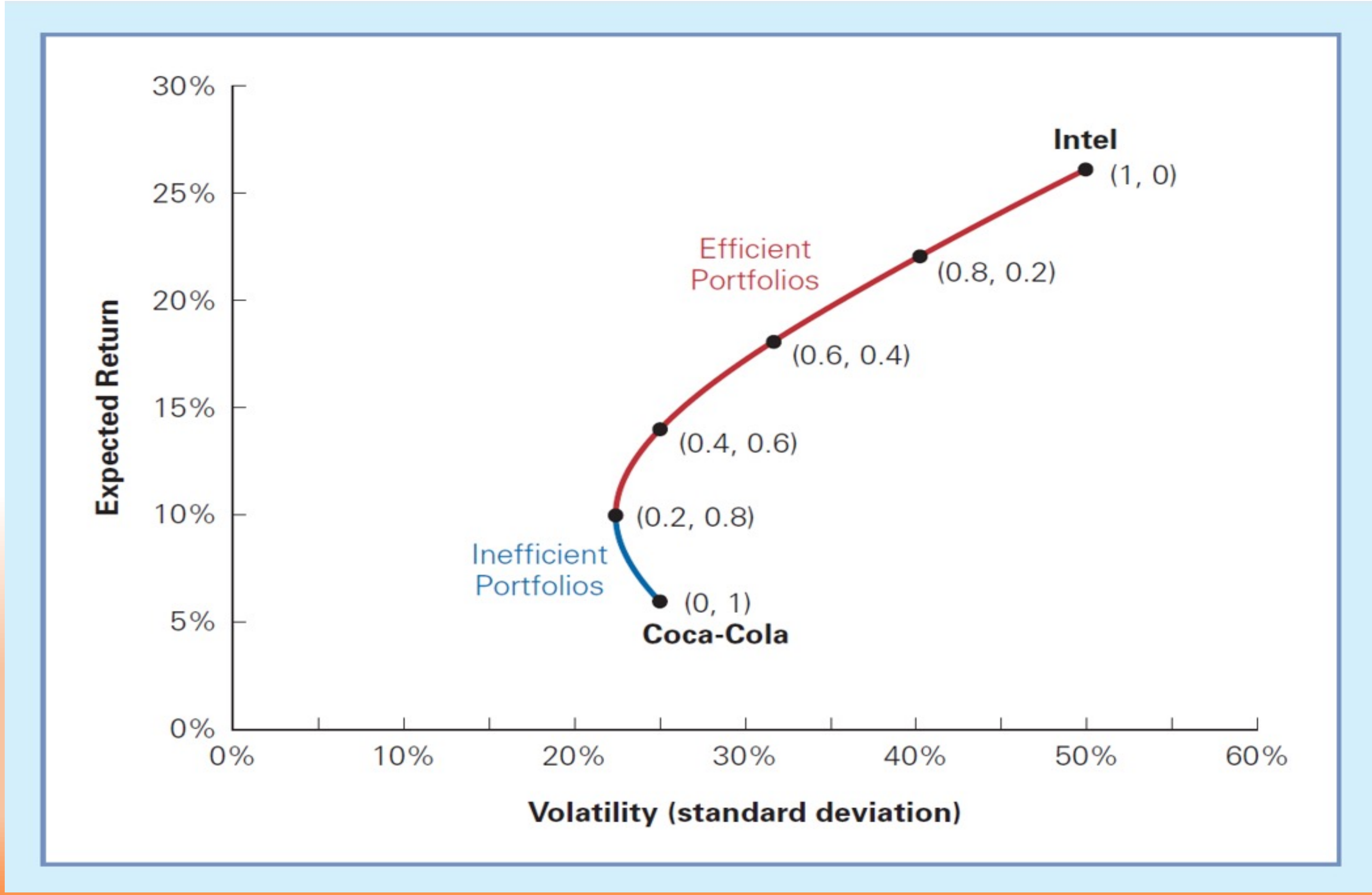
Expected Returns and Volatility for Different Portfolios of Two Stocks

Portfolio Weights		Expected Return (%)	Volatility (%)
x_I	x_C	$E[R_P]$	$SD[R_P]$
1.00	0.00	26.0	50.0
0.80	0.20	22.0	40.3
0.60	0.40	18.0	31.6
0.40	0.60	14.0	25.0
0.20	0.80	10.0	22.4
0.00	1.00	6.0	25.0

- Plot the combinations shown in the table.

2.3 Efficient portfolios

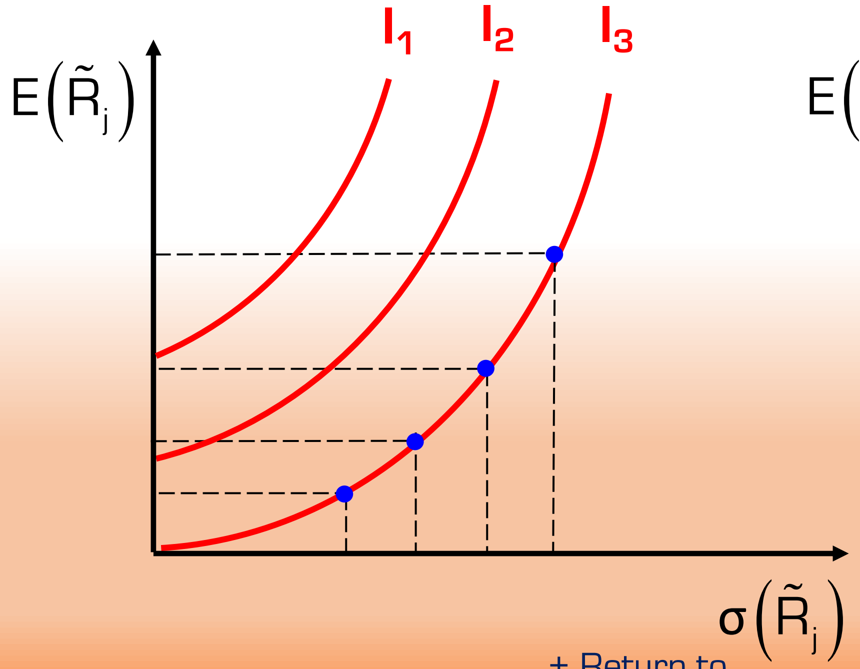
2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (XII)



2.3 Efficient portfolios

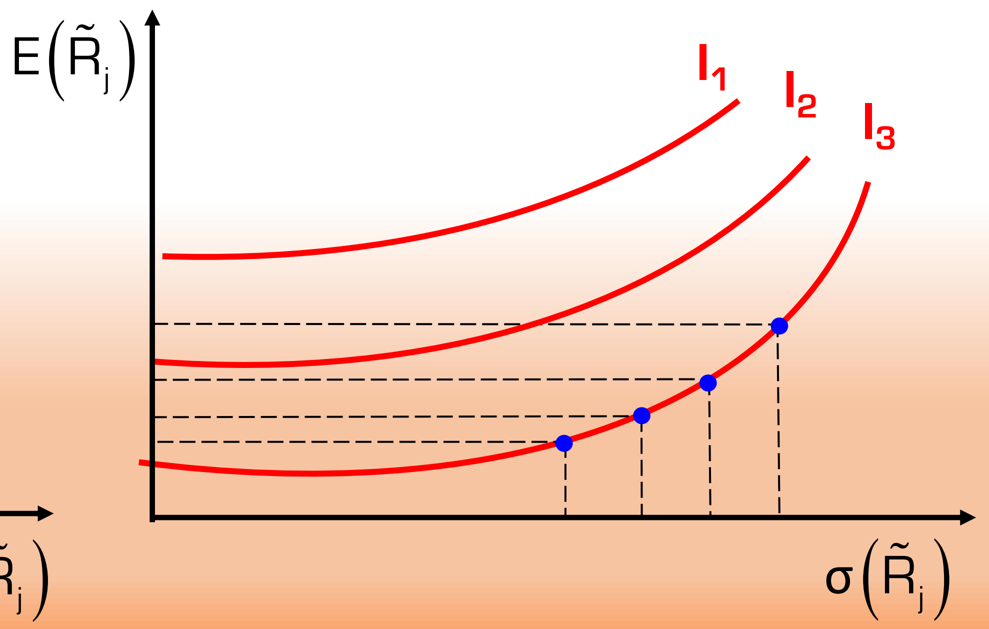
2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (XII)

– **PHASE 3: SPECIFY INVESTOR PREFERENCES USING INDIFFERENCE CURVES**
  - The indifference curves are increasing and convex.
  - Their shape will depend on the investor's level of risk aversion.



Very risk averse investor

+ Return to compensate for risk



Investor not averse to risk

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (XIII)

– **PHASE 4: DETERMINING THE OPTIMAL PORTFOLIO**

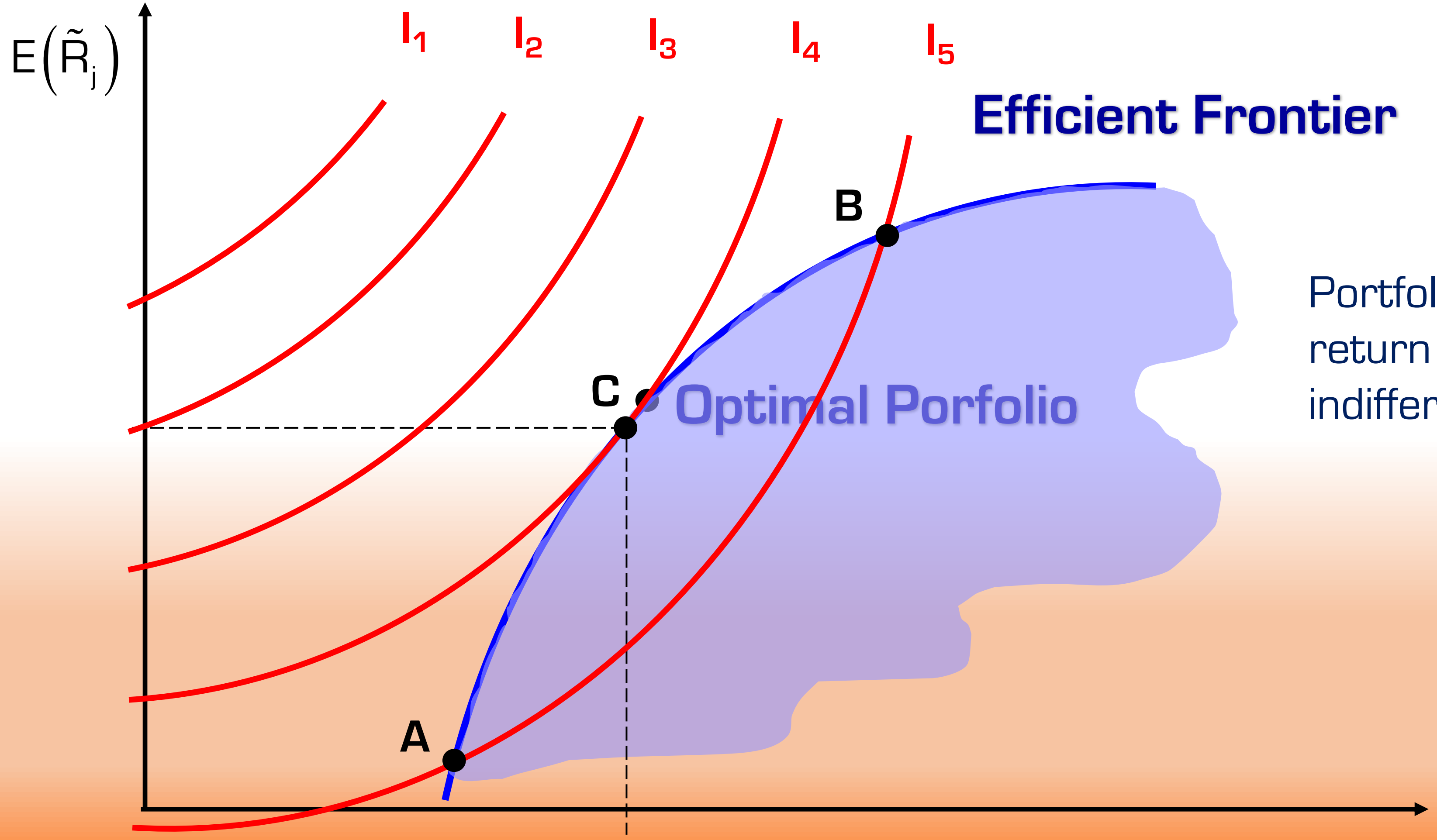
- Investors will choose the efficient portfolio that best suits their preferences (maximizes their expected utility).

- By superimposing the efficient frontier onto the investor's indifference curves, we can identify the **Optimal Portfolio**.

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (XIV)

- Graphically:

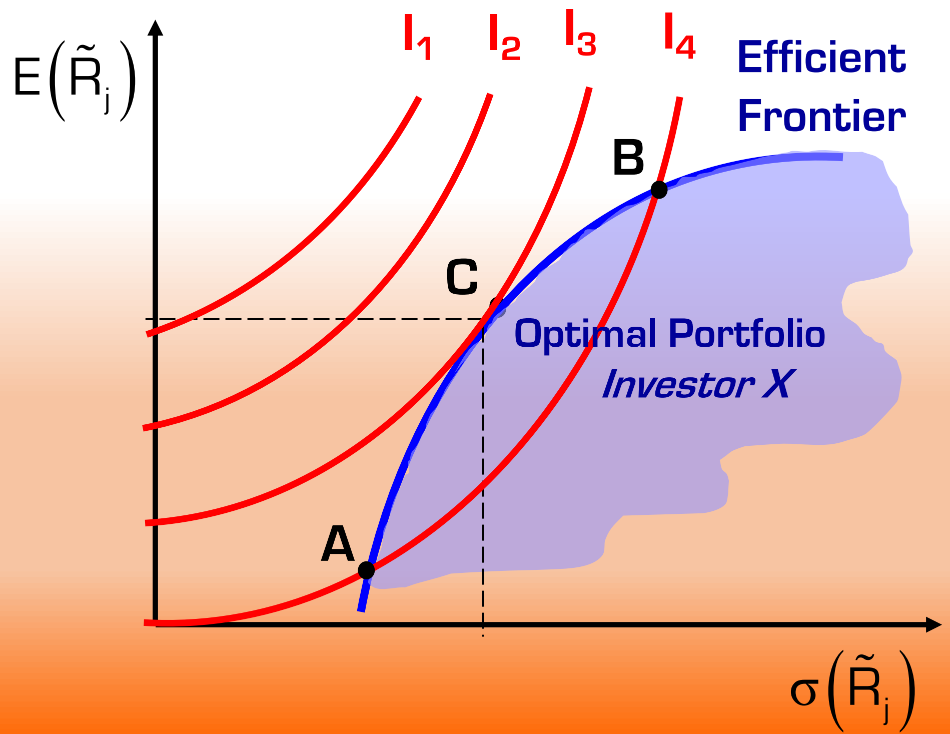


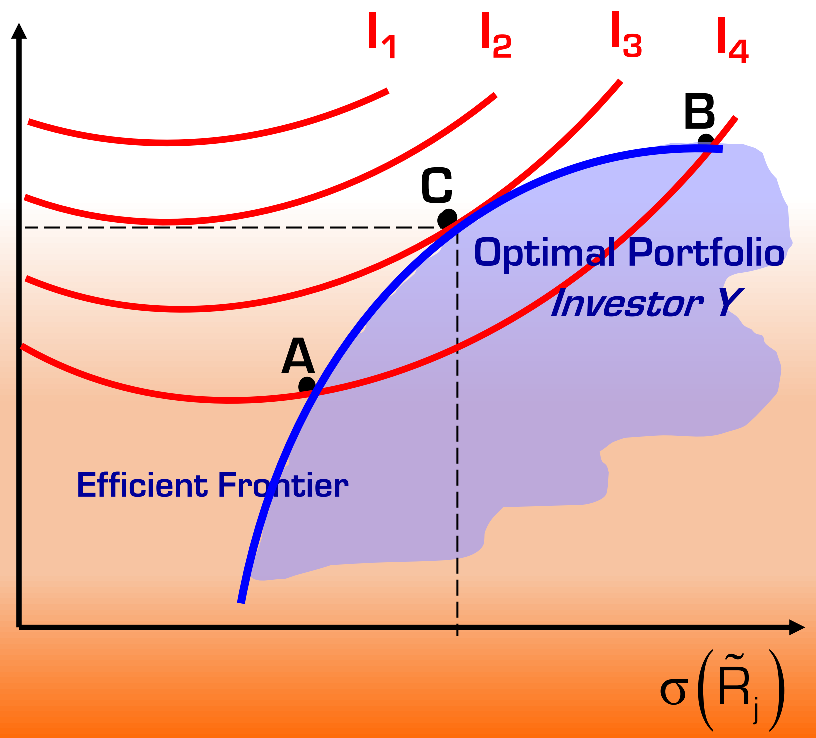
Portfolio C provides the greatest possible return since it is on the highest in the indifference curve.

2.3 Efficient portfolios

2.3.1. Markowitz Model: Obtaining the Optimal Portfolio (XV)

- **The Optimal Portfolio:**
  - is guaranteed given the convexity of the indifference curves and the concavity of the efficient frontier.
  - is unique and different for each investor and depends on their level of risk aversion.





2.3 Efficient portfolios

2.3.2. Tobin Model: Introduction of the risk-free asset (I)

- The **TOBIN Model** introduces a risk-free asset F into the Markowitz Model so that:
  - one can lend and borrow the desired amount of money.
  - the return of the ALR is true: $E(R_F)=R_F$.
  - the risk is zero : $\sigma(R_F)=0$.
  - its covariance with the other assets in the portfolio is null: $\sigma(R_F, R_i)=0$.

- The **Mixed Portfolio P** involves:
  - investing in risky assets → this is a Markowitz E Efficient Portfolio to which some of the budget is allocated, ω.
  - investing in the risk-free asset → the rest of the available budget is allocated, $(1-\omega)$.

2.3 Efficient portfolios

2.3.2. Tobin Model: Introduction of the risk-free asset (II)

– The formal ex-post expression of the Mixed Portfolio P will be:

$$\tilde{R}_P = \omega \tilde{R}_E + (1-\omega) R_F \quad (2.6)$$

– The **Expected Return** (ex-ante) of the Mixed Portfolio P will be :

Return *ex–ante*

$$E(\tilde{R}_P) = \omega E(\tilde{R}_E) + (1-\omega) R_F \quad (2.7)$$

- The Risk of the Mixed Portfolio P will be:

$$\sigma(\tilde{R}_P) = \sigma_P^2 = \omega^2 \sigma_E^2 \quad (2.8)$$

Risk

$$\sigma_P = \omega \sigma_E \quad (2.9)$$

2.3 Efficient portfolios

2.3.2. Tobin Model: Introduction of the risk-free asset (III)

– Solving for ω and substituting in Eq. (2.7), we obtain:

$$\omega = \frac{\sigma_P}{\sigma_E} \quad (2.10)$$

$$E(\tilde{R}_P) = \frac{\sigma_P}{\sigma_E} E(\tilde{R}_E) + \left(1 - \frac{\sigma_P}{\sigma_E}\right) R_F \quad (2.11)$$

– Regrouping the terms, finally we obtain the expression of a linear relationship:

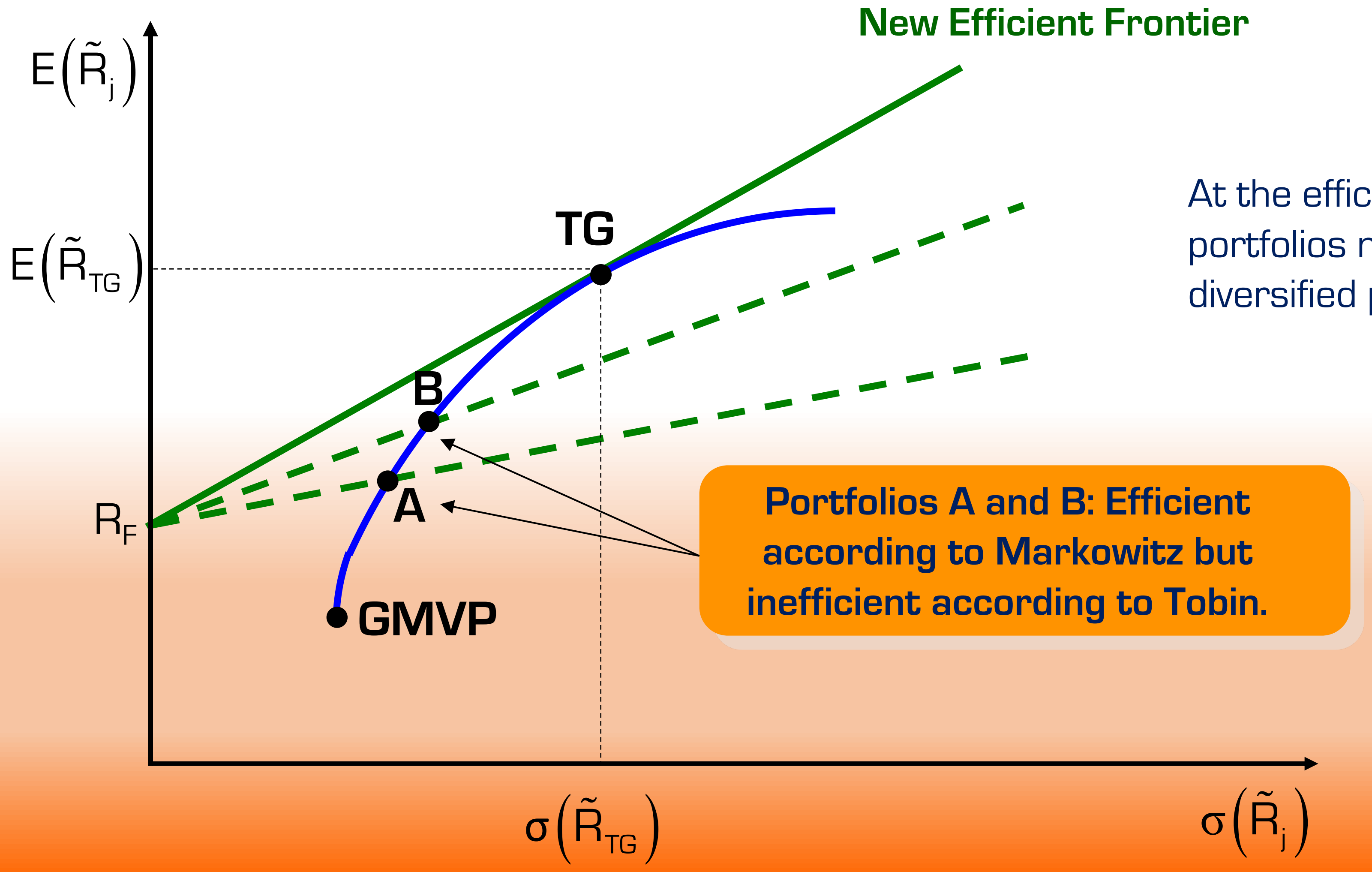
Tobin Model

$$E(\tilde{R}_P) = R_F + \left(\frac{E(\tilde{R}_E) - R_F}{\sigma_E}\right)\sigma_P \quad (2.12)$$

2.3 Efficient portfolios

2.3.2. Tobin Model: Introduction of the risk-free asset (IV)

– Representing expression (2.12) graphically, we obtain:



At the efficient frontier are efficient portfolios made up of RF and TG, which are diversified portfolios.

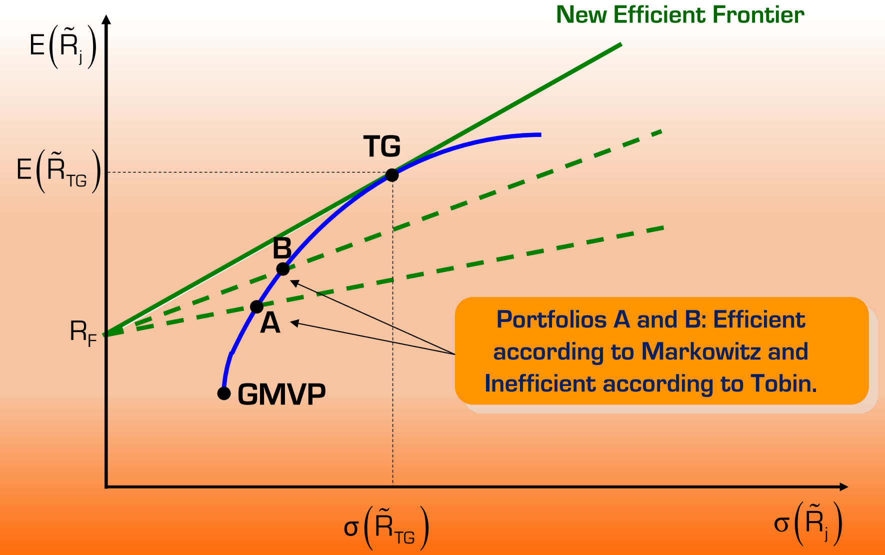
2.3 Efficient portfolios

2.3.2. Tobin Model: Introduction of the risk-free asset (IV)

– A rational and risk-averse investor will invest in a mixed portfolio located on the R_F line

–TG has become the **New Efficient Frontier**; therefore:

- Portfolios located on the R_F –TG line provide the maximum expected return for each level of risk.
- Portfolios located on the R_F –TG line provide the minimum risk for a certain level of return.
- The combinations between R_F and the TG portfolio (which we call the tangent portfolio) render the portfolios located on the previous efficient frontier inefficient and thus become the New Efficient Frontier.



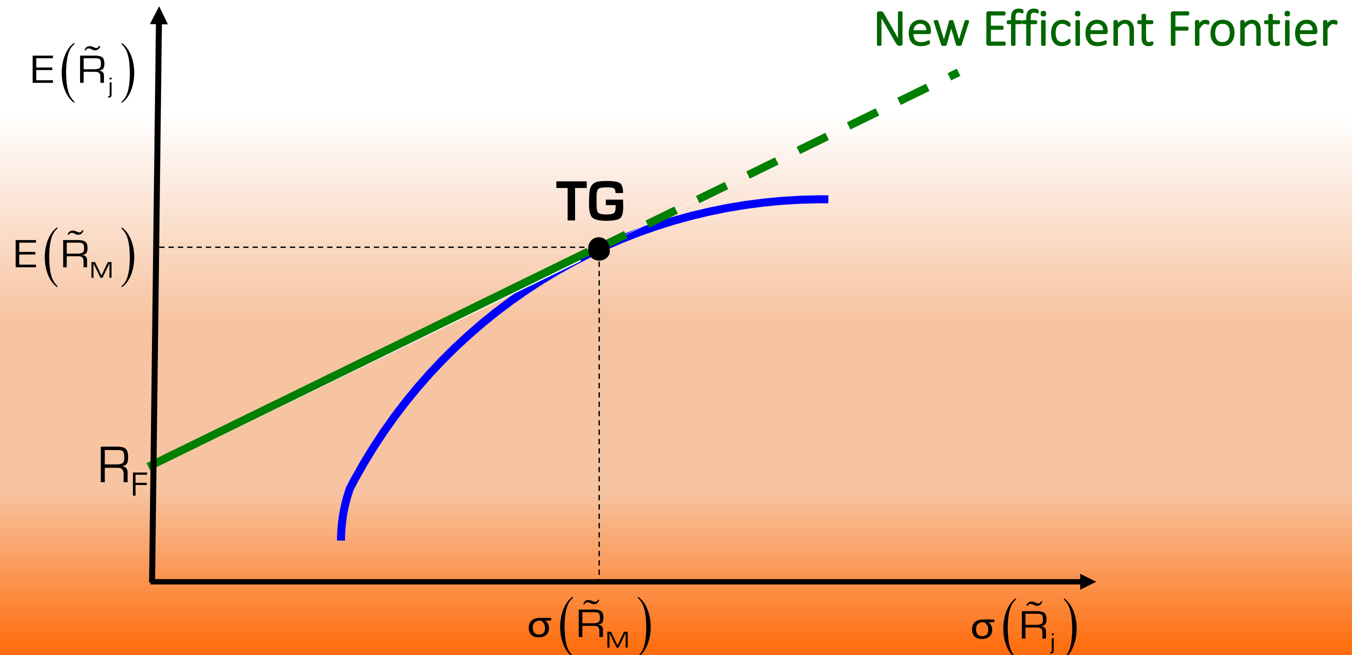
2.3 Efficient portfolios

2.3.2. Tobin Model: Introduction of the risk-free asset (V)

In **Efficient Portfolios** the investor can:

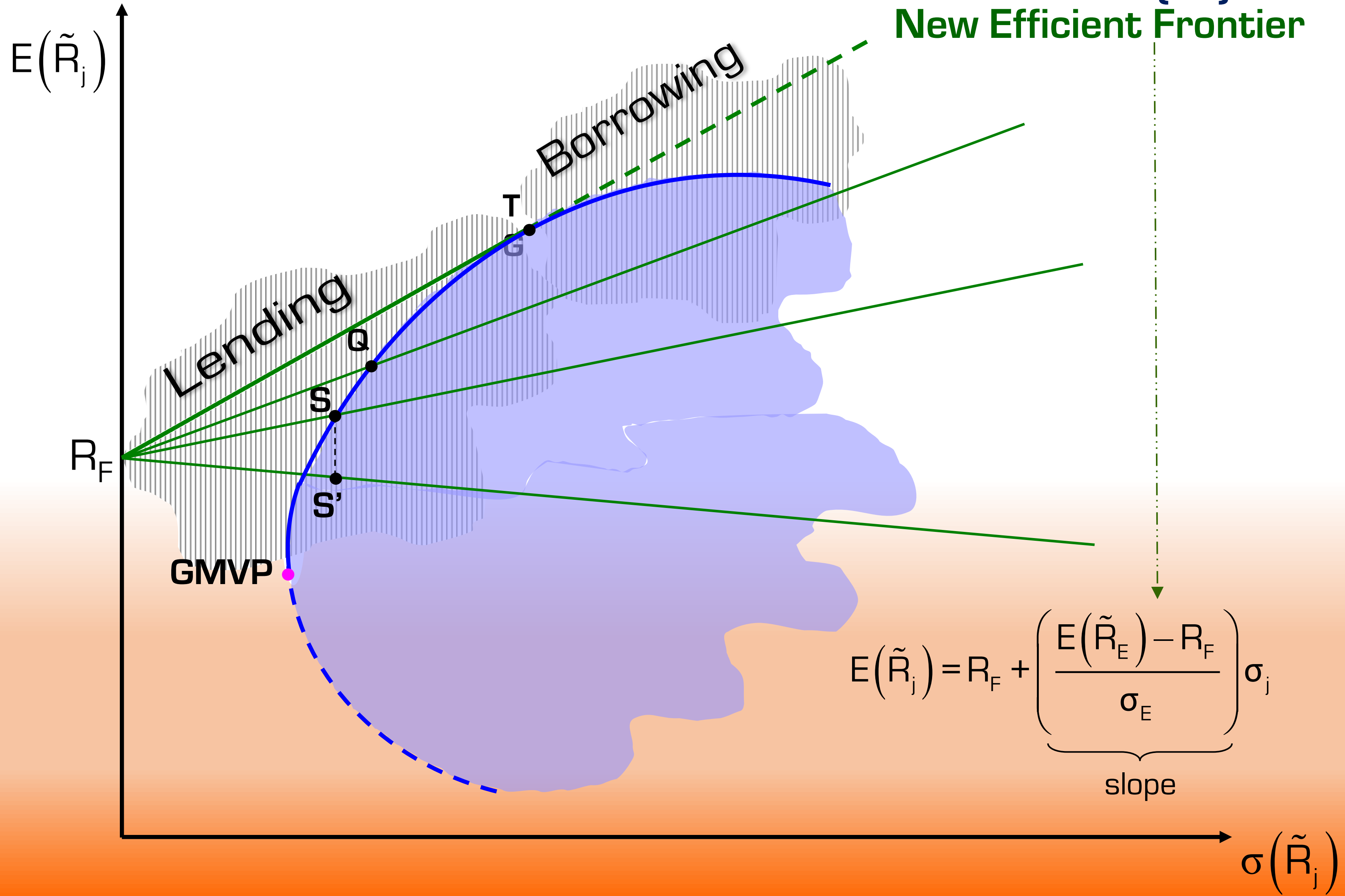
– allocate their entire budget to risk-free asset R_F.

- allocate their entire budget to risky assets: TG Portfolio.
- allocate part of the budget to be lent: section R_F–TG (continuous line).
- invest in TG a higher amount than they have. They must borrow at the R_F rate: section from TG (discontinuous line).



2.3 Efficient portfolios

2.3.2. Tobin Model: Introduction of the risk-free asset (VI)



$$E(\tilde{R}_j) = R_F + \underbrace{\left(\frac{E(\tilde{R}_E) - R_F}{\sigma_E}\right)}_{\text{slope}} \sigma_j$$

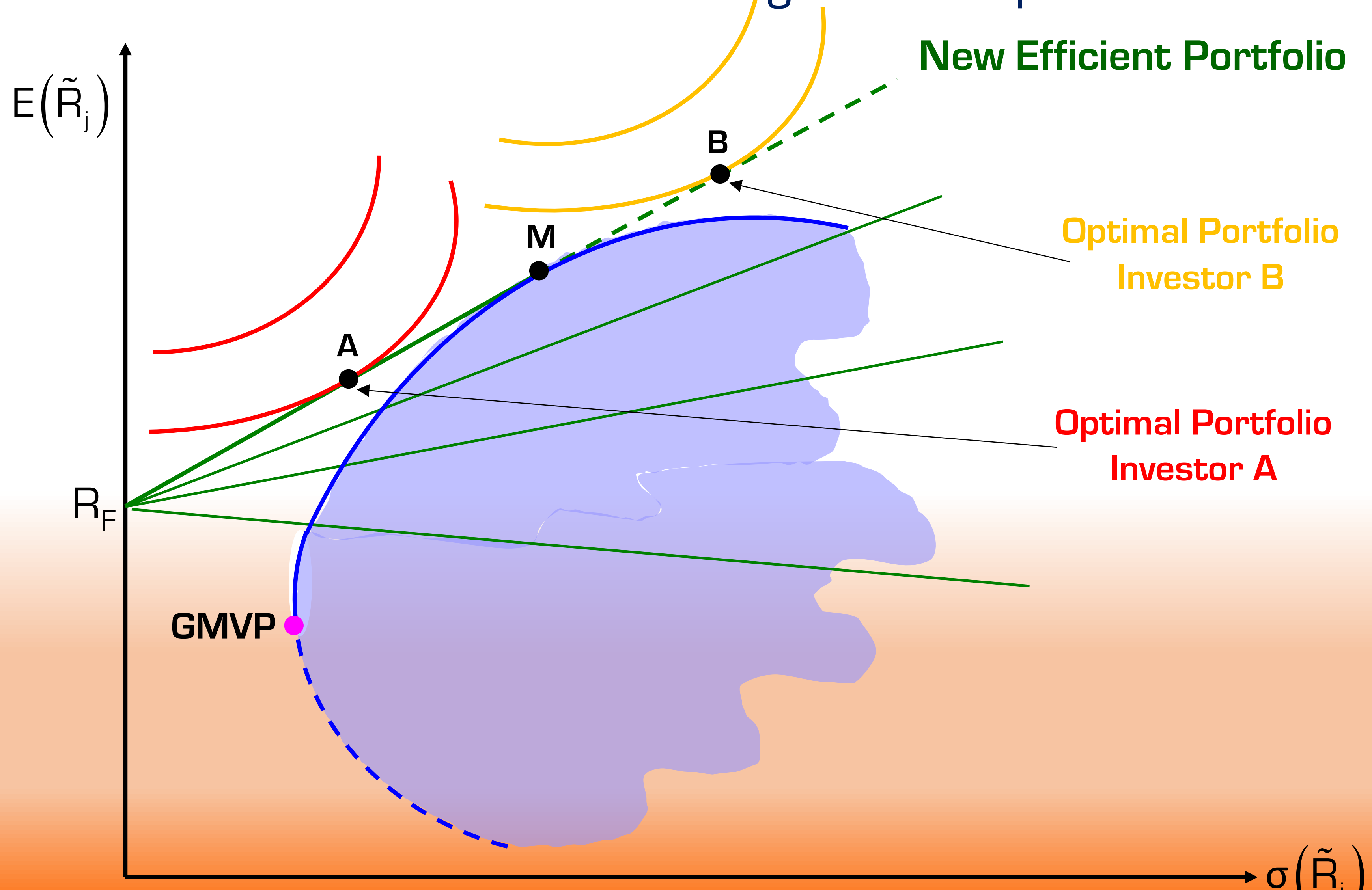
2.3 Efficient portfolios

2.3.2. Tobin Model: Obtaining a New Optimal Portfolio (I)

- For investors to find their **OPTIMAL PORTFOLIO**, they need to know:
  - The New Efficient Frontier.
  - The map of indifference curves.
- By superimposing both of these elements, investors find their Optimal Portfolio, which is also different for each investor → since it responds to the profitability-risk ratio appropriate to their risk aversion.

2.3 Efficient portfolios

2.3.2. Tobin Model: Obtaining a New Optimal Portfolio (II)



2.4 Capital allocation and the separation property

– **Separation principle**:

- Efficient investors will need to divide their budget between two funds or investment alternatives :
  - Investing in the Tangent Portfolio (TG) representative of all risky securities
  - Investing in the risk-free asset (F).

- Depending on their aversion to risk, investors will choose a certain portfolio and position themselves along the line.

Unit 2. The Markowitz Portfolio Selection Model

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