Quantum Field Theory under external conditions

Gravitation and electrodynamics

By

Silvia Pla García

Vniversitat de València

Departament de Física Teòrica
Universitat de València

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Advisor:
Prof. José Navarro-Salas

Abstract

The semiclassical approximation in Quantum Field Theory is commonly used to study the propagation of quantized fields in classical backgrounds and accounts for fascinating non-perturbative quantum phenomena such as the spontaneous creation of particles induced by (non-trivial) external configurations. In this context, the computation of vacuum expectation values of physical observables becomes a complex issue, and advanced renormalization techniques are required to tame the new ultraviolet divergences caused by the external backgrounds. In curved spacetimes, all these interesting features are commonly studied within the Quantum Field Theory in Curved Spacetime framework, initiated in the early 60s and developed until nowadays. In semiclassical electrodynamics, different analytical methods, usually expressed in the modern language of Quantum Electrodynamics (QED), are employed to account for relevant non-perturbative quantum effects.

This work aims to explore the interconnections between these two approaches, describing the underlying physics behind the semiclassical theory in a unified way. We analyze the intertwining relation between particle creation and quantum anomalies and study the backreaction problem in two-dimensional electrodynamics, investigating the range of validity of the semiclassical approach in a self-consistent way. We also work with different asymptotic expansions for the heat-kernel and the effective action in curved spacetimes and QED, focusing on its non-linear behavior. For the one-loop QED effective action, we find a new, resumed, asymptotic expansion that encapsulates in a non-perturbative factor all terms containing the field-strength invariants.

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Author's Declaration

I declare that the work presented in this thesis was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Program.

The contents of chapters two, three, four, five and six are based on the following papers. Credit copyright American Physical Society.

1. *Role of gravity in the pair creation induced by electric fields*; A. Ferreiro, J. Navarro-Salas and S. Pla, Phys. Rev. D **98** 045015 (2018).
2. *Translational anomaly of chiral fermions in two dimensions*; P. Beltrán-Palau, J. Navarro-Salas, and S.Pla, Phys. Rev. D **99** 105008 (2019).
3. *Breaking of adiabatic invariance in the creation of particles by electromagnetic fields*; P. Beltrán-Palau, A. Ferreiro, J. Navarro-Salas, and S.Pla, Phys.Rev.D **100** 085014 (2019).
4. *Adiabatic regularization for Dirac fields in time-varying electric backgrounds*; P. Beltrán-Palau, J. Navarro-Salas, and S.Pla, Phys. Rev. D **101** 105014 (2020).
5. *R-summed form of adiabatic expansions in curved spacetime*; A. Ferreiro, J. Navarro-Salas, and S. Pla, Phys. Rev. D **101** 105011 (2020).
6. *Pair production due to an electric field in 1+1 dimensions and the validity of the semiclassical approximation*; S. Pla, I. M. Newsome,

R. S. Link, P. R. Anderson, and J. Navarro-Salas, Phys.Rev.D **103** 105003 (2021).

7. $(\mathcal{F}, \mathcal{G})$*-summed form of the QED effective action*; J. Navarro-Salas and S. Pla, Phys. Rev. D **103** L081702 (2021).

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Valencia, 27 April 2022
Silvia Pla García.

Resumen en castellano

La fusión entre nuestra visión actual del microcosmos – La Teoría Cuántica de Campos – y la del macrocosmos – la relatividad general y su interpretación de la gravedad en términos de un espacio-tiempo curvo – sigue siendo uno de los problemas más incómodos en física fundamental. No obstante, es innegable que la física teórica ha conseguido enormes éxitos durante las últimas décadas. Por un lado, el descubrimiento del Bosón de Higgs [1] ha supuesto la culminación del Modelo Estándar de la física de partículas; Por otro, la reciente detección de ondas gravitatorias [2, 3] ha evidenciado la fortaleza de la teoría de la relatividad general de Einstein, más de 100 años después de su formulación. Además, partiendo de distintos problemas abiertos y con la esperanza de tener nuevos datos procedentes de la nueva generación de experimentos y observaciones, se han logrado construir modificaciones y extensiones muy interesantes de estas dos formas complementarias de entender la naturaleza [4, 5].

Aunque aún no tenemos una teoría cuántica definitiva de la gravedad, sí que es posible construir una descripción semiclásica autoconsistente de la misma, donde estudiar la propagación de campos cuánticos en espaciotiempos curvos y explorar así nuevos efectos (cuánticos) de la gravedad. El lector puede consultar, por ejemplo, las referencias [6, 7, 8], donde encontrará una excelente descripción de este campo de estudio. La aproximación semiclásica se puede entender como una versión efectiva de una teoría totalmente cuántica de la gravedad, con un rango de validez limitado.

Uno de los beneficios de esta propuesta es que incluye de forma natural efectos no perturbativos fascinantes, como la creación de partículas inducida por campos gravitatorios extremos. Por ejemplo, la presencia de un campo gravitatorio dependiente del tiempo, como el que describe un universo en expansión, permite la creación espontánea de partículas a partir del vacío [9, 10, 11, 12]. Este mecanismo puede estar detrás de las anisotropías observadas en el fondo cósmico de microondas (CBM) y puede ser crucial para explicar la creación explosiva de partículas elementales en la época del recalentamiento, justo después del big bang [13, 14]. En un colapso gravitatorio que culmina en un agujero negro, este proceso también se activa, generando una radiación térmica constante, conocida como radiación de Hawking [15, 16, 17].

En electrodinámica cuántica (QED), la aproximación semiclásica también resulta extremadamente útil para entender este mecanismo. En 1951, Schwinger calculó la parte imaginaria de la acción efectiva a un bucle para un campo eléctrico constante y homogéneo [18], con el fin de evaluar la amplitud de persistencia del vacío a partir de la fórmula $|\langle out|in\rangle|^2 = \exp(-2\mathrm{Im}\Gamma(A_\mu))$. En d dimensiones espacio-temporales, la parte imaginaria resulta ser [19]

$$\frac{2\,\mathrm{Im}\,\Gamma(A_\mu)}{VT} = \frac{2}{(2\pi)^d}\sum_{n=1}^{\infty}\left(\frac{qE}{n}\right)^{(d+1)/2}\exp\left(\frac{-n\pi m^2}{qE}\right)\;, \tag{1}$$

donde VT representa el volumen del espacio-tiempo. Cabe mencionar que la probabilidad de persistencia del vacío es exactamente uno menos la probabilidad de crear pares. El factor exponencial en la ecuación anterior muestra la naturaleza no perturbativa de este fenómeno,[1] que podría ser detectado en los laboratorios en un futuro no muy lejano [21, 22].

Una segunda ventaja de la aproximación semiclásica es que predice uno de los resultados fundamentales de la teoría cuántica de campos: la existencia de anomalías cuánticas. En este contexto, el cálculo de valores esperados de vacío de distintos observables físicos es una tarea complicada, y se necesitan técnicas avanzadas de renormalización para eliminar las nuevas divergencias ultravioleta causadas por los campos externos. Como contrapartida, este proceso genera resultados finitos y no ambiguos, conocidos como anomalías cuánticas, que indican la ruptura cuántica de una simetría clásica. La anomalías axiales son particularmente relevantes, puesto que se pueden relacionar directamente con el proceso de creación de partículas [23, 24]. Por ejemplo, la acción clásica de un campo de Dirac sin masa ψ es invariante bajo transformaciones quirales. Esto implica, por el teorema de Noether, que la corriente axial $J^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$ debe ser conservada. La teoría cuántica rompe esta ley de conservación, produciendo el siguiente resultado no nulo [25, 26]

$$\partial_\mu \left\langle J^\mu_A \right\rangle = -\frac{q^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,. \tag{2}$$

En términos de creación de partículas, podemos decir que una cantidad mínima de partículas debe ser creada para preservar esta no conservación de la quiralidad.

La aproximación semiclásica también ofrece un marco coherente en el que se puede estudiar la influencia de distintos efectos cuánticos sobre un

[1] Para una revisión histórica consultar [20].

campo clásico mediante las ecuaciones semiclásicas

$$\partial_\mu F^{\mu\nu} = J^\nu_{\rm class} + \langle J^\nu \rangle_{\rm ren} \, , \tag{3}$$

$$G_{\mu\nu} = 8\pi G \left(T^{\rm class}_{\mu\nu} + \langle T_{\mu\nu} \rangle_{\rm ren} \right) \, . \tag{4}$$

Este análisis resulta determinante para obtener, por ejemplo, geometrías corregidasinducidas por efectos de polarización del vacío [27], o para explorar el efecto que producen las partículas creadas por un campo dependiente del tiempo (p.ej., un campo eléctrico) sobre el mismo campo [28].

Esta tesis pretende investigar todos estos fenómenos cuánticos desde una perspectiva renovada, explorando las interconexiones entre la gravedad y electrodinámica semiclásicas. El contenido de la tesis se divide en tres partes y se puede resumir como sigue.

Teoria Cuántica de Campos en *backgrounds* dependientes del tiempo

La primera parte de la tesis corresponde a los Capítulos 2, 3 y 4 y pretende estudiar algunas de las características más importantes de la teoría cuántica de campos en presencia de campos externos. Nos centramos en configuraciones homogéneas con dependencia temporal arbitraria.

En el Capítulo 2 revisamos algunos conceptos básicos de la teoría cuántica de campos en espacios curvos. Este capítulo debe ser entendido como una introducción que permitirá al lector familiarizarse con la notación y con las ideas más importantes de la tesis. Consideramos la generalización más simple de la relatividad especial: un campo escalar cuantizado propagándose en un espaciotiempo homogéneo y dependiente del tiempo en cuatro dimensiones (3 espaciales + 1 temporal), caracterizado por el intervalo $ds^2 = dt^2 - a(t)^2 d\vec{x}^2$. Ilustramos dos problemas fundamentales que deben ser tenidos en cuenta i) la ausencia de un vacío privilegiado y sus consecuencias, como por ejemplo, la creación espontánea de partículas a partir del vacío [9, 10], y ii) la necesidad de usar técnicas de renormalización avanzada, que deben ser aplicadas de forma sistemática para lidiar con las nuevas divergencias ultravioleta que aparecen en este contexto. Introducimos el método de regularización adiabática, el más eficiente para esta clase de configuraciones externas [29, 30, 31, 32].

En el Capítulo 3 estudiamos cómo cambia el marco anterior cuando incluimos un segundo campo dependiente del tiempo. En este caso, un campo electromagnético. Por conveniencia, en lugar de considerar un campo escalar en cuatro dimensiones, examinamos un campo de Dirac propagándose en dos dimensiones espaciotemporales. Este ejemplo ha sido previamente considerado en la referencia [33]. Nuestra contribución consiste en ir un paso más allá, profundizando en las diferencias y similitudes entre un universo en expansión, caracterizado por el factor de escala $a(t)$ y un campo eléctrico dependiente del tiempo, representado por el potencial vector $A(t)$. Nos centramos en la extensión del método de regularización adiabática mostrando que debe existir una jerarquía entre los dos campos externos para que dicho método sea consistente. El factor de escala debe ser una función de orden adiabático cero, mientras que el potencial vector debe ser considerado de orden uno. Justificamos esta propuesta desde tres perspectivas distintas: conservación de la energía [34, 35], equivalencia con otros métodos de renormalización [36] y reproducción de las anomalías cuánticas esperadas.

Finalmente, en el Capítulo 4, y usando las herramientas y los métodos introducidos en el capítulo anterior, estudiamos algunas de las predicciones más importantes de la teoría semiclásica en electrodinámica, centrando nuestro análisis en campos externos dependientes del tiempo. Como hemos indicado anteriormente, una de las ventajas de la aproximación semiclásica es que proporciona una descripción sencilla y directa del fenómeno de creación de partículas. En el lenguaje canónico, decimos que una solución a la ecuación de Dirac que inicialmente es de frecuencia positiva, evoluciona en una superposición de soluciones de frecuencias positivas y negativas para $t \to \infty$. Decimos entonces que se han creado partículas a partir del vacío [9]. En el lenguaje moderno de QED, la existencia de una parte imaginaria no nula en la acción efectiva a un bucle indica que el vacío es inestable, y por tanto permite la posibilidad de que se creen partículas [37, 18]. Como pasa con este fenómeno, el marco semiclásico también proporciona una forma directa de entender las anomalías cuánticas. En este capítulo, nos centramos en la anomalía quiral bidimensional que existe para campos de Dirac sin masa [38],

$$\partial_\mu \left\langle J^\mu_A \right\rangle_{\rm ren} = -\frac{q}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}\,, \tag{5}$$

y estudiamos sus consecuencias fenomenológicas en términos de creación de partículas. Descubrimos que la invariancia adiabática esperada para el número de partículas se rompe en los casos en los que surge la anomalía quiral. Cuando estudiamos campos cuánticos propagándose en un universo en expansión, se puede demostrar que la densidad de partículas creadas tiene una propiedad muy interesante: es un invariante adiabático. Esto significa que cuando la expansión del universo es extremadamente lenta, no se crean partículas. En este capítulo mostramos que, aunque esta afirmación resulta ser cierta para campos de Dirac masivos propagándose en el seno de un campo eléctrico dependiente del tiempo, para campos de Dirac sin masa deja de serlo: una mínima cantidad de partículas debe ser creada para que se cumpla la no conservación de la corriente quiral [39]. También señalamos la existencia de una nueva anomalía cuántica para la componente

$\mu 1$ del tensor energía momento canónico para los dos sectores de Weyl que emerge en el mismo contexto que la anomalía quiral y que fue estudiada por primera vez en la referencia [40].

El problema de la *backreaction*

La segunda parte de esta tesis corresponde al Capítulo 5. En este capítulo vamos un paso más allá en nuestro análisis y consideramos un campo eléctrico clásico *dinámico*, que puede interactuar con las partículas creadas por su propio decaimiento. Seguimos trabajando con campos de Dirac en dos dimensiones espaciotemporales, ahora en Minkowski, pero mejoramos el modelo añadiendo una fuente clásica externa J_C^μ que inicie el proceso de creación de partículas y permita, por tanto, interacciones posteriores. La contribución de estas partículas en el campo clásico está encapsulada en el valor esperado de vacío de la corriente de Dirac $\langle J_Q^\mu \rangle_{\rm ren} = -q\langle \bar{\psi}\gamma^\mu\psi\rangle_{\rm ren}$ que puede ser directamente introducido en las ecuaciones semicásicas de Maxwell

$$\partial_\mu F^{\mu\nu} = J_C^\mu + \langle J_Q^\mu \rangle_{\rm ren} \,. \tag{6}$$

Obtenemos y analizamos soluciones numéricas a las ecuaciones semicásicas para la corriente externa $J_C^1(t) = -E_0\delta(t)$, que clásicamente genera un campo eléctrico constante con amplitud E_0 para $t > 0$. Investigamos cómo cambia esta imagen clásica al considerar efectos de *backreaction* generados por las partículas creadas mediante el mecanismo de Schwinger. También estudiamos la transferencia de energía entre el campo clásico y las partículas. La interacción entre el campo eléctrico y las partículas creadas produce como resultado final las conocidas oscilaciones de plasma. El límite $m \to 0$ requiere especial atención, ya que en este caso podemos obtener soluciones analíticas a las ecuaciones semiclásicas de Maxwell. Este problema ha sido explorado en la literatura desde diferentes puntos de vista y usando diversas aproximaciones (para más información, consultar las referencias introducidas al comienzo del Capítulo 5). Nosotros seguimos la referencia [28].

A continuación, estudiamos la validez de las soluciones semiclásicas en este contexto. Esta vez, trabajamos con la corriente asintóticamente constante $J^1_C = -qE_0/(1+qt)^2$. Nuestra forma de abordar el problema consiste en estudiar perturbaciones lineales a las soluciones de las ecuaciones semiclásicas de Maxwell δE mediante la ecuación de respuesta lineal. Para ello, adaptamos la propuesta de las referencias [41] y [42] para gravedad semiclásica e inflación caótica respectivamente como sigue: Construimos soluciones aproximadas (y homogéneas) de la ecuación de respuesta lineal y estudiamos su evolución temporal. Si estas soluciones crecen en exceso, podemos decir que la aproximación semiclásica deja de ser válida. Analizamos distintas soluciones en términos del campo crítico $E_{\rm crit} = m^2/q$ y la amplitud externa E_0. Nuestro análisis muestra que cuando $E_0 \sim E_{\rm crit}$, la aproximación semiclásica resulta ser poco precisa después de que se haya creado la primera ráfaga de partículas. Por otro lado, para $E_0 \gg E_{\rm crit}$, la aproximación semiclásica es aceptable durante un período más largo de tiempo. Destacamos que este límite corresponde al caso ultra relativista, donde las partículas creadas son prácticamente no masivas. Para el límite $m \to 0$, las perturbaciones lineales son estables, y la anomalía axial determina la dinámica del proceso.

Expansiones asintóticas

La última parte de esta tesis corresponde al Capítulo 6 y está dedicada a explorar distintas propiedades de un tipo de expansiones asintóticas que se utilizan comúnmente en la teoría cuántica de campos semiclásica, y que se llevan a cabo mediante el formalismo del tiempo propio. Primero, estudiamos con detalle la expansión asintótica de DeWitt-Schwinger de la función de Green de Feynman y la comparamos con la expansión adiabática estándar introducida en los capítulos anteriores. Nos centramos en campos escalares propagándose en cuatro dimensiones espaciotemporales. Aunque estas dos expansiones se obtienen en contextos muy distintos, se puede mostrar que son equivalentes, como se indica en las referencias [43, 44, 45]. Una vez entendida esta equivalencia, explicamos una propiedad no perturbativa muy interesante de la expansión de DeWitt-Schwinger: se puede sumar en todos los términos que contienen la curvatura escalar $R(x)$ [46, 47]. La nueva expansión generada después de esta suma parcial no contiene ningún término que se hace cero cuando la curvatura escalar se sustituye por cero: toda la dependencia en $R(x)$ queda capturada en un prefactor exponencial. Nuestra contribución consiste en mostrar que existe una propiedad similar para la expansión adiabática que se consigue con una redefinición del orden cero $\omega \to \bar{\omega} = (\frac{k^2}{a^2} + m^2 + (\xi - \frac{1}{6})R)^{1/2}$, como indicamos en la referencia [48].

Finalmente, exploramos la posibilidad de encontrar una factorización similar en electrodinámica cuántica. En este caso, nos centramos en la expansión asintótica en tiempo propio de la acción efectiva a un bucle, tanto para campos escalares como para campos de Dirac [49]. Esta expansión es equivalente, salvo derivadas totales, a la expansión (coincidente) de DeWitt-Schwinger. Encontramos que, como ocurre en gravedad, es posible sumar todos los términos que contienen los invariantes electromagnéticos $\mathcal{F} = \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$ and $\mathcal{G} = \frac{1}{4}\tilde{F}_{\mu\nu}(x)F^{\mu\nu}(x)$ de forma que, la nueva expansión resumada no contiene ningún término que se hace cero cuando los invariantes son reemplazados por cero [50, 51]. Sorprendentemente, el prefactor que contiene toda la dependencia en $\mathcal{F}$ y $\mathcal{G}$ es exactamente el Lagrangiano de Euler-Heisenberg, pero con dependencia arbitraria en las coordenadas espaciotemporales. Siguiendo la notación de [7], esta factorización se puede expresar como

$$\mathcal{L}^{(1)}_{\text{scalar}} = \int_0^\infty \frac{ds}{s} e^{-im^2 s} \left[\det \left(\frac{esF(x)}{\sinh(esF(x))} \right) \right]^{1/2} \bar{g}(x; is) \,, \tag{7}$$

$$\begin{aligned} \mathcal{L}^{(1)}_{\text{spinor}} = -\frac{1}{2} \int_0^\infty \frac{ds}{s} e^{-im^2 s} & \left[\det \left(\frac{esF(x)}{\sinh(esF(x))} \right) \right]^{1/2} \\ & \times \operatorname{tr}[e^{-\frac{1}{2} esF_{\mu\nu}(x)\sigma^{\mu\nu}}] \, \bar{h}(x; is) \,, \end{aligned} \tag{8}$$

con $F = F^{\mu}_{\ \nu}$ y $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, y donde los coeficientes de las expansiones $\bar{h}(x; is)$ y $\bar{g}(x; is)$ no contienen ningún término que se hace cero cuando $\mathcal{F}$ y $\mathcal{G}$ son reemplazados por cero. Analizamos algunas consecuencias físicas de esta factorización. Por un lado, encontramos expresiones exactas del Lagrangiano a un bucle para algunas configuraciones externas. También analizamos sus potenciales implicaciones en términos de creación de partículas y discutimos sobre la posibilidad de encontrar una factorización similar cuando se incluye también un campo gravitatorio externo.

En el Capítulo 7 resumimos las ideas principales de este trabajo y explicamos futuras perspectivas del mismo.

Metodología y entrenamiento

Los métodos empleados en esta tesis son teóricos, pero también computacionales. Entre ellos se incluyen: investigación bibliográfica, cálculo de observables físicos dentro del marco de la teoría cuántica de campos espaciotiempos curvos, extensión de métodos específicos a diferentes áreas de investigación (p. ej., de gravitación a electrodinámica), resolución de ecuaciones diferenciales y análisis sistemático de sus soluciones, optimización de relaciones algorítmicas para calcular los términos que siguen al primer order en diferentes expansiones y relación de ideas para generar nuevos resultados.

Para producir los resultados presentados en este trabajo hemos utilizado software de cálculo avanzado. En particular, *Mathematica* (incluyendo el paquete xAct) se ha utilizado principalmente para simplificar resultados analíticos y generar relaciones recursivas y *MATLAB* se ha utilizado con fines numéricos. Las herramientas utilizadas en esta tesis provienen de diferentes áreas de la física y las matemáticas, como teoría clásica y cuántica de campos, electrodinámica y teoría gauge, relatividad general, cosmología, álgebra lineal, ecuaciones diferenciales, análisis real y complejo, geometría diferencial o análisis funcional. También ha sido necesaria una fuerte colaboración entre diferentes investigadores (nacionales e internacionales).

El análisis de la validez de la aproximación semiclásica presentado en el Capítulo 5 ha requerido soluciones numéricas muy precisas de las ecuaciones de *backreaction* [ver Eqs. (5.4), (5.5) y (5.7)]. Para obtenerlas, hemos utilizado recursos informáticos de alto rendimiento (*High Performance Computers* - HPC). En particular, hemos utilizado el *Distributed Environment for Academic Computing* (DEAC) Cluster de la Universidad de Wake Forest. Para aprovechar al máximo la potencia de este sistema, hemos resuelto en paralelo nuestras ecuaciones para distintos valores de la amplitud E_0 utilizando simultaneamente varios núcleos. En cuanto a la resolución numérica de las ecuaciones de *backreaction*, conviene mencionar algunos detalles técnicos: Hemos reescalado las ecuaciones en términos de variables y parámetros adimensionales, hemos discretizado el momento $k \to k_n$ asociado a los modos $h_k^{I,II}$. Hemos transformado la ecuación semiclásica de Maxwell (de segundo grado) en dos ecuaciones de primer grado. Hemos acotado la integral asociada a la corriente inducida $\langle J_Q \rangle_{\rm ren}$ [ver Eq. (5.8)] y la hemos convertido en suma $\int_{-\infty}^{\infty} dk \to \int_{K_{\min}}^{K_{\min}} dk \to \Delta k \sum_{n=1}^{N}$, donde

$\Delta k = k_n - k_{n-1}$ y $N = (K_{\max} - K_{\min})/\Delta k + 1$. En total, nuestro sistema contiene $4N + 2$ ecuaciones a resolver (4 por cada modo mas las dos asociadas al campo eléctrico). Para los cálculos más precisos $4N + 2 \sim 10^5$. Los métodos específicos empleados en el resto de capítulos se han comentado a lo largo del texto introductorio.

Con el fin de ampliar mis conocimientos básicos y como parte de la formación doctoral he asistido a diferentes cursos y escuelas. Estos cursos han abordado distintos temas de interés dentro de la física teórica, por ejemplo, teorías de gravedad modificada, anomalías en teoría cuántica de campos, sistemas Hamiltonianos con ligaduras, colapso gravitatorio y agujeros negros, *Machine learning* o teoría de grupos. Por otro lado he mejorado mis habilidades de comunicación asistiendo y participando en diferentes conferencias internacionales (p.ej., Marcel Grossmann Meeting, APS April meeting, GR22-Amaldi13), talleres y seminarios.

Conclusiones y futuras direcciones

Esta tesis resume los resultados centrales de la investigación llevada a cabo por la autora, en colaboración con su supervisor y otros colaboradores científicos, durante los últimos cuatro años y medio. Esta investigación se centra fundamentalmente en explorar aspectos no perturbativos de la teoría cuántica de campos dentro del marco semiclásico.

En el Capítulo 3 hemos contribuido a la mejora del esquema de regularización/renormalización adiabática que, en contraste con la literatura anterior, es consistente cuando están presentes al mismo tiempo un campo eléctrico y otro gravitatorio. Hemos puesto a prueba la solidez de la propuesta presentada en la referencia [33] usando tres argumentos distintos. A saber, conservación de la energía, equivalencia con el método de DeWitt-Schwinger y obtención de las anomalías cuánticas esperadas [34, 35, 36]. En el Capítulo 4 hemos estudiado ampliamente la fuerte relación existente entre las anomalías quirales y el proceso subyacente de creación de partículas. En particular, hemos encontrado que la invariancia adiabática esperada del número de partículas se rompe para algunas configuraciones externas. Estas condiciones son las mismas en las que emerge la anomalía quiral. En este caso decimos que una mínima cantidad de partículas debe ser creada independientemente de la forma específica de los campos externos (tanto gravitatorios como electromagnéticos). También hemos encontrado que la anomalía quiral para campos de Dirac en dos dimensiones viene acompañada por una nueva anomalía traslacional para los dos sectores de Weyl [39, 40].

Este análisis ha motivado el trabajo presentado en el Capítulo 5, donde hemos investigado extensamente el problema de la *backreaction* (es decir, el efecto de las partículas creadas sobre el campo clásico que las produce) en el modelo de Schwinger masivo en electrodinámica bidimensional. Para ello, hemos resuelto numéricamente las ecuaciones semiclásicas de *backreaction*. Hemos encontrado algunos límites especiales donde el punto de vista semiclásico es preciso: el límite $m \to 0$, relacionado con la anomalía quiral, y el límite $m \to \infty$, en virtud del teorema de desacoplo [28]. Finalmente, en el Capítulo 6 hemos trabajado con diferentes expansiones asintóticas para el "heat-kernel" y para la acción efectiva en espacios curvos y en QED. En la Sección 6.2 hemos presentado la equivalencia encontrada entre la expansión resumada de DeWitt-Schwinger y una nueva expansión adiabática resumada para campos escalares en espaciotiempos cosmológicos [48]. Por otro lado, en la Sección 6.3 hemos propuesto una nueva expansión asintótica resumada para la acción efectiva a un bucle en QED, que encapsula todos los términos que contienen los invariantes electromagnéticos $\mathcal{F}$ y $\mathcal{G}$ en un factor global [50, 51].

La investigación presentada en esta tesis puede desembocar en distintos proyectos.

El trabajo desarrollado en la Sección 6.3 puede ser generalizado de dos formas independientes. Primero, extendiendo la factorización encontrada a campos externos no abelianos, donde esperamos un resultado similar para la acción efectiva. En paralelo, debemos comprobar si para la acción efectiva asociada a campos de Dirac en espaciotiempos curvos es posible encontrar una segunda factorización (además de la factorización exponencial asociada a la curvatura escalar) similar a la encontrada en QED. En segundo lugar, debemos explorar las consecuencias fenomenológicas de nuestra factorización cuando se incluye también gravedad. Un detalle interesante que podría ser analizado es la aparición de una corrección logarítmica en la acción efectiva de QED debido al prefactor gravitatorio R-sumado y sus efectos potenciales en la propagación de la luz.

Una segunda dirección de investigación consiste en analizar en mayor detalle el problema del vacío cuántico en universos en expansión, introducido en el Capítulo 2. Una forma de abordar el problema puede ser utilizar la simetria conforme que emerge cerca del big bang (esta opción se puede justificar también mediante la hipótesis de curvatura de Weyl [52]). Puede ser interesante explorar esta posibilidad teniendo en mente su aplicación en una supuesta etapa preinflacionaria y estudiar su impacto en las perturbaciones primordiales a gran escala. También puede resultar interesante explorar en detalle las relaciones entre distintos métodos de renormalización en espaciotiempos curvos. En particular, entre el método de Hadamard y el método de regularización adiabática para campos escalares cargados (e incluyendo un campo externo electromagnético). Se puede proceder de forma similar a la Sección 6.1. Entender estas interconexiones nos permitirá entender mejor la correspondencia entre la condición de Hadamard y la condición de regularidad adiabática para descartar estados de vacío no físicos.

El trabajo presentado en el Capítulo 5 también se puede extender fácilmente de dos formas complementarias. En primer lugar, recordemos que para estudiar la validez de la aproximación semiclásica hemos utilizado soluciones aproximadas a la ecuación de respuesta lineal. Un análisis más riguroso debería involucrar soluciones numéricas exactas a esta ecuación [53]. En segundo lugar, un estudio más realista del problema de la *backreaction* requiere que se extienda nuestro análisis numérico a cuatro dimensiones espaciotemporales. Para el caso con un campo eléctrico puro dependiente del tiempo, podemos utilizar el esquema propuesto en la referencia [36], donde se desarrolla el método de regularización adiabática para campos de Dirac en cuatro dimensiones propagándose el seno de campos eléctricos dependientes del tiempo. No obstante, una segunda opción más atractiva podría ser el caso en el que, además de un campo eléctrico dependiente del tiempo, se tiene un campo magnético constante en la misma dirección. En este segundo caso podría ser interesante ver las situaciones en las que la anomalía quiral resulta relevante para comprender la dinámica del sistema [de acuerdo con Eq. (4.57)].

El formalismo general de teoría cuántica de campos en espacios curvos, así como sus conexiones con una posible teoría de gravedad cuántica, son también posibles áreas de investigación a explorar. En particular, podría ser interesante aprender cómo la teoría cuántica de campos en espacios curvos emerge como una aproximación a bajas energías en el contexto de teorías efectivas y renormalización de Wilson. El modelo de Schwinger, que es exactamente resoluble en la teoría cuántica completa, puede ser usado como un modelo de prueba antes de abordar este reto más ambicioso.

Contents

Chapter 1

Introduction

The merger of our fundamental description of the microcosmos – the Quantum Theory of Fields – and that of the macrocosmos –general relativity and the curved spacetime description of gravity – is still the most uncomfortable problem of fundamental physics. However, it is undeniable that theoretical physics has made enormous progress over the last decades. On the one hand, the discovery of the Higgs Boson [1] supposed the culmination of the Standard Model of particle physics; on the other, the detection of gravitational waves [2, 3] exhibited the strength of Einstein's General Relativity more than 100 years after its formulation. Furthermore, and without diminishing these very positive milestones, long-standing open theoretical issues, together with the hope for new data coming from the next generation of experiments and observations has become a starting point to build very interesting modifications and extensions of these two complementary descriptions of Nature [4, 5].

Although a definitive quantum description of gravity has not been found yet, it is possible to build a self-consistent semiclassical description, where one studies the propagation of quantum fields in curved spacetimes to explore new (quantum) effects of gravitation. See, for example, Refs. [6, 7, 8] for excellent and comprehensive textbooks on this topic. This approach is usually regarded as a truncated and effective version of a fully quantized theory, with a limited range of validity.

One of the benefits of this proposal is that it naturally accounts for fascinating non-perturbative quantum phenomena, such as the creation of particles induced by strong gravitational fields. For example, the presence of a time-dependent gravitational field, as that describing the expanding Universe, permits the spontaneous creation of particles out of the vacuum [9, 10, 11, 12]. This mechanism could be behind the observed CMB anisotropies, and can be crucial to account for the explosive creation of elementary particles in the reheating epoch [13, 14] – this time induced by the inflaton field – . In a gravitational collapse that ends in a black hole, this process is also activated, generating a constant thermal radiation [15, 16, 17].

In quantum electrodynamics (QED), the semiclassical approximation becomes also extremely useful to capture this non-perturbative mechanism. In 1951, Schwinger computed the imaginary part of the one-loop effective action for a homogeneous and constant electric field [18], to evaluate the vacuum persistence amplitude from the formula $|\langle out|in\rangle|^2 = \exp(-2\mathrm{Im}\Gamma(A_\mu))$. In d spacetime dimensions it reads [19]

$$\frac{2\,\mathrm{Im}\,\Gamma(A_\mu)}{VT} = \frac{2}{(2\pi)^d}\sum_{n=1}^{\infty}\left(\frac{qE}{n}\right)^{(d+1)/2}\exp\left(\frac{-n\pi m^2}{qE}\right) , \qquad (1.1)$$

where VT represents the volume of the spacetime. We note that the vacuum persistence probability is equal to one minus the total probability of creating pairs. The exponential factor in Eq. (1.1) shows the non-perturbative nature of the process.[1] This fascinating effect is likely to be experimentally detected in a near future [21, 22].

A second advantage of the semiclassical approach is that it also encodes one of the main results of the theory of quantum fields: the existence of quantum anomalies. In this context, the computation of vacuum expectation values of physical observables becomes a complex issue, and advanced renormalization techniques are required to tame new ultraviolet divergences caused by the external fields. As a counterpart, this process generates finite and unambiguous results, known as quantum anomalies, that signal the quantum breaking of a classical symmetry. Axial anomalies become particularly relevant, since they can be directly related with particle creation [23, 24]. For example, the classical action for a massless Dirac field ψ is invariant under chiral transformations. This implies, via Noether's theorem, that the axial current $J_A = \bar{\psi}\gamma^\mu\gamma^5\psi$ is conserved. The quantum theory breaks this conservation law, giving the following non-vanishing result [25, 26]

$$\partial_\mu \left\langle J^\mu_A \right\rangle = -\frac{q^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,. \tag{1.2}$$

In terms of particle creation, one can say then that a minimum amount of particles should be created to preserve the non-conservation of chirality.

The semiclassical approximation also offers a coherent framework to study the influence of different quantum effects into the classical background via the semiclassical field equations,

$$\partial_\mu F^{\mu\nu} = J^\nu_{\text{class}} + \langle J^\nu \rangle_{\text{ren}}\,, \tag{1.3}$$

$$G_{\mu\nu} = 8\pi G\left(T^{\text{class}}_{\mu\nu} + \langle T_{\mu\nu}\rangle_{\text{ren}}\right)\,. \tag{1.4}$$

[1] For a historical perspective, see [20].

This analysis becomes crucial to obtain, for example, quantum-corrected geometries induced by vacuum polarization effects [27], or to explore the backreaction of the created particles by time-dependent background fields [28].

This thesis aims to investigate all these interesting quantum phenomena from a renewed perspective, exploring the interconnections between semi-classical gravity and electrodynamics. The content of the thesis is divided in three parts and can be summarized as follows.

Quantum Field Theory in time-dependent backgrounds

The first part of the thesis corresponds to Chapters 2, 3 and 4 and it is devoted to studying some of the main features of the Theory of Quantum Fields when investigated in the presence of external backgrounds. We focus on homogeneous configurations with an arbitrary time-dependence.

In Chapter 2 we review some basics about Quantum Field Theory in Curved Spacetimes. This chapter has to be thought of as an introduction that will allow the reader to become familiar with the notation and the most important ideas that appear throughout this thesis. We consider the simplest generalization of the special relativity case: a quantized scalar field propagating in a four-dimensional homogeneous and time-dependent spacetime characterized by the interval $ds^2 = dt^2 - a(t)^2 d\vec{x}^2$. We illustrate two fundamental issues that have to be taken under consideration i) the absence of a preferred quantum vacuum and its consequences, i.e., the spontaneous creation of particles out of the vacuum [9, 10], and ii) the need for advanced renormalization techniques, that have to be systematically applied to deal with the ultraviolet (UV) divergences that naturally arise in this context. We introduce the adiabatic regularization method, which turns out to be the most efficient renormalization technique in homogeneous and time-dependent backgrounds [29, 30, 31, 32].

In Chapter 3 we study how the previous picture changes when including a second, time-dependent, electromagnetic background. For future convenience, instead of considering the standard four-dimensional scalar field, we examine a spin-$\frac{1}{2}$ field in two spacetime dimensions. This setup was previously considered in Ref. [33]. Our contribution is to go deeper into the differences and similarities between an expanding FLRW background, characterized by the scale factor $a(t)$, and an electric field background, represented by the potential vector $A(t)$. We focus on the extension of the adiabatic regularization method and show that there must be an adiabatic hierarchy between these backgrounds to have a consistent regularization method. The scale factor $a(t)$ should be a function of adiabatic order zero, while the potential vector $A(t)$ should be a function of adiabatic order one. We give support to this statement from three different perspectives: energy conservation [34, 35], equivalence with other regularization methods [36] and reproduction of the expected quantum anomalies.

Finally, in Chapter 4, and taking advantage of the tools and methods introduced in Chapter 3, we study some of the main semiclassical predictions in electrodynamics for time-dependent backgrounds. As stated above, one of the advantages of the semiclassical framework is that it provides a very nice and direct description of the phenomenon of particle creation. In the canonical language, an early-times positive-frequency solution to the Dirac equation evolves, at late-times, into a superposition of positive- and negative-frequency solutions. Therefore, we say that particles are created out of the vacuum [9]. In the modern language of QED, a non-zero imaginary part of the one-loop effective action accounts for the quantum instability of the vacuum and the associated particle creation effect [37, 18]. As happens with particle creation, the semiclassical framework also provides a direct way to study and understand quantum anomalies. We focus on the two-dimensional chiral anomaly that exists for massless Dirac fields [38],

$$\partial_\mu \left\langle J_A^\mu \right\rangle_{\text{ren}} = -\frac{q}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}\,, \tag{1.5}$$

and study its phenomenological consequences in terms of particle creation. We discover that the expected adiabatic invariance of the particle number is broken in the cases where the chiral anomaly emerges. We also point out the existence of a new quantum anomaly for the $\mu 1$ component of the canonical stress-energy tensor of both Weyl sectors that emerges in the same context as the chiral one and that was studied for the first time in Ref. [40].

The backreaction problem

The second part of the thesis corresponds to Chapter 5. In this chapter, we go a step further in our analysis and consider a *dynamical* electromagnetic background field that can interact with the particles created by its own decay. We keep working with two-dimensional Dirac fields (now in Minkowski spacetime), but we upgrade the preceding model by including a classical, external source J_C^μ that initiates the particle creation process and, therefore, allows posterior interactions. The potential contribution of the created particles on the (time-dependent) background is encapsulated in the vacuum expectation value of the Dirac current $\langle J_Q^\mu\rangle_{\text{ren}} = -q\langle\bar{\psi}\gamma^\mu\psi\rangle_{\text{ren}}$ that can be directly included in the (semiclassical) Maxwell equations

$$\partial_\mu F^{\mu\nu} = J_C^\mu + \langle J_Q^\mu\rangle_{\text{ren}}\,. \tag{1.6}$$

We obtain and analyze numerical solutions to the semiclassical equations for the time-dependent classical source $J_C(t) = -E_0\delta(t)$, that classically corresponds to a constant electric field with amplitude E_0 for $t > 0$. We investigate how this picture changes when considering backreaction effects coming from the created particles via the Schwinger effect. The interaction between the electric field and the created particles results in the well-known plasma oscillations. We also study the energy transfer between the electric field and the particles. Special attention is given to the massless limit $m \to 0$, where analytical solutions to the semiclassical Maxwell equations can be computed. This problem has been previously explored in the literature from different perspectives and using other approaches (see references at the beginning of Chapter 5). We follow Ref. [28].

The second step in the backreaction analysis is to study the validity of our semiclassical solutions. This time we work with the asymptotically constant classical current $J_C = -qE_0/(1 + qt)^2$. We approach this complicated problem by studying linear perturbations to the solutions of Maxwell's semiclassical equations δE via the linear response equation. We adapt the analysis of Refs. [41] and [42] for semiclassical gravity and chaotic inflation respectively as follows: we build approximated (and homogeneous) solutions to the linear response equation and study its growth with time. For the cases where these solutions overgrow in time, we say that the semiclassical approximation breaks down. We analyze different solutions in terms of a critical scale $E_{\text{crit}} = m^2/q$ and the external characteristic amplitude E_0. We see that when $E_0 \sim E_{\text{crit}}$, the semiclassical approximation loses accuracy after the first burst of particles is created. On the other hand, for $E_0 \gg E_{\text{crit}}$ the semiclassical approximation our criterion is satisfied for a longer period. We note that this limit corresponds to the ultra-relativistic limit for which the created particles become (almost) massless. For the massless limit, linear perturbations are stable, and the axial anomaly determines the dynamics of the system.

Asymptotic expansions

The last part of the thesis corresponds to Chapter 6 and it is devoted to exploring some properties of a general class asymptotic expansions that are widely used in semiclassical Quantum Field Theory, and that are carried out via the proper-time formalism. First, we study with some detail the DeWitt-Schwinger asymptotic expansion of the Feynman Green's function and compare it with the standard adiabatic expansion introduced in previous chapters for scalar fields propagating in four-dimensional flat FLRW universes. Although they are worked out using completely different techniques, we see that they turn out to be equivalent, as pointed out in Refs. [43, 44, 45]. Once the equivalence between these two different methods is understood, we explain a very interesting, non-perturbative property of the DeWitt-Schwinger expansion: it can be summed in all terms containing the scalar curvature $R(x)$ [46, 47]. The new expansion generated after the summation does not contain any term that vanishes when the scalar curvature is replaced by zero: all the dependence on R is encapsulated in an overall exponential factor. We show that a similar property exists for the adiabatic expansion by implementing a redefinition of the leading order $\omega \to \bar{\omega} = (\frac{k^2}{a^2}+m^2+(\xi - \frac{1}{6})R)^{1/2}$, as explained in Ref. [48].

Finally, we explore the possibility of finding a similar non-perturbative factorization in Quantum Electrodynamics. This time, we focus on the proper-time expansion of the one-loop QED effective action for both, scalar and spin-$\frac{1}{2}$ fields [49]. This expansion is equivalent, up to total derivatives, to the (coincident) DeWitt-Schwinger expansion. We find that, as happens in the gravitational case, it is possible to sum all terms containing the electromagnetic invariants $\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}(x)$ and $\mathcal{G} = \frac{1}{4}\tilde{F}_{\mu\nu}F^{\mu\nu}(x)$ in such a way that the new, $(\mathcal{F}, \mathcal{G})$-summed expansion does not contain any term that vanishes when the invariants are replaced by zero [50, 51]. Surprisingly, the overall factor that encapsulates the $(\mathcal{F},\mathcal{G})$-dependence is exactly the Euler-Heisenberg Lagrangian but now, with an arbitrary dependence on the spacetime coordinates. Following the notation of [7], this factorization reads

$$\mathcal{L}^{(1)}_{\text{scalar}} = \int_0^\infty \frac{ds}{s} e^{-im^2 s} \left[\det \left(\frac{esF(x)}{\sinh(esF(x))} \right) \right]^{1/2} \bar{g}(x;is) \,,$$

$$\mathcal{L}^{(1)}_{\text{spinor}} = -\frac{1}{2} \int_0^\infty \frac{ds}{s} e^{-im^2 s} \left[\det \left(\frac{esF(x)}{\sinh(esF(x))} \right) \right]^{1/2} \times \text{tr}[e^{-\frac{1}{2} esF_{\mu\nu}(x)\sigma^{\mu\nu}}] \, \bar{h}(x;is) \,, \tag{1.7}$$

with $F = F^{\mu}_{\ \nu}$ and $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, and where the coefficients of the expansions $\bar{h}(x;is)$ and $\bar{g}(x;is)$ do not contain any term that vanishes when $\mathcal{F}$ and $\mathcal{G}$ are replaced by zero. We analyze some physical consequences of this factorization. We find exact expressions of the one-loop effective Lagrangian for some external configurations, analyze its potential implications in terms of particle creation, and discuss the possibility of finding a similar factorization when including also an external gravitational field.

In Chapter 7 we summarize the main ideas of this work and explain possible future prospects of our work.

1.1 Methodology and training

The methods employed in this thesis are theoretical and also computational. These include: bibliographic research, computation of physical observables within the Quantum Field Theory in Curved Spacetime framework, extension of specific methods to different areas of research (e.g., from gravitation to electrodynamics), solve differential equations and complete a systematic analysis of their solutions, optimize algorithmic relations to compute next-to-leading order terms of different expansions, and relate ideas to generate new results. To produce the results presented in this work we have used advanced calculus software. In particular, the *Mathematica* software (including the xAct package) was mainly used to simplify analytic results and generate recursive relations and the *MATLAB* software was used for numerical purposes. The specific physical and mathematical tools utilized in this thesis come from different areas of physics and mathematics such as classical and quantum field theory, electrodynamics and gauge theory, general relativity, cosmology, linear algebra, differential equations, real and complex analysis, differential geometry or functional analysis. It was also required strong collaboration between different (national and international) researchers.

The validity analysis performed in Chapter 5 required very precise numerical solutions to the backreaction equations [see Eqs. (5.4), (5.5) and (5.7)]. To obtain these solutions, we have used High Performance Computing (HPC) resources. In particular, we have used the Distributed Environment for Academic Computing (DEAC) Cluster of Wake Forest University. To take advantage of the power of this resource, we have solved in parallel our backreaction equations for various values of the external amplitude E_0 using simultaneously different nodes. Regarding the numerical resolution of the backreaction equations, it is important to mention some technical details: We have rescaled the equations to express them in terms of dimensionless variables and parameters. We have discretized the momentum $k \to k_n$ associated with the mode functions $h_k^{I,II}$. We have transformed the semiclassical Maxwell equation (second-order) into two first-order differential equations. We put cut-offs to the momentum integral of the induced electric current $\langle J_Q \rangle_{\rm ren}$ [see Eq. (5.8)] and we have transformed it into a sum $\int_{-\infty}^{\infty} dk \to \int_{K_{\rm min}}^{K_{\rm min}} dk \to \Delta k \sum_{n=1}^{N}$, where $\Delta k = k_n - k_{n-1}$ and $N = (K_{\rm max} - K_{\rm min})/\Delta k + 1$. In total, our system contains $4N + 2$ equations (4 for each mode and 2 for the electric field). For the most precise computations $4N + 2 \sim 10^5$. Specific methods for the other chapters have already been commented along the introductory text.

In order to extend my basic knowledge and as a part of the Ph.D. training, I have assisted in different courses and schools. I have also improved my communication skills by attending and participating in different conferences, workshops, and seminars.

Before starting with the main content of this work, let us clarify some details first: throughout this thesis, we follow the conventions of Ref. [7]. The matter fields are quantized via standard canonical quantization. We work in the Heisenberg picture, where the quantum operators carry all the dynamical dependence. We use units such that $\hbar = c = 1$.

Chapter 2

Quantum Field Theory in Curved Spacetime: vacuum choices and renormalization

In this chapter, we review two of the main issues that appear in the theory of quantized fields in curved spacetime: the absence of a preferred vacuum state and the need for specific renormalization techniques to evaluate vacuum expectation values of physical operators. The concepts, mathematical techniques, and conventions detailed throughout this chapter will be important to better understand the main ideas of this thesis. However, if the reader is familiarized with this framework, they can safely jump to the next chapter.

The theory of quantum fields in curved spacetimes is a semiclassical theory where one can study the influence of gravity in different quantum processes. In this framework, the gravitational field is regarded as a classical background field while the matter degrees of freedom are quantized. As one can imagine, it is a natural extension of standard Quantum Field Theory in Minkowski. However, although local quantities are easily extended to curved spacetime, global concepts lose their uniqueness [54]. In particular, there is not a unique definition for the vacuum state. This ambiguity leads to one of the most important results of this framework: the spontaneous creation of particles out of the vacuum. This effect was firstly discovered by L. Parker [9, 10, 11, 12] in the context of expanding universes. A few years later it was further explored by S. Hawking to study the problem of particle creation in the vicinity of black holes [15, 16].

The computation of vacuum expectation values of physical operators in curved spacetimes is also a complex issue that requires the application of involved regularization and renormalization techniques to deal with the new UV divergences generated by spacetime curvature in a way consistent with general covariance. In the second part of this chapter, we show how to deal with this issue in homogeneous and isotropic spacetimes by means of the adiabatic regularization prescription.

2.1 Vacuum choices

To understand the issue of the vacuum choice in curved spacetimes, let us first review the Minkowski vacuum. Consider a free quantized scalar field ϕ propagating in Minkowski spacetime. It satisfies the Klein-Gordon equation

$$(\Box + m^2)\phi = 0\,. \tag{2.1}$$

The general solution of (2.1) can be written as a linear combination of positive and negative energy solutions

$$\phi(t,\vec{x}) = \int d^3k (A_{\vec{k}} f_{\vec{k}}(t,\vec{x}) + A^\dagger_{\vec{k}} f^*_{\vec{k}}(t,\vec{x})) \tag{2.2}$$

where $A^\dagger_{\vec{k}}$ and $A_{\vec{k}}$ are the usual creation and annihilation operators satisfying the commutation relations $[A_{\vec{k}}, A^\dagger_{\vec{k}'}] = \delta(\vec{k}-\vec{k}')$ and $[A_{\vec{k}}, A_{\vec{k}'}] = 0 = [A^\dagger_{\vec{k}}, A^\dagger_{\vec{k}'}]$, and where

$$f_{\vec{k}}(t,\vec{x}) = \frac{1}{\sqrt{2(2\pi)^3\omega}} e^{i\vec{k}\,\vec{x}} e^{-i\omega t}, \tag{2.3}$$

with $\omega^2 = k^2 + m^2$. The functions $f_{\vec{k}}(t,\vec{x})$ form a complete orthonormal basis with respect to the Klein-Gordon product. [1] With this basis choice we are at the same time, defining the vacuum state of the system, namely

$$A_{\vec{k}}|0_M\rangle = 0\,, \qquad \forall k. \tag{2.4}$$

It is important to note here that, the mode expansion in terms of $\vec{k}$ is not necessary, but convenient [i.e., we can choose other quantum numbers (ω, l, m)]. However, to split the solution in terms of positive and negative frequency solutions $\sim \frac{e^{\mp i\omega t}}{\sqrt{\omega}}$ is *fundamental*. This choice has a physical meaning since $|0_M\rangle$ is also the ground state of the (time-independent) Hamiltonian. It is also the only choice consistent with Poincaré invariance.

Consider now a free, quantized, scalar field ϕ propagating in a FLRW universe

$$ds^2 = dt^2 - a(t)^2 d\vec{x}^2\,. \tag{2.5}$$

The (generalized) Klein-Gordon equation now reads

$$(\Box + m^2 + \xi R)\phi = 0\,, \tag{2.6}$$

with ξ a dimensionless constant and R the scalar curvature. As in Minkowski, we can split the solution of (2.6) as a linear combination of orthogonal solutions

$$\phi(t,\vec{x}) = \int \frac{d^3k}{\sqrt{2(2\pi)^3 a^3}} (A_{\vec{k}}\, e^{i\vec{k}\,\vec{x}} h_k(t) + A^\dagger_{\vec{k}}\, e^{-i\vec{k}\,\vec{x}} h^*_k(t)) \tag{2.7}$$

[1] $(f_{\vec{k}}, f_{\vec{k}'}) = i \int d^3x (f^*_{\vec{k}} \partial_t f_{\vec{k}'} - f_{\vec{k}'} \partial_t f^*_{\vec{k}}) = \delta(\vec{k} - \vec{k}')$

where the complex function $h_k(t)$ satisfies

$$\ddot{h}_k + [\omega_k(t)^2 + \sigma(t)]h_k = 0\,. \tag{2.8}$$

$\omega_k^2 = k^2/a^2 + m^2$ is the physical frequency of the scalar field and $\sigma(t) = (6\xi - \frac{3}{4})\frac{\dot{a}^2}{a^2} + (6\xi - \frac{3}{2})\frac{\ddot{a}}{a}$ can be interpreted as defining a natural frequency scale of the spacetime. The mode function $h_k(t)$ satisfies a second-order ordinary differential equation that has two linearly independent solutions. The most general solution is a linear combination of these two solutions. To *choose a vacuum* implies selecting one solution among all these possibilities. In Minkowski spacetime, we argued that there exists a preferred choice, namely $h_k = \frac{1}{\sqrt{\omega_k}}e^{-i\omega_k t}$, that determines uniquely the vacuum state of the system $A_{\vec{k}}|0_M\rangle = 0$. However, if we do not restrict ourselves to Minkowski spacetime, we do not have, in general, a natural choice for $h_k(t)$, and we need to find a criterion to choose a preferred basis. As we will shortly see, one of the consequences of this ambiguity is the creation of particles out of the vacuum, driven by the expansion of the universe.

Although the existence of this inherent ambiguity for general backgrounds, there are some special cases where we can naturally choose a preferred basis. It can be done as follows.

1. Asymptotically flat regions.

Let us consider first a very simple situation where it is possible to naturally fix the vacuum: a spacetime with asymptotically flat regions. To this end we can think about a generic flat FLRW universe that is characterized by a scale factor which is asymptotically bounded at early and late times, namely

$$a(t \to -\infty) = a_{\rm in}\,, \qquad a(t \to \infty) = a_{\rm out}\,. \tag{2.9}$$

In Figure 1 is represented a possible scale factor that satisfy the condition above.

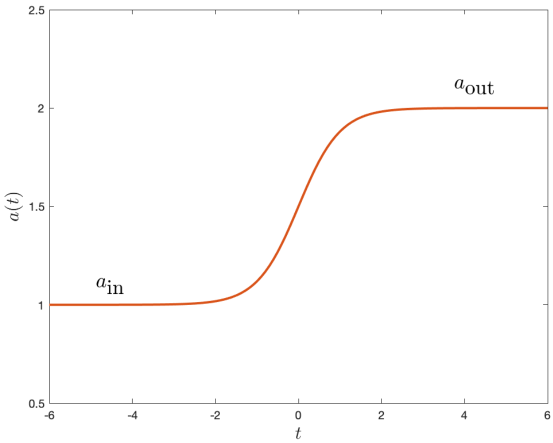

Figure 2.1: Asymptotically bounded scale factor $a(t)$ represented in arbitrary time units.

In this spacetime, a natural way to fix the vacuum is to select the modes that, at early times, behave like positive-frequency solutions, namely

$$h_k^{\mathrm{in}} \sim \frac{1}{\sqrt{\omega_{\mathrm{in}}}} e^{-i\,\omega_{\mathrm{in}} t}\,, \tag{2.10}$$

where $\omega_{\mathrm{in}} = \sqrt{\frac{k^2}{a_{\mathrm{in}}^2} + m^2}$. However, there is a second possibility. We can also fix the modes by requiring that they behave as positive frequency solutions at late times

$$h_k^{\mathrm{out}} \sim \frac{1}{\sqrt{\omega_{\mathrm{out}}}} e^{-i\,\omega_{\mathrm{out}} t}\,, \tag{2.11}$$

where $\omega_{\text{out}} = \sqrt{\frac{k^2}{a_{\text{out}}^2} + m^2}$. The interesting point here is that these two choices are not equivalent. If we choose h_k^{in} as initial condition for our modes, after time evolution we find

$$h_k \quad \sim \quad \frac{\alpha_k}{\sqrt{\omega_{\text{out}}}} e^{-i\,\omega_{\text{out}} t} + \frac{\beta_k}{\sqrt{\omega_{\text{out}}}} e^{i\,\omega_{\text{out}} t} . \tag{2.12}$$

The early times positive-frequency solutions have evolved into a mixture of positive- and negative-frequency modes. The coefficients α_k and β_k are the so-called Bogoliugov coefficients [7, 6], that are studied in detail in Chapter 4, in the context of time-dependent electric backgrounds. It can be shown that the coefficient β_k is related with the particle number density at late times, i.e., $\langle N^{\text{out}} \rangle \sim \int d^3k\, n_k$ with $n_k = |\beta_k|^2$ (note that at early times $\langle N^{\text{in}} \rangle = 0$). The physical interpretation of two inequivalent vacuum choices becomes clear: particles are created out of the vacuum.

2. De Sitter spacetime.

Let us consider now the De Sitter spacetime. It can be considered as a particular FLRW universe with scale factor $a = e^{Ht}$. It is a maximally symmetric spacetime where, instead of the Poincaire symmetry that holds in Minkowski spacetime, we have the $SO(1, 4)$ symmetry. In particular, it is not difficult to prove that the De Sitter metric is invariant under the transformation [55]

$$t \to t' = t + t_0 \ , \qquad \vec{x} \to \vec{x}' = e^{-Ht_0}\vec{x} \ . \tag{2.13}$$

In this context, the general solution to the mode equation (2.8) is

$$h_k(t) = \sqrt{\frac{\pi}{2H}} [E_k H_\nu^{(1)}(kH^{-1}e^{-Ht}) + F_k H_\nu^{(2)}(kH^{-1}e^{-Ht})] \ , \tag{2.14}$$

with $\nu^2 = 9/4 - (m^2 + 12\xi H^2)/H^2$ and where $H_\nu^{(1,2)}(z)$ are the Hankel functions of the first and second kind respectively. To choose a vacuum means to determine the constants E_k and F_k. Imposing invariance of the two-point correlation function under (2.13) implies $E_k = E$ and $F_k = F$. That is, the coefficients do not depend on k. Furthermore, imposing a Minkowski-like behaviour for $k \to \infty$, which in De Sitter spacetime implies

$$h_k(t) \sim \frac{1}{\sqrt{ke^{-Ht}}} e^{i(kH^{-1}e^{-Ht})} , \tag{2.15}$$

we immediately find $E = 1$ and $F = 0$. Therefore, we say that in De Sitter spacetime there is a natural vacuum $|0_{BD}\rangle$, which is usually called the Bunch-Davies vacuum [56, 57]. We note that there is still an important controversy regarding the stability of de Sitter spacetime (see, for instance [58]).

The previous examples illustrate three essential characteristics of the (curved spacetime) quantum vacuum. It has to respect the symmetries of the spacetime.[2] In general, it is not unique, and one of the consequences of this statement is that, in some situations, particles can be created out of the vacuum. Moreover, for consistency, it has to approach the flat spacetime behavior at an appropriate rate [59]. This is a critical condition, required to end up with finite quantities after *renormalization*.

It is well known that vacuum expectation values of quadratic operators in quantum field theory are ultraviolet divergent. These divergences can be systematically removed through the process of renormalization [23, 60].

[2] We note that when the spacetime is not maximally symmetric, this condition does not single out an unique vacuum.

When dealing with free quantized fields propagating in Minkowski spacetime, divergent terms can be easily removed by means of normal ordering. However, when computing the same expectation values for free fields in curved spacetimes, normal ordering is not enough, and new ultraviolet terms arise because of the curvature of the spacetime. In this context, it is required to work with advanced renormalization techniques to obtain finite and meaningful physical quantities. For homogeneous and time-dependent spacetimes, a very simple and illustrative method can be used to renormalize physical observables: the adiabatic regularization method. We will devote the following section to reviewing this renormalization technique, explaining its main features.

2.2 Adiabatic regularization

With the the mode expansion of the field, we can easily compute the vacuum expectation value of relevant physical operators. For simplicity, let us focus on the vacuum expectation value of the two-point function at coincidence $\langle 0|\phi(x)\phi(x)|0\rangle \equiv \langle\phi^2\rangle$. From (2.7) it is not difficult to arrive to

$$\langle\phi^2\rangle = \int \frac{d^3k}{2(2\pi)^3 a(t)^3}|h_k(t)|^2 = \frac{1}{(2\pi)^2 a(t)^3}\int_0^\infty dk k^2 |h_k(t)|^2 . \tag{2.16}$$

This quantity is ultraviolet divergent.[3] To obtain a finite, physical value, we have to perform appropriate subtractions (i.e. to eliminate the divergent part in order to get a meaningful result), namely

$$\langle\phi^2\rangle_{\rm ren} = \frac{1}{(2\pi)^2 a(t)^3}\int_0^\infty dk k^2 [|h_k(t)|^2 - SUBTRACTIONS\ ] . \tag{2.17}$$

As one can imagine, to correctly determine the subtraction terms is, in general, a complex task. However, in FLRW universes, a handy tool can be

[3] Note: in Minkowski $|h_k|^2 \sim \frac{1}{\omega_k}$ and therefore $\frac{1}{4\pi^2}\int_0^\Lambda dk k^2 |h_k|^2 \sim \frac{\Lambda^2}{8\pi^2} + \frac{m^2}{16\pi^2}\ln(\frac{2\Lambda^2}{m^2})$.

used to this end: the adiabatic expansion of the scalar modes $h_k(t)$.

We say that a physical process is adiabatic when the rate of change of the parameters controlling the system is slow. In the context of an expanding universe, it means that the expansion of the universe is very smooth (even if the total amount of expansion is high). This condition can be expressed intuitively as

$$\omega_k \gg \frac{\dot{a}}{a} \, . \tag{2.18}$$

Therefore, the adiabatic expansion of the field modes can be interpreted as an expansion in the number of derivatives of the scale factor in such a way that time-derivatives of the background field $a(t)$ increase the adiabatic order of the expansion. In this language we can say that that $a(t)$ is a function of adiabatic order zero, $\frac{\dot{a}}{a}$ is of adiabatic order one, $\frac{\ddot{a}}{a}$ and $\frac{\dot{a}^2}{a^2}$ are of adiabatic order 2, and so on.

For scalar fields, the adiabatic expansion is based on the Wentzel-Kramers-Brillouin (WKB) ansatz

$$h_k(t) \sim \frac{1}{\sqrt{W_k}} e^{-i\int^t W_k(t')dt'} , \tag{2.19}$$

where the function W_k admits an adiabatic expansion of the form

$$W_k = \sum_{n=0}^{\infty} \omega_k^{(n)} \, , \tag{2.20}$$

and where the super-index (n) refers to the adiabatic order of the coefficient $\omega^{(n)}$, that will be a function of the scale factor and its derivatives (up to order n). The key point of the process is to correctly determine the leading order. In this case, and because of the adiabatic condition (2.18), we demand the (expected) Minkowski-like behaviour

$$h_k(t) \sim \frac{1}{\sqrt{\omega_k(t)}} e^{-i\int^t \omega_k(t)dt'} \tag{2.21}$$

which implies $\omega_k^{(0)} = \omega = \sqrt{\frac{k^2}{a^2} + m^2}$. The next-to-leading orders are obtained by iteration from the mode equation. Introducing (2.19) in the mode equation (2.8) we find

$$W_k^2 = \omega^2 + \sigma + \frac{3}{4}\frac{\dot{W}_k^2}{W_k^2} - \frac{1}{2}\frac{\ddot{W}_k}{W_k} \ . \tag{2.22}$$

Inserting the adiabatic expansion (2.20) and grouping terms of the same adiabatic order we obtain, by iteration, higher order terms from the lower ones. As stated above, time derivatives increase the adiabatic order of a function so that $\dot{\omega}_k^{(n)}$ is a function of order $(n+1)$. We also note that $\sigma \equiv \sigma(\frac{\dot{a}^2}{a^2}, \frac{\ddot{a}}{a})$ is a function of adiabatic order two. Furthermore, it can be shown that $\omega_k^{(2n+1)} = 0$. Following this process we find that the first next-to-leading orders are

$$\omega^{(2)} = \frac{1}{2\omega^3}\left\{\sigma\omega^2 + \frac{3}{4}\dot{\omega}^2 - \frac{1}{2}\omega\ddot{\omega}\right\} , \tag{2.23}$$

$$\omega^{(4)} = \frac{1}{2\omega^3}\left\{2\sigma\omega\omega^{(2)} - 5\omega^2(\omega^{(2)})^2 + \frac{3}{2}\dot{\omega}\dot{\omega}^{(2)} - \frac{1}{2}(\omega\ddot{\omega}^{(2)} + \omega^{(2)}\ddot{\omega})\right\} \tag{2.24}$$

Once we have the adiabatic expansion of the field modes, we can expand the integrand of different physical observables. In particular, the two-point function can be (adiabatically) expanded in terms of W_k as

$$|h_k|^2_{\text{ad}} \sim (W_k^{-1})^{(0)} + (W_k^{-1})^{(2)} + (W_k^{-1})^{(4)} + ... \tag{2.25}$$

The important characteristic of the adiabatic expansion, and the reason why it can be used for renormalization is as follows.

For large momentum k, the expansion above (2.25) reads

$$(W^{-1}_{k\to\infty})^{(0)} \sim \frac{a}{k} - \frac{m^2a^3}{2k^3} + \frac{3m^4a^5}{8k^5} - \frac{5m^6a^7}{16k^7} + \mathcal{O}(k^{-9}) \quad (2.26)$$

$$(W^{-1}_{k\to\infty})^{(2)} \sim -\frac{(\xi-\frac{1}{6})R\,a^3}{2k^3} + \frac{b_5^{(2)}}{k^5} + \frac{b_7^{(2)}}{k^7} + \mathcal{O}(k^{-9}) \quad (2.27)$$

$$(W^{-1}_{k\to\infty})^{(4)} \sim +\frac{b_5^{(4)}}{k^5} + \frac{b_7^{(4)}}{k^7} + \mathcal{O}(k^{-9}) \quad (2.28)$$

where the coefficients $b_n^{(i)}$ depend on the scale factor a, its derivatives, the mass of the scalar field m and the scalar coupling ξ. We observe that, the higher is the adiabatic order, the more convergent it is. In particular, we see that the ultraviolet divergences of (2.25) are captured in the leading order terms $n = 0, 2$.[4] It can be argued [7] that the UV divergences of the adiabatic expansion are exactly the expected divergences of the observable under investigation. This is so because, for a physically acceptable state (for instance, the *in* or *out* vacua in example 1), its high frequency behaviour has to approach to the Minkowskian behaviour (at an appropriated rate), and the adiabatic expansion, by definition [see Eq. (2.21)], satisfies this requirement. Therefore, this expansion can be regarded as an asymptotic expansion ($k \to \infty$) that is shared by an infinite number of (acceptable) quantum states $h_k(t)$. In conclusion: the adiabatic expansion captures all expected divergences of an observable in its first terms, and hence, it can be used for renormalization.

The adiabatic renormalization method is a very efficient way to obtain the finite expectation values of quadratic field quantities in FLRW universes (and other time-dependent backgrounds). It is a mode-by-mode (under the

[4]Note that $(W_k^{-1})^{(2)}$ contains a UV divergence that was not present in Minkowski space time, where $R = 0$.

integral sign) subtraction process that can be summarized in the following steps:

1. Make an adiabatic expansion of the mode function $h_k(t)$.

2. Compute the spectrum of the observable (e.g., $\langle\phi_k^2\rangle$) in terms of this expansion:
$$|h_k|^2_{\rm ad} \sim (W_k^{-1})^{(0)} + (W_k^{-1})^{(2)} + ... \tag{2.29}$$

3. Subtract the first terms of the adiabatic expansion to the observable
$$\langle\phi^2\rangle_{\rm ren} = \frac{1}{(2\pi)^2 a(t)^3}\int_0^\infty dk\, k^2\Big(|h_k|^2 - (W_k^{-1})^{(0)} - (W_k^{-1})^{(2)}\Big). \tag{2.30}$$
The number of subtractions depends on the scaling dimension of the observable. It is equivalent to the "degree of divergence" of the observable (for instance, 2 for $\langle\phi^2\rangle$ and 4 for $\langle T_{\mu\nu}\rangle$).

4. If the state is acceptable, the observable (e.g., $\langle\phi^2\rangle_{\rm ren}$) is finite at the end of the process.

Here, we have explicitly shown how to compute the renormalized vacuum expectation value of the two-point function. However, the same process applies to compute other relevant observables, as the stress-energy tensor $\langle T^{\mu\nu}\rangle_{\rm ren}$ (in this case, the adiabatic subtractions have to be performed up to and including the 4th adiabatic order). As we can see, the adiabatic regularization method is a very efficient way to compute renormalized quantities.

This method was first proposed for scalar fields in expanding universes in Refs. [29, 30, 31] and further upgraded in Ref. [32] (see Refs. [6, 8, 7] for extended reviews). It was also expanded to spin-$\frac{1}{2}$ in Refs. [61, 62, 63], including the case when a classical scalar background field is also present.

It was adapted for electric homogeneous backgrounds [64, 65, 66], and later improved to make it consistent with gravity [33, 34, 36]. It was also extended to include an arbitrary mass scale μ in the subtraction terms in Ref. [67]. For scalar and spin-$\frac{1}{2}$ fields in FLRW cosmologies it was proved to be equivalent to the DeWitt-Schwinger expansion [44, 45].

This chapter introduced some of the main features of quantum field theory in curved spacetimes. We have analyzed the issue of the vacuum choice in the context of expanding universes and illustrated how this ambiguity leads to an extraordinary consequence: the creation of particles out of the vacuum. We have also presented the adiabatic regularization method as a very efficient procedure to renormalize physical quantities in homogeneous and time-dependent backgrounds. In the following chapters, we will deeply analyze the extension of these ideas to the context of semiclassical electrodynamics. We will see how the adiabatic method can be upgraded in this setup and the main differences with respect to the gravitational case.

Chapter 3

Adiabatic regularization in electromagnetic backgrounds

This chapter aims to explain how the standard gravitational picture for homogeneous and time-dependent backgrounds changes when including a second, homogeneous and time-dependent electromagnetic background field. We work with a quantized spin-$\frac{1}{2}$ in two spacetime dimensions. After presenting the principal features of this model, we explore two potential possibilities to perform an adiabatic expansion of the Dirac modes in terms of the background fields.

In most of the literature about this topic, it was implicitly assumed that the potential vector $A(t)$ is a function of adiabatic order zero (see, for example, [64, 68, 66]). However, we show here that the adiabatic method is consistent if and only if there is an adiabatic hierarchy between the external backgrounds. The scale factor $a(t)$ must be a function of zero adiabatic order, while the potential vector $A(t)$ must be a function of adiabatic order one, as stated in Refs. [33, 34, 35]. We use three different arguments to support this claim. First, we show that only if $A(t)$ is of adiabatic order one the adiabatic prescription is compatible with energy conservation [34]. Second, we show that only for $A(t)$ of adiabatic order one, the adiabatic regularization method can be equivalent to the DeWitt-Schwinger renormalization prescription [36]. Finally, we show that only if $A(t)$ is of adiabatic order one, are the expected quantum anomalies reproduced via the adiabatic prescription. We note that, with some extra assumptions, the validity of the adiabatic order zero choice for $A(t)$ can be recovered in Minkowski spacetime. This is why this important feature has been unnoticed so far in the literature.

3.1 The model

Let us consider a quantized spin-$\frac{1}{2}$ field ψ in a two-dimensional FLRW universe $ds^2 = dt^2 - a(t)^2 dx^2$ coupled with a classical, homogeneous electric field so that $E = E(t)$ in a given reference frame. It can be described in terms of the potential vector $A_\mu = (0, -A(t))$.[1] Since we have included the electromagnetic interaction, we have to deal with two background fields. This fact leads to an ambiguity in the adiabatic regularization prescription that must be solved consistently. The Dirac equation for the spin-$\frac{1}{2}$ field reads

$$(i\underline{\gamma}^\mu D_\mu - m)\psi = 0\,, \tag{3.1}$$

where D_μ is defined as

$$D_\mu \psi = (\nabla_\mu - \Gamma_\mu - iqA_\mu)\psi\,, \tag{3.2}$$

with Γ_μ the spin connection (see Ref. [7]) and $\underline{\gamma}^\mu$ the spacetime-dependent gamma matrices that satisfy the anticomutation relations $\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = 2g^{\mu\nu}$.

[1]This choice of the potential vector corresponds to the Lorenz gauge $\partial_\mu A^\mu = 0$.

They are related with the Minkowskian gamma matrices by $\underline{\gamma}^0 = \gamma^0$ and $a\underline{\gamma}^1 = \gamma^1$. For the case under investigation, it is not difficult to arrive to $\underline{\gamma}^\mu \Gamma_\mu = -\frac{\dot{a}}{2a}\gamma_0$ [33]. From now on, we will use the Weyl representation for the two-dimensional gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

Since the background fields are homogeneous, one can expand the Dirac field in modes as

$$\psi = \int_{-\infty}^{\infty} dk [B_k u_k(t,x) + D^\dagger v_k(x,t)] \,, \tag{3.4}$$

where the two independent spinor solutions can be written as

$$u_k(t,x) \;=\; \frac{e^{ikx}}{\sqrt{2\pi a}} \begin{pmatrix} h_k^I(t) \\ -h_k^{II}(t) \end{pmatrix} \,, \tag{3.5}$$

$$v_k(t,x) \;=\; \frac{e^{-ikx}}{\sqrt{2\pi a}} \begin{pmatrix} h_{-k}^{II*}(t) \\ h_{-k}^{I*}(t) \end{pmatrix} \,, \tag{3.6}$$

and where B_k and D_k are the creation and annihilation operations, that fulfill the usual anticommutation relations. Inserting the mode expansion (3.4) into the Dirac equation, we immediately find

$$\dot{h}_k^I - \frac{i}{a}(k + qA)\, h_k^I - imh_k^{II} = 0\,, \tag{3.7}$$

$$\dot{h}_k^{II} + \frac{i}{a}(k + qA)\, h_k^{II} - imh_k^{I} = 0\,, \tag{3.8}$$

together with the normalization condition

$$|h_k^I|^2 + |h_k^{II}|^2 = 1, \tag{3.9}$$

required to ensure the consistency of the theory.[2]

[2]The normalization condition is needed to recover the usual Dirac product $(u_k, u_{k'}) = \int dxa u_k^\dagger u_{k'} = \delta(k - k')$, $(u_k, v_{k'}) = \int dxa u_k^\dagger v_{k'} = 0$.

In this context, the most relevant physical observables are the two point function $\bar{\psi}\psi$, with $\bar{\psi} = \psi^\dagger\gamma^0$, the electric current $J^\mu = -q\bar{\psi}\gamma^\mu\psi$ and the symmetric stress-energy tensor $T^{\mu\nu} = \frac{i}{4}(\bar{\psi}\gamma^\mu \overleftrightarrow{D}^\nu\psi + \bar{\psi}\gamma^\nu \overleftrightarrow{D}^\mu\psi)$. As in the purely gravitational case, the vacuum expectation values of these observables are plagued by ultraviolet divergences that have to be eliminated in a consistent way. Because of the nature of the problem, the most useful method is the adiabatic regularization prescription, explained in Section 2.2 for scalar fields in FLRW universes. However, a new difficulty arises here: there are two background fields, and an adiabatic hierarchy between them may exist to ensure the consistency of the method. We already know that the scale factor $a(t)$ should be a function of adiabatic order zero. We have two possibilities for the potential vector $A(t)$. It can be a function of adiabatic order zero or adiabatic order one.

$A(t)$ of adiabatic order one

Let us start with the case where $A(t)$ is a function of adiabatic order one. This choice was recently proposed in [33] and analyzed for four-dimensional scalar fields and for a two-dimensional spin-$\frac{1}{2}$ field. Later, it was extended to four-dimensional spin-$\frac{1}{2}$ fields in Minkowski spacetime in Ref. [36]. In our case, and motivated by the Minkowskian solutions

$$h_k^{I,II} = \pm\sqrt{\frac{\omega \mp k}{2\omega}} e^{-i\omega t}\,, \tag{3.10}$$

we propose the following ansatz for the adiabatic expansion of the Dirac modes

$$h_k^I = \sqrt{\frac{\omega - \frac{k}{a}}{2\omega}} F_k(t) e^{-i\int^t \Omega_k(t')dt'}, \quad h_k^{II} = -\sqrt{\frac{\omega + \frac{k}{a}}{2\omega}} G_k(t) e^{-i\int^t \Omega_k(t')dt'}, \tag{3.11}$$

where, $\omega = \sqrt{\frac{k^2}{a^2} + m^2}$ and where the real function Ω_k and the complex functions F_k and G_k can be adiabatically expanded as

$$\Omega(t) = \sum_{n=0}^{\infty} \omega^{(n)}(t), \quad F(t) = \sum_{n=0}^{\infty} F^{(n)}(t), \quad G(t) = \sum_{n=0}^{\infty} G^{(n)}(t). \qquad (3.12)$$

For simplicity, we have omited the subindex k. We note here that, since $A(t)$ is a function of adiabatic order one, it does not appear in the leading order of the adiabatic expansion. Inserting the ansatz (3.11) into the mode equations and the normalization condition (3.9) we end up with the following system of equations

$$\left(\omega - \tfrac{k}{a}\right)\left(\dot{F} - i\Omega F - \frac{i}{a}\left(k + qA\right)F\right) + im^2 G = 0, \qquad (3.13)$$

$$\left(\omega + \tfrac{k}{a}\right)\left(\dot{G} - i\Omega G + \frac{i}{a}\left(k + qA\right)G\right) + im^2 F = 0, \qquad (3.14)$$

$$(\omega - \tfrac{k}{a})|F|^2 + (\omega + \tfrac{k}{a})|G|^2 = 2\omega. \qquad (3.15)$$

Now, introducing the adiabatic expansions (3.12) in the previous system and grouping the terms of the same adiabatic order, we can directly obtain the nth adiabatic order terms from lower adiabatic order terms once the leading order is specified. For $\Omega^{(0)} = \omega$ and $F^{(0)} = G^{(0)} = 1$ we directly obtain (for $n \geq 1$)

$$\begin{aligned} \omega^{(n)} \quad &= \frac{\left(\omega - \frac{k}{a}\right)}{2\omega}\left[\dot{F}_y^{(n-1)} - \sum_{i=1}^{n-1} \omega^{(n-i)} F_x^{(i)} - \frac{qA}{a} F_x^{(n-1)}\right] \qquad (3.16) \\ &+ \frac{\left(\omega + \frac{k}{a}\right)}{2\omega}\left[\dot{G}_y^{(n-1)} - \sum_{i=1}^{n-1} \omega^{(n-i)} G_x^{(i)} + \frac{qA}{a} G_x^{(n-1)}\right] \\ &+ \frac{\dot{a}}{a}\frac{km^2}{4a\omega^3}\left[F_y^{(n-1)} - G_y^{(n-1)}\right], \end{aligned}$$

$$F_x^{(n)} = \frac{1}{2\omega}\left[\dot{F}_y^{(n-1)} - \sum_{i=1}^{n}\omega^{(n-i)}F_x^{(i)} - \frac{qA}{a}F_x^{(n-1)}\right] \tag{3.17}$$
$$-\frac{\left(\omega - \frac{k}{a}\right)}{4\omega}\sum_{i=1}^{n-1}\left(F_x^{(i)}F_x^{(n-i)} + F_y^{(i)}F_y^{(n-i)}\right)$$
$$-\frac{\left(\omega + \frac{k}{a}\right)}{4\omega}\sum_{i=1}^{n-1}\left(G_x^{(i)}G_x^{(n-i)} + G_y^{(i)}G_y^{(n-i)}\right)$$
$$+\frac{\left(\omega + \frac{k}{a}\right)}{4\omega^3}\frac{\dot{a}}{a}F_y^{(n-1)},$$

$$F_y^{(n)} = G_y^{(n)} - \frac{(\omega - k_3)}{m^2}\left[\dot{F}_x^{(n-1)} + \sum_{i=1}^{n-1}\omega^{(n-i)}F_y^{(i)} + \frac{qA}{a}F_y^{(n-1)}\right] \tag{3.18}$$
$$-\frac{\dot{a}}{a}\frac{kF_x^{(n-1)}}{2a\omega^2}.$$

where $F_x = \mathrm{Re}(F)$, $F_y = \mathrm{Im}(F)$, $G_x = \mathrm{Re}(G)$ and $\mathrm{G}_y = Im(G)$. It is easy to see that $G(k, qA)$ satisfies the same equation than $F(-k, -qA)$, hence we take $G^{(n)}(k, qA) = F^{(n)}(-k, -qA)$. We note that there is an ambiguity in the imaginary part of $G^{(n)}$ and $F^{(n)}$ that does not affect our physical observables. For simplicity, and without loss of generality, we have eliminated it by choosing $\mathrm{Im}(G^{(n)}) = -\mathrm{Im}(F^{(n)})$. The origin of the ambiguity has been clarified in [69]. Furthermore, the natural condition $\mathrm{Im}(G^{(n)}) = -\mathrm{Im}(F^{(n)})$ has appeared to be related, at least for a cosmology setup, to an underlying CPT symmetry [70, 71].

As an example, we explicitly give the first next-to-leading order of the adiabatic expansion, namely

$$\omega^{(1)} = \frac{kqA}{a^2\omega}, \tag{3.19}$$

$$F^{(1)} = -\frac{qA(\omega + \frac{k}{a})}{2a\omega^2} - i\frac{k\dot{a}}{4a^2\omega^2}\,, \tag{3.20}$$

$$G^{(1)} = +\frac{qA(\omega - \frac{k}{a})}{2a\omega^2} + i\frac{k\dot{a}}{4a^2\omega^2}\,. \tag{3.21}$$

The next orders are obtained directly from the algorithmic relations above and can be found, for example, in [33].

$A(t)$ of adiabatic order zero

Let us consider now the case where $A(t)$ is a function of adiabatic order zero. This choice was first assumed in [64, 68, 66, 72] to study the backreaction problem in strong electric fields in Minkowski spacetime and in most of the subsequent papers on this topic. In this case, the proposed ansatz to perform the adiabatic expansion is as follows

$$h_k^I = \sqrt{\frac{w-p}{2w}}F(t)e^{-i\int^t \Omega(t')dt'}, \tag{3.22}$$

$$h_k^{II} = -\sqrt{\frac{w+p}{2w}}G(t)e^{-i\int^t \Omega(t')dt'}, \tag{3.23}$$

with $p = \frac{k+qA}{a}$, $w = \sqrt{p^2+m^2}$ and where the real function Ω and the complex functions F and G can be expanded adiabatically as in (3.12). We recall that here p is a function of adiabatic order zero. Therefore, $\dot{p}$ is of adiabatic order 1, $\ddot{p}$ and $\dot{p}^2$ are of adiabatic order two and so on. Inserting the ansatz (3.22) and (3.23) into the mode equations and the normalization condition we arrive at

$$(w-p)\left(\dot{F} - i(\Omega+p)F\right) + im^2G - \frac{m^2\dot{p}}{2w^2}F = 0\,, \tag{3.24}$$

$$(w+p)\left(\dot{G} - i(\Omega-p)G\right) + im^2F + \frac{m^2\dot{p}}{2w^2}G = 0, \tag{3.25}$$

$$(w-p)|F|^2+(w+p)|G|^2 \;=\; 2w\,. \qquad (3.26)$$

Now, introducing the adiabatic expansions (3.12) in the previous system and fixing the leading order to $\omega^{(0)}=w$ and $F^{(0)}=G^{(0)}=1$, we can obtain the next-to-leading orders by iteration. The solutions to the (algebraic) system of equations for $\Omega^{(n)}$, $F_x^{(n)}$ and $F_y^{(n)}$ are given by (3.16), (3.17) and (3.18) with the changes $\omega\to w$, $\frac{k}{a}\to p$, $A\to 0$ and $\frac{\dot a}{a}\to -\dot p$. As in the previous case, we take $G^{(n)}(p)=F^{(n)}(-p)$. We also find an ambiguity in the imaginary part of $F^{(n)}$ and $G^{(n)}$ that can be easily resolved by imposing the condition $\mathrm{Im}(G^{(n)})=-\mathrm{Im}(F^{(n)})$. The first next-to-leading order terms of the adiabatic expansion when $A(t)$ is considered a function of adiabatic order zero are

$$\omega^{(1)}=0\,,\quad F^{(1)}=i\frac{\dot p}{4w^2}\,,\quad G^{(1)}=-i\frac{\dot p}{4w^2}\,. \qquad (3.27)$$

We have presented two alternative and apparently non-equivalent ways of performing the adiabatic expansion of the Dirac field modes when two background fields are present. In the next section, we argue that $A(t)$ *must* be a function of adiabatic order one. We use three different arguments.

3.2 Adiabatic orders and consistency of the method

Energy conservation

When we assign an adiabatic order to the background fields, we are, at the same time, defining the leading order of the adiabatic expansion. We have already seen that when $A(t)$ is considered a function of adiabatic order zero, the leading order of adiabatic expansion reads $\omega_k^{(0)}=\sqrt{\frac{(k+qA)^2}{a^2}+m^2}$. On the other hand, when $A(t)$ is interpreted as a function of adiabatic order one, the leading order is $\omega_k^{(0)} = \sqrt{\frac{k^2}{a^2} + m^2}$. These different choices result in two different expressions for the renormalized vacuum expectation values of physical operators. For example, for the electric current $\langle J^1 \rangle_{\rm ren}$ we find [35]

$$\left\langle J^1 \right\rangle_{ren}^{A\sim O(0)} = q \int \frac{dk}{2\pi a} \left(\left|h_k^{II}\right|^2 - \left|h_k^{I}\right|^2 - \frac{k + qA}{a\sqrt{(k+qA)^2/a^2 + m^2}} \right), \quad (3.28)$$

$$\left\langle J^1 \right\rangle_{ren}^{A\sim O(1)} = q \int \frac{dk}{2\pi a} \left(\left|h_k^{II}\right|^2 - \left|h_k^{I}\right|^2 - \frac{k}{a\omega} - \frac{m^2 qA}{a\omega^3} \right), \quad (3.29)$$

with $\omega^2 = \frac{k^2}{a^2} + m^2$. Since we are working in a two-dimensional setting, the subtractions for the electric current have to be performed up to and including the first adiabatic order. These two vacuum expectation values seem different. However, it can be easily shown that

$$\begin{aligned} \Delta \left\langle J^1 \right\rangle_{\rm ren} &= \left\langle J^1 \right\rangle_{\rm ren}^{A\sim O(0)} - \left\langle J^1 \right\rangle_{\rm ren}^{A\sim O(1)} \\ &= q \int_{-\infty}^{-\infty} \frac{dk}{2\pi a} \left[\frac{k}{a\omega} - \frac{(k+qA)}{a\sqrt{(k+qA)^2/a^2 + m^2}} + \frac{m^2 qA}{a\omega^3} \right] = 0 \,, \end{aligned} \quad (3.30)$$

giving the impression that both choices are completely equivalent. However, as we will shortly see, this is not the case.

Consider now the backreaction effect of the created particles on the background field. This phenomenon can be studied via the semiclassical Maxwell equation $\nabla_\mu F^{\mu\nu} = \langle J^\nu \rangle_{\rm ren}$,[3] In our system, these equations reduce to a single equation

$$\dot{E} = -\langle J^1 \rangle_{\rm ren} \,. \quad (3.31)$$

In this circumstances, the energy of the system has to be conserved, namely

$$\nabla_\mu \left\langle T^{\mu\nu} \right\rangle_{\rm ren} + \nabla_\mu T^{\mu\nu}_{\rm elec} = 0 \,, \quad (3.32)$$

[3] This effect will be studied in detail in Chapter 5.

where $T^{elec}_{\mu\nu} = \frac{1}{2}E^2 g_{\mu\nu}$. The conservation of the 0th component can be expanded as

$$\nabla_\mu \langle T^{\mu 0}\rangle_{\rm ren} + \nabla_\mu T^{\mu 0}_{\rm elec} = \partial_0 \langle T_{00}\rangle_{\rm ren} + \frac{\dot a}{a}\langle T_{00}\rangle_{\rm ren} + \frac{\dot a}{a^3}\langle T_{11}\rangle_{\rm ren} + \partial_0 T^{\rm elec}_{00} = 0 \,. \tag{3.33}$$

If $A(t)$ is of adiabatic order one, the non-vanishing components of the renormalized stress-energy tensor are given by [35]

$$\langle T_{00}\rangle_{\rm ren} = \frac{1}{2\pi a}\int_{-\infty}^{\infty} dk\ i\left[h_k^{II}\dot h_k^{II*} + h_k^I \dot h_k^{I*}\right] + \omega + \frac{kqA}{\omega} + \frac{m^2 q^2 A^2}{2a^2\omega^3} - \frac{k^2 m^2 \dot a^2}{8a^4\omega^5}\,, \tag{3.34}$$

$$\langle T_{11}\rangle_{\rm ren} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk (k+qA)\left(|h_k^I|^2 - |h_k^{II}|^2\right) + \frac{k^2}{a\omega} + \frac{km^2qA}{a\omega^3} + \frac{kqA}{a\omega} - \frac{m^4\ddot a}{4\omega^5}\frac{m^2\ddot a}{4\omega^3} + \frac{5m^6\dot a^2}{8a\omega^7} - \frac{3m^4\dot a^2}{4a\omega^5} + \frac{m^2\dot a^2}{8a\omega^3} + \frac{3m^4q^2A^2}{2a\omega^5} - \frac{m^2q^2A^2}{2a\omega^3}\,. \tag{3.35}$$

We note that in two dimensions, the stress-energy tensor must be renormalized by subtracting up to and including the terms of adiabatic order two. It is strictly required to guarantee the consistency of the renormalization procedure in curved spacetimes [7]. In Table 3.1 we show the number of adiabatic orders that have to be subtracted to obtain finite quantities for both the stress-energy tensor and the electric current in different spacetime dimensions. Inserting Eqs. (3.34) and (3.35) into the conservation equation we obtain, after some algebra,

$$\nabla_\mu \langle T^{\mu 0}\rangle_{\rm ren} + \nabla_\mu T^{\mu 0}_{\rm elec} = \frac{\dot A}{a}\left(\frac{\ddot A}{a} - \frac{\dot A \dot a}{a^2} - \langle J^1\rangle_{\rm ren}\right) = 0\,, \tag{3.36}$$

where the term in parentheses is precisely the semiclassical Maxwell equation (3.31) with $\langle J^1\rangle_{\rm ren} \equiv \langle J^1\rangle_{\rm ren}^{A\sim O(1)}$. However, if we repeat the same procedure with $A(t)$ of adiabatic order zero (i.e., compute the components of the stress-energy tensor and check the conservation equation), we see that the energy is no longer conserved

$$\nabla_\mu \left\langle T^{\mu 0} \right\rangle_{\rm ren} + \nabla_\mu T^{\mu 0}_{\rm elec} = \frac{\dot{A}}{a} \langle J^1 \rangle^{(2)} \neq 0 \,, \tag{3.37}$$

where $\langle J^1 \rangle^{(2)}$ is the second adiabatic order of the electric current $\left\langle J^1 \right\rangle_{\rm ren}^{A \sim O(0)}$, that cannot be properly absorbed into the renormalization subtractions [note that this implicitly means $\Delta\langle T_{00}\rangle_{\rm ren} \neq 0$]. This argument was first developed for two-dimensional scalar fields in Ref. [34] and then further extended to two-dimensional fermions in [35]. It has been also reanalyzed for scalar fields propagating in d spacetime dimensions [73].

As a final comment, we would like to stress that in absence of gravity both choices turn out to be equivalent. It is because when the gravitational field is not present, we have some extra freedom in the regularization procedure. In particular, for $A(t) \sim O(0)$, the stress-energy tensor can be renormalized by subtracting only the 0th adiabatic order (instead of the second adiabatic order as in curved spacetime). When doing so, the conservation of the stress-energy tensor is restored, and the equivalence between the subtraction terms of the two possibilities [see Eq. (3.30)] holds for the main observables of the theory. In Table 3.2 we show the number of adiabatic orders that have to be subtracted to obtain finite values for both the stress-energy tensor and the electric current when $A(t)$ is of adiabatic order zero and $a(t) = 1$. Although we have explicitly discussed the equivalence for spin-$\frac{1}{2}$ fields in two dimensions, it also holds in four spacetime dimensions [36].

Required subtraction order in presence of gravity	$d=4$	$d=3$	$d=2$
$\langle J^\mu \rangle_{\mathrm{ren}}$	3	2	1
$\langle T^{\mu\nu} \rangle_{\mathrm{ren}}$	4	3	2

Table 3.1: Required number of adiabatic subtractions for the stress-energy tensor and the electric current in curved spacetimes and for different spacetime dimensions.

Required subtraction order without gravity for $A(t) \sim \mathcal{O}(0)$	$d=4$	$d=3$	$d=2$
$\langle J^\mu \rangle_{\mathrm{ren}}$	2	1	0
$\langle T^{\mu\nu} \rangle_{\mathrm{ren}}$	2	1	0

Table 3.2: Required number of adiabatic subtractions for the stress-energy tensor and the electric current in different spacetime dimensions, and for when $A(t)$ of adiabatic order zero in Minkowski spacetime $a(t) = 1$.

DeWitt coefficients

The second argument that we develop here is that only when $A(t)$ is considered a function of adiabatic order one can the adiabatic expansion be equivalent to the DeWitt-Schwinger renormalization scheme.

The DeWitt-Schwinger method is a point-splitting renormalization technique based on the proper-time expansion of the Feynman Green's function $G_F(x, x')$ for general spacetimes. This asymptotic expansion identifies consistently the divergent terms of $G_F(x, x')$ and therefore it can be used for renormalization [74, 75, 76] (see [77] for its momentum representation version). In Ref. [44] it was shown that the adiabatic and the DeWitt-Schwinger renormalization schemes are equivalent for scalar and spin-$\frac{1}{2}$ fields in four-dimensional FLRW backgrounds. The same equivalence holds in two spacetime dimensions. In terms of the spin-$\frac{1}{2}$ two-point function (at coincidence), this equivalence reads

$$^{(2n)}\langle\bar{\psi}\psi\rangle_{DS} = {}^{(2n)}\langle\bar{\psi}\psi\rangle_{\mathrm{Ad}} \tag{3.38}$$

where the super index refers to the (adiabatic) order of the expansion, meaning that the equality is satisfied order by order. One may expect the same equivalence when including a time-dependent electromagnetic background field described by the potential vector $A_\mu = (0, A(t))$. Here, we are not going to prove this expected equivalence but to show that it only holds if $A(t)$ is of adiabatic order one.

The proper-time (DeWitt-Schwinger) asymptotic expansion of the two-point function for a two-dimensional spin-$\frac{1}{2}$ field reads

$$^{(2n)}\langle\bar{\psi}\psi\rangle_{DS} = -im\int_0^\infty \frac{ds}{(4\pi i s)} e^{-im^2 s}\sum_{k=0}^{n} \mathrm{tr}E_k(x)(is)^k\,. \tag{3.39}$$

We note that the DeWitt-Schwinger coefficient E_n is a coefficient of adiabatic order $2n$, this is why we have included the super-index $2n$ in $\langle\bar{\psi}\psi\rangle_{DS}$. The first coefficients of the expansion are, in a covariant form [78, 79]

$$E_0 = I\,, \qquad E_1 = \frac{1}{6}RI - Q\,, \tag{3.40}$$

and

$$\begin{aligned} E_2 \;=\; & \left(-\frac{1}{30}\Box R + \frac{1}{72}R^2 - \frac{1}{180}R^{\mu\nu}R_{\mu\nu} + \frac{1}{180}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right) I \\ & + \frac{1}{12}W^{\mu\nu}W_{\mu\nu} + \frac{1}{2}Q^2 - \frac{1}{6}RQ + \frac{1}{6}\Box Q\,, \end{aligned} \tag{3.41}$$

where $W_{\mu\nu} = -iqF_{\mu\nu}I - \frac{1}{4}R_{\mu\nu\rho\sigma}\underline{\gamma}^\rho\underline{\gamma}^\sigma$ and $Q = \frac{1}{4}RI - \frac{i}{2}qF_{\mu\nu}\underline{\gamma}^\mu\underline{\gamma}^\nu$. For $n \geq 1$ the proper-time integrals are finite and can be computed without any difficulty.

On the other hand, the adiabatic expansion of the two-point function is given by

$$
{}^{(n)}\langle\bar{\psi}\psi\rangle_{Ad} = \int_{-\infty}^{\infty}\frac{dk}{2\pi a}\sum_{j=0}^{n}\left(h_k^I h_k^{*II} + h_k^{II} h_k^{*I}\right)^{(j)} = \int_{-\infty}^{\infty}\frac{dk}{2\pi a}\sum_{j=0}^{n}(\bar{\psi}\psi)_k^{(j)}\,, \tag{3.42}
$$

where $\left(h_k^I h_k^{*II} + h_k^{II} h_k^{*I}\right)^{(j)}$ represents the jth adiabatic order of the mode expansion of the two point function, that can be directly obtained from the adiabatic expansion of the mode functions h_k^I and h_k^{II} [see Eqs. (3.11) and (3.12)]. The first orders of this expansion are (assuming $A(t)$ of adiabatic order one)

$$
(\bar{\psi}\psi)_k^{(0)} = -\frac{m}{\omega}\,, \tag{3.43}
$$

$$
(\bar{\psi}\psi)_k^{(1)} = \frac{mkqA}{a^2\omega^3}\,, \tag{3.44}
$$

$$
(\bar{\psi}\psi)_k^{(2)} = \frac{3m^3q^2A^2}{2a^2\omega^5} - \frac{mq^2A^2}{a^2\omega^3} + \frac{5\dot{a}^2m^5}{8a^2\omega^7} - \frac{7\dot{a}^2m^3}{8a^2\omega^5} + \frac{\dot{a}^2m}{4a^2\omega^3} - \frac{m^3\ddot{a}}{4a\omega^5} + \frac{m\ddot{a}}{4a\omega^3}\,. \tag{3.45}
$$

For $n \geq 1$, the momentum integral of the adiabatic expansion is finite and can be easily integrated and directly compared with the deWitt coefficients. Therefore, and for simplicity, we restrict our comparison to the finite terms of both asymptotic expansions. It is direct to see that, for $A(t)$ of adiabatic order one

$$
\langle\bar{\psi}\psi\rangle^{(1)} = 0\,, \tag{3.46}
$$

$$
\langle\bar{\psi}\psi\rangle^{(2)} = \frac{1}{4\pi m}\left(\frac{\ddot{a}}{3a}\right) = -\frac{\mathrm{tr}\,E_1}{4\pi m}\,, \tag{3.47}
$$

$$
\langle\bar{\psi}\psi\rangle^{(3)} = 0\,, \tag{3.48}
$$

$$
\langle\bar{\psi}\psi\rangle^{(4)} = \frac{1}{4\pi m^3}\left(-\frac{\dot{a}^2\ddot{a}}{30a^3} + \frac{\ddot{a}^2}{15a^2} - \frac{a^{(4)}}{30a} + \frac{2q^2\dot{A}^2}{3a^2} + \frac{a^{(3)}\dot{a}}{30a^2}\right) \tag{3.49}
$$

$$= \quad -\frac{\operatorname{tr} E_2}{4\pi m^3} \,.$$

In other words, when $A(t)$ is of adiabatic order one we obtain the expected equivalence between the DeWitt-Schwinger and adiabatic renormalization schemes (3.38).

On the contrary, if $A(t)$ is of adiabatic order zero we find

$$\langle \bar{\psi}\psi \rangle^{(2)} = \frac{\ddot{a}}{12\pi a m} + \frac{q\dot{A}^2}{6\pi m^3 a^2} \neq -\frac{\operatorname{tr} E_1}{4\pi m} \,. \tag{3.50}$$

and the equivalence is explicitly broken.

Quantum anomalies

The third argument that we point out is that only if $A(t)$ is of adiabatic order one we can reproduce the trace anomaly when both the gravitational and the electromagnetic fields are present. This conclusion can be directly inferred from the previous subsection. However, and because of the importance of quantum anomalies, we find it convenient to treat it in a separated subsection.

The existence of quantum anomalies, i.e., the breaking of a classical symmetry in the quantized version of the theory, is one of the main results of the Theory of Quantum Fields. In this framework, the removal of the intrinsic UV divergences that appear through the process of renormalization can give rise to finite and *unambiguous* results known as quantum anomalies.

In this sense, for a new renormalization method to be consistent, it has to be able to reproduce these expected quantum anomalies. In particular, it should be able to reproduce the trace anomaly. For a (two-dimensional)

Dirac field, the trace of the stress-energy tensor can be written as

$$T_\mu^\mu = m\bar{\psi}\psi \,. \tag{3.51}$$

It is easy to see that in the classical theory the trace vanishes in the massless limit. However, this is not the case after quantization. For a renormalization prescription to be consistent, the subtractions must be performed up to and including a given order. In the particular case of adiabatic regularization in two dimensions, we have to subtract the zeroth, the first, and the second adiabatic order when we working with the stress-energy tensor. However, we have already seen that the second adiabatic order of the two-point function $m\langle\bar{\psi}\psi\rangle^{(2)}$ turns out to be finite [see Eq. (3.47)] and independent of the mass. It means that, in the massless limit and for $A(t)$ of adiabatic order one, we find

$$\langle T_\mu^\mu\rangle_{\rm ren}^{m=0} = -\lim_{m\to 0} m\langle\bar{\psi}\psi\rangle^{(2)} = -\frac{\ddot{a}}{12\pi a} = -\frac{R}{24\pi} \,. \tag{3.52}$$

This result coincides with the expected two-dimensional trace anomaly [80]. On the contrary, if $A(t)$ is of adiabatic order zero, we obtain a term proportional to (3.50), which is incompatible with the trace anomaly unless we do not consider the electromagnetic background field (i.e. $A(t) = \dot{A}(t) = 0$). This result can also be obtained for scalar fields in two, and four spacetime dimensions [33]. In summary, only when $A(t)$ is of adiabatic order one the trace anomaly is recovered.

Chapter 4

Particle creation and quantum anomalies

In the previous chapter, we studied how to compute renormalized quantities in the context of Quantum Field Theory under (time-dependent) external conditions. This is an unavoidable requirement to properly understand the underlying physics behind the semiclassical theory. We devote this chapter to studying the astonishing consequences that this framework naturally provides. We investigate two fundamental issues, i) particle creation and ii) quantum anomalies.

In the context of particle creation, we analyze the spontaneous creation of spin-$\frac{1}{2}$ particles caused by the time evolution of an electromagnetic background field in two spacetime dimensions in terms of the well-known Bogoliugov transformations [7, 6]. The frequency-mixing approach to predict particle creation was first discovered in the context of isotropically expanding universes [9, 10], and quantities mathematically equivalent to the so-called Bogoliubov coefficients were independently introduced.

As happens in expanding universes, time-varying electric fields can also create particles [81] (see also [18]). This effect is of particular interest since it might be experimentally detected in a (not far) future [82]. It can be also relevant in astrophysical and cosmological scenarios [83, 84, 85], and in non-equilibrium processes [86, 87]. In the first part of the chapter, we explain how to characterize this effect and some properties of the creation process. For cosmological backgrounds, it was shown that the particle number has an important property: it is an adiabatic invariant [6, 88]. We expose that, although for a massive spin-$\frac{1}{2}$ field propagating in an electromagnetic background this statement is still true, the adiabatic invariance of the particle number is broken when the Dirac field is massless. This breaking is directly linked with the chiral anomaly, that is studied in the second part of the chapter. This content is based on Ref. [39].

As we have previously stressed, the existence of quantum anomalies is one of the most important results of the theory of quantum fields. We can construct a very simple example of these quantum anomalies in two-dimensional electrodynamics. A massless Dirac field interacting with an external electromagnetic field has a chiral anomaly [38],

$$\partial_\mu \left\langle J^\mu_A \right\rangle_{\text{ren}} = -\frac{q}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} \, . \tag{4.1}$$

This was first discovered in the analysis of a four-dimensional quantized Dirac field ψ in the presence of an electromagnetic background [25, 89, 26]. In the second part of the chapter, we analyze how this quantum anomaly naturally emerges within the adiabatic framework. We point out the existence of a new quantum anomaly that materializes in the same context as the chiral anomaly: the translational anomaly. This analysis is based on Ref. [40]. We also explain in detail the close relationship between particle creation and the chiral anomaly in QED_2. To finish the chapter, we briefly show how, in four spacetime dimensions, the adiabatic invariance of the particle number is also broken when the four-dimensional chiral anomaly enters the game.

Before starting with the main ideas of the chapter, let us illustrate the idea of particle creation with a straightforward example. Consider a spin-$\frac{1}{2}$ field ψ propagating in an electric background $E(t)$ in two spacetime dimensions, that tends to zero at both early and late times. For simplicity we can consider the Sauter pulse [90]

$$E(t) = -E_0 \cosh^{-2}(\omega_0 t)\,. \tag{4.2}$$

For the Dirac field, and because the electric background is asymptotically zero at early and late times, we can naturally choose as initial state the Minkowski vacuum. At early times ($t \to -\infty$), all physical observables are zero. In particular, for the particle number and the electric current we find $\langle N_{\rm in}\rangle = 0$ and $\langle J^1\rangle_{\rm ren} = 0$. As time evolves, the electric field increases and then decreases, returning to its initial value at late times. At this point, a natural question arises: what is the late times' value of these observables? As we will shortly see they are not zero,

$$\langle N_{\rm out}\rangle \neq 0\,, \qquad \langle J^1\rangle_{\rm ren} \neq 0\,. \tag{4.3}$$

It means that particles have been created out of the vacuum. In the following sections, we see how these ideas emerge in the semiclassical framework and the close relation between the chiral anomaly and particle production.

4.1 Particle creation

To start with our analysis, let us characterize the process of particle creation for spin-$\frac{1}{2}$ particles interacting with homogeneous, time-dependent electromagnetic backgrounds $E(t)$ described by the potential vector $A_\mu = (0, A(t))$. We use the model described in Chapter 3 with $a(t) = 1$. The spinor field ψ satisfy the Dirac equation $(i\gamma^\mu D_\mu - m)\psi = 0$, and because of the nature of the background, it can be expanded in modes as

$$\psi = \int_{-\infty}^{\infty} dk[B_k u_k(t,x) + D_k^\dagger v_k(x,t)]\,, \tag{4.4}$$

where B_k, $B_k^\dagger$, D_k, and $D_k^\dagger$ are the usual creation and anhilation operators. The modes u_k and v_k are parametrized as in (3.5) and (3.6) in terms of two time-dependent functions h_k^I and h_k^{II} satisfying Eqs. (3.7) and (3.8). With these choices, we are implicitly choosing a basis of solutions to expand the Dirac field. However, this choice is not unique, and we can expand the Dirac field in terms of a new basis

$$\psi = \int_{-\infty}^{\infty} dk[b_k U_k(t,x) + d_k^\dagger V_k(x,t)]\,, \tag{4.5}$$

with b_k, $b_k^\dagger$, d_k and $d_k^\dagger$ the creation and annihilation operators associated with the new basis and where

$$U_k(t,x) = \frac{e^{ikx}}{\sqrt{2\pi}} \begin{pmatrix} g_k^I(t) \\ -g_k^{II}(t) \end{pmatrix}, \qquad V_k(t,x) = \frac{e^{-ikx}}{\sqrt{2\pi}} \begin{pmatrix} g_{-k}^{II*}(t) \\ g_{-k}^{I*}(t) \end{pmatrix}, \tag{4.6}$$

with g_k^I and g_k^{II} satisfying also (3.7) and (3.8). The (linear) relation that must exist between these two bases can be parametrized by means of the Bogoliugov coefficients α_k and β_k (see, for example [7]). In terms of the creation and annihilation operators, the relation is given by

$$\begin{aligned} b_k &= \alpha_k B_k + \beta_k^* D_{-k}^\dagger\,, \\ d_k &= \alpha_{-k} D_k - \beta_{-k}^* B_{-k}^\dagger\,. \end{aligned} \tag{4.7}$$

while, in terms of the mode functions it reads

$$h_k^I = \alpha_k g_k^I - \beta_k g_k^{II*}\,, \tag{4.8}$$

$$h_k^{II} = \alpha_k g_k^{II} + \beta_k g_k^{I*}\,. \tag{4.9}$$

The normalization condition (3.9) imposes the following relation between the Bogoliubov coefficients

$$|\alpha_k|^2 + |\beta_k|^2 = 1\,. \tag{4.10}$$

Let us now give some physical meaning to these two different bases. Consider a bounded electric profile. It is characterized by an electric field that is asymptotically zero $E(t \to \pm\infty) \sim 0$ and by a potential vector is asymptotically bounded. For simplicity, we can take $A(t \to -\infty) \sim 0$ and $A(t \to \infty) \sim A_0$. An example of this type of electric profile is the Sauter pulse (see, for example, Figure 4.1 in the next section). We can fix the modes $h_k^{I,II}$ by requiring that they behave as positive-frequency solutions at early times[1], namely

$$h_k^{I,II}(t \to -\infty) \sim \pm\sqrt{\frac{\omega_{\text{in}} \mp k}{2\omega_{\text{in}}}}e^{-i\omega_{\text{in}}t}\,. \tag{4.11}$$

where $\omega_{\text{in}} = \omega = \sqrt{k^2 + m^2}$. This is the natural solution at early times. On the other hand, we can fix the modes $g_k^{I,II}$ by requiring that they behave as a positive-frequency solutions at late times, that is,

$$g_k^{I,II}(t \to \infty) \sim \pm\sqrt{\frac{\omega_{\text{out}} \mp (k + qA_0)}{2\omega_{\text{out}}}}e^{-i\omega_{\text{out}}t}\,. \tag{4.12}$$

where $\omega_{\text{out}} = \sqrt{m^2 + (k + qA_0)^2}$. The interesting point is that the solutions that at early times behave as positive-frequency solutions (that is, $h_k^{I,II}$) do not evolve into positive-frequency solutions at late times, but into a linear

[1]At early and late times the background is Minkowski-like because of the characteristics of the electric profile.

combination of positive- and negative-frequency solutions:

$$h_k^{I,II}(t \to \infty) \sim \pm\alpha_k \sqrt{\frac{\omega_{\rm out} \mp (k+qA_0)}{2\omega_{\rm out}}} e^{-i\omega_{\rm out}t} + \beta_k \sqrt{\frac{\omega_{\rm out} \pm (k+qA_0)}{2\omega_{\rm out}}} e^{i\omega_{\rm out}t}. \tag{4.13}$$

That is, a linear combination of the $g_k^{I,II}$ solutions, as expected from Eqs. (4.8) and (4.9). The advantage of this setup is that now we can give a physical meaning to the Bogoliugov coefficients. Consider the vacuum characterized by the positive-frequency solutions at early times $|0\rangle$. Note that, by definition, the vacuum expectation value of the particle number density at early times $\langle N_k^{\rm in}\rangle = \langle B_k^\dagger B_k\rangle + \langle D_k^\dagger D_k\rangle$ is zero, $\langle N_k^{\rm in}\rangle = 0$. At late times, we can also define the number density operator $N_k^{\rm out} = b_k^\dagger b_k + d_k^\dagger d_k$, and in terms of the Bogoliugov coefficients we easily find

$$\langle N_{\rm out}\rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \langle N_k^{\rm out}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(|\beta_k|^2 + |\beta_{-k}|^2\right), \tag{4.14}$$

where $|\beta_k|^2$ accounts for particles and $|\beta_{-k}|^2$ for antiparticles. Furthermore, at late times it is also possible to obtain an approximated expression for the renormalized value of the electric current in terms of the Bogoliugov coefficients (for a detailed computation see Ref. [39])

$$\langle J^1\rangle_{\rm ren} \sim \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{k}{\omega}\left(|\beta_k|^2 - |\beta_{-k}|^2\right). \tag{4.15}$$

4.2 Breaking of the adiabatic invariance

So far, we have acquired the necessary tools to study a very interesting property of the particle number that holds in a cosmological context: it is an adiabatic invariant. It means that when the expansion of the universe is adiabatic [see Eq. (2.18)], namely $\frac{\dot{a}}{a} \to 0$, particles are not created, $|\beta_k|^2 \to 0$. It happens even if the total expansion rate $a_{\rm final}/a_{\rm initial}$ is large and irrespective of the value of the mass of the created particles.

In a cosmological setup, an adiabatic expansion of a two-dimensional universe can be characterized by a conformal scale factor of the form $a(\tau)^2 = 1+B(1+\tanh(\rho\tau))$, where τ is the conformal time $d\tau = a^{-1}dt$, ρ is the adiabaticity parameter and B is a positive constant. The adiabatic limit of the scale factor is found when $\rho \to 0$, which corresponds to an infinitely slow expansion. For this scale factor, the square of the total expansion rate is $a^2_{\text{final}}/a^2_{\text{initial}} = (1+2B)$ is independent of ρ. On the other hand, the (conformal) Hubble rate is $\frac{a'}{a} = \frac{1}{2a^2}B\rho\cosh(\rho t)^{-2}$. We easily see that it tends to zero for $\rho \to 0$. The parameter ρ characterizes a family of scale factors that differ in their instantaneous expansion rate (a'/a) but share the same total expansion rate. In this context, is very easy to see that, when computing the (late times) particle production associated with a scalar field propagating in this type of universe $|\beta_k|^2$, it vanishes in the adiabatic limit

$$\lim_{\rho\to 0} |\beta_k|^2 \to 0\,, \tag{4.16}$$

regardless of the value of the mass of the created particles. For a detailed analysis of this example, see Refs. [6, 39].

In this section, we extend the adiabatic analysis to electromagnetic backgrounds. As in the previous section, we work with spin-$\frac{1}{2}$ fields in two spacetime dimensions. We find this case of particular interest because for massless Dirac fields, there is a chiral anomaly (4.1) that directly influences the physics of the system. One may think that if the chiral charge is not conserved, therefore, particles should be necessarily created to produce the (expected) quantum anomaly. Consequently, one could expect a breaking of the adiabatic invariance in this scenario: massless particles should be created even if the background change rate is very slow.

Let us test this statement with a very illustrative example. Consider a Sauter-type electromagnetic background characterized by the bounded potential

$$A(t) = \frac{A_0}{2}(1 + \tanh \rho t)\,. \tag{4.17}$$

Therefore, the electric field is as follows

$$E(t) = -\frac{\rho A_0}{2} \cosh^{-2}(\rho t)\,. \tag{4.18}$$

The potential vector $A(t)$ plays a similar role as the scale factor $a(\tau)^2$ in the cosmological example. The electric profile is the well-known Sauter pulse, but now it is expressed in such a way that it allows to study the adiabatic limit directly in terms of the adiabatic parameter ρ. In Figure 4.1 are represented the potential vector $A(t)$ and the electric field $E(t)$ for different values of the adiabatic parameter ρ. We see that the potential vector is bounded at early and late times. The adiabatic limit is an extremely slow time evolution for the potential vector, which is characterized by the limit $\rho \to 0$. We note that, although $E(t) \to 0$ when $\rho \to 0$, the integral

$$\int_{-\infty}^{+\infty} E_{\rho_1}(t)dt = \int_{-\infty}^{+\infty} E_{\rho_2}(t)dt = cte = -qA_0\,. \tag{4.19}$$

remains constant: when the time evolution is slow, the effect of the electric field lasts more in time. This is what properly characterizes the adiabatic limit.

Let us study now the spin-$\frac{1}{2}$ particle number at late times. For this electric profile, the mode equations (3.7) and (3.8) can be easily solved in terms of the hypergeometric functions $F(a, b, c; x)$ [91]. Imposing the early-times asymptotic condition for the modes (4.11) we arrive to

$$h_k^I(t) = \sqrt{\frac{\omega_{\rm in} - k}{2\omega_{\rm in}}} \left(\frac{A(t)}{A_0}\right)^{-i\frac{\omega_{in}}{2\rho}} \left(1 - \frac{A(t)}{A_0}\right)^{i\frac{\omega_{\rm out}}{2\rho}}$$

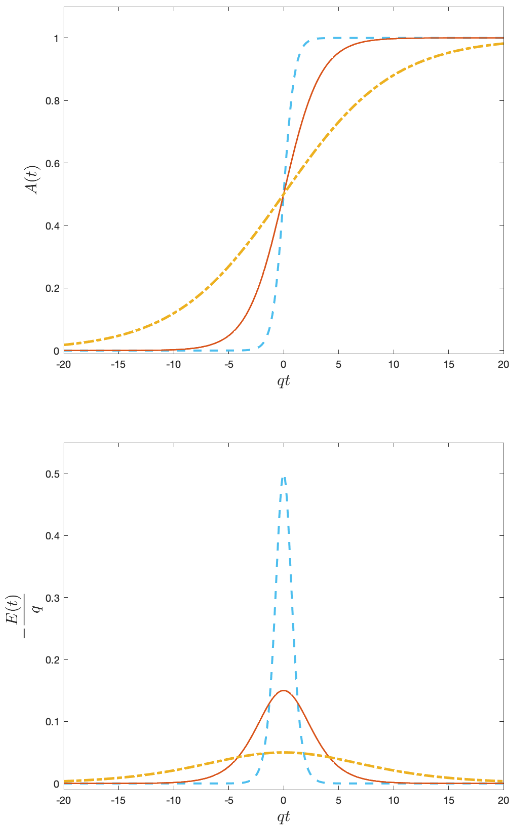

Figure 4.1: Potential vector (up) and electric field (down) for the Sauter-type profile and for $A_0 = 1$. We have represented the time evolution of this profile for three different values of the adiabatic parameter ρ/q. The blue, dashed, line represents the case $\rho/q = 1$, the orange line represents $\rho/q = 0.3$ and the yellow, dashed-dotted line represents $\rho/q = 0.1$.

$$\times F\left(i\frac{\omega_- + qA_0/2}{\rho}, 1 + i\frac{\omega_- - qA_0/2}{\rho}, 1 - i\frac{\omega_{\rm in}}{\rho}; \frac{A(t)}{A_0}\right), \quad (4.20)$$

where $\omega_{\rm in}^2 = m^2 + k^2$, $\omega_{\rm out}^2 = m^2 + (k + qA_0)^2$ and $\omega_\pm^2 = \frac{1}{2}(\omega_{\rm out} \pm \omega_{\rm in})$. The solution for h_k^{II} is the same with the changes $k \to -k$ and $A_0 \to -A_0$. Note that the sign of the quotient $A(t)/A_0$ remains unchanged.

As explained in Section 4.1, at late times the mode functions evolve into a superposition of positive- and negative-frequency solutions [see Eq. (4.13)], where the coefficient β_k accounts for the created particles. Given the exact solution for $h_k^{I,II}$, and the asymptotic form of the late-times modes (4.12) we can use Eq. (4.8) [at late-times] to isolate β_k. After some algebra we find

$$|\beta_k|^2 = \frac{\cosh(\pi \frac{qA_0}{\rho}) - \cosh(2\pi \frac{\omega_-}{\rho})}{2\sinh(\pi \frac{\omega_{\rm in}}{\rho})\sinh(\pi \frac{\omega_{\rm out}}{\rho})}. \quad (4.21)$$

We can now proceed to compute the adiabatic limit $\rho \to 0$. For massive fermions we find that when $\rho \to 0$ the coefficient $|\beta_k|^2$ behaves as

$$|\beta_k|^2 \sim e^{-\frac{\pi}{\rho}\delta} \quad (4.22)$$

where $\delta = 2\omega_+ - |qA_0|$. We note that this function has a minimum at $k = -\frac{qA_0}{2}$, with value $\delta_{\rm min} = \sqrt{(qA_0)^2 + 4m^2} - |qA_0| > 0$. Therefore,

$$\lim_{\rho \to 0} |\beta_k|^2 \to 0\,. \quad (4.23)$$

That is, massive particles are not created in the adiabatic limit. This picture completely changes when considering massless particles. In this case, and for an arbitrary ρ, we find

$$\lim_{m \to 0} |\beta_k|^2 = 1\,, \qquad \text{for}\, k \in (-qA_0, 0)\,. \quad (4.24)$$

and zero elsewhere. We say then that the adiabatic invariance of the particle number is broken in the massless case. The (late-times) particle number becomes (4.14)

$$\langle N^{\rm out}\rangle = \int_{-\infty}^{\infty}\frac{dk}{2\pi}(|\beta_k|^2+|\beta_{-k}|^2) = \int_{-|qA_0|}^{|qA_0|}\frac{dk}{2\pi} = \frac{|qA_0|}{\pi}\,. \tag{4.25}$$

We can also compute the renormalized vacuum expectation value of the electric current for the massless Dirac field. Starting from (3.29) and computing its time derivative, we directly find

$$\partial_t\langle J^1\rangle_{\rm ren} = \frac{2qm}{\pi}\int dk\,{\rm Im}\Big(h_k^I h_k^{II*}\Big) - \frac{q^2}{\pi}\dot{A}\,. \tag{4.26}$$

In the massless limit the first term vanish and we end up with the following, non-vanishing result

$$\langle J^1\rangle_{\rm ren} = -\frac{q^2}{\pi}A\,. \tag{4.27}$$

This result is in agreement and generalizes for an arbitrary t the (approximated) expression of the late-times electric current given in (4.15), that becomes exact for massless particles.

As we will shortly see, the breaking of the adiabatic invariance is directly connected with the existence of an axial anomaly. To better understand this proposal, let us introduce first and with some detail the idea of quantum anomalies in the semiclassical framework.

4.3 Chiral and Translational anomalies

To study the emergence of quantum anomalies for spin-$\frac{1}{2}$ fields interacting with an external electric background in two-dimensional Minkowski spacetime, let us take a look at the classical action of the Dirac field

$$\mathcal{S}_{\rm class} = \int d^2x\left(\frac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{D}_\mu\,\psi - m\bar{\psi}\psi\right)\,. \tag{4.28}$$

It is easy to see that it is invariant under the transformation $\psi \to e^{-i\epsilon}\psi$. It ensures via Noether's theorem that the current $J_V^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved. If the external field es homogeneous, the theory (4.28) has also a translational invariance under the x coordinate, that is $x \to x + \epsilon$. This implies that the (classical) canonical stress-energy tensor $T_c^{\mu\nu}$ obeys the following conservation law:

$$\partial_\mu T_c^{\mu 1} = 0\,, \tag{4.29}$$

regardless of the value of the mass.[2]

For a massless Dirac field, the theory is also invariant under chiral transformations $\psi \to e^{-i\epsilon\gamma^5}\psi$. This symmetry guarantees that the axial current $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is conserved, i.e., $\partial_\mu J_A^\mu = 0$. Furthermore, in this case, the Dirac field can be decomposed into two independent Weyl spinors $\psi_{R,L} = \frac{I\pm\gamma^5}{2}\psi$, so that, the right (left) currents $J_{R,L}^\mu = \frac{1}{2}(J_V^\mu \pm J_A^\mu)$ are also conserved

$$\partial_\mu J_{R,L}^\mu = 0\,. \tag{4.30}$$

We note that, in the free theory and in null coordinates $x^\pm = t \pm x$, the Weyl equations for the (R, L) sectors read $\partial_+\psi_R = 0 = \partial_-\psi_L$. The solutions to these equations are plane waves that move to the right (left). Thus, we refer to the particles (or antiparticles) associated with ψ_R as right-moving fermions, while the particles associated with ψ_L will be left-moving fermions [23]. In this context, the (canonical) stress-energy tensor can also be decomposed into its right and left components $T_c^{\mu\nu} = T_{c\,R}^{\mu\nu} + T_{c\,L}^{\mu\nu}$, where

$$T_{c\,R,L}^{\mu\nu} = \tfrac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{\partial^\nu}\left(\tfrac{I\pm\gamma^5}{2}\right)\psi\,, \tag{4.31}$$

[2] Note that $\partial_\mu T_c^{\mu\nu} \sim (\partial^\nu A_\mu) J_V^\mu$. If the external background is homogeneous, then $\partial_1 A^\mu$ is zero.

and because of the underlying symmetries, the $\nu = 1$ component of each chiral sector is separately conserved, i.e.,

$$\partial_\mu T^{\mu 1}_{c\,R,L} = 0 \,. \tag{4.32}$$

Some of these classical symmetries break down after quantization. To analyze this anomalous behavior, let us go back for a moment to the massive theory. Quantizing the Dirac field and expanding it in modes as in Eqs. (3.4), (3.5) and (3.6), we can express the vacuum expectation values of the relevant observables in terms of the time-dependent complex functions h^I_k and h^{II}_k. In terms of these functions, the formal vacuum expectation values of the relevant components of $J^\mu_{R,L}$ and $T^{\mu\nu}_{c\,R,L}$ read

$$\langle J^0_R \rangle = \frac{q}{2\pi}\int_{-\infty}^{\infty} |h^I_k|^2 \,, \quad \langle J^0_L \rangle = \frac{q}{2\pi}\int_{-\infty}^{\infty} |h^{II}_k|^2, \tag{4.33}$$

$$\langle T^{01}_{c\,R} \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dkk|h^I_k|^2 \,, \quad \langle T^{01}_{c\,L} \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dkk|h^{II}_k|^2. \tag{4.34}$$

These quantites are ultraviolet divergent and have to be renormalized. In this context, the most appropriated method turns out to be the adiabatic regularization method, introduced in the previous chapter. In two dimensions, the (R, L) currents have to be renormalized up to and including the first adiabatic order, while the the components of the stress-energy tensor have to be renormalized up to second adiabatic order. Following the recipe given in Section 3.1 we easily arrive to

$$\langle J^0_{R,L} \rangle_{\rm ren} = \frac{q}{2\pi}\int_{-\infty}^{\infty} dk \left(|h^{I,II}_k|^2 - \frac{\omega \mp k}{2\omega} \pm \frac{qm^2}{2\omega^3} A \right), \tag{4.35}$$

and

$$\langle T^{01}_{c\,R,L} \rangle_{\rm ren} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dkk \left(|h^{I,II}_k|^2 - \frac{\omega \mp k}{2\omega} \mp \frac{3km^2q^2A^2}{4\omega^5} \right), \tag{4.36}$$

where $\omega = \sqrt{k^2 + m^2}$, and where, for simplicity, we have assumed that $A(t)$ vanish at early times. Note that the last term of Eqs. (4.35) and (4.36) is finite and can be easily integrated in the momentum space. We can now proceed to evaluate (on shell) the time derivatives of the quantities above, i.e.,

$$\partial_t \left\langle J^0_{R,L} \right\rangle_{ren} = \mp \frac{m}{\pi} \int \mathrm{Im}\left(h_k^{II} h_k^{I*}\right) dk \pm \frac{q}{2\pi} \dot{A} \, , \tag{4.37}$$

and

$$\partial_t \left\langle T^{01}_{c\,R,L} \right\rangle_{ren} = \pm \frac{m}{\pi} \int_{-\infty}^{\infty} k \, \mathrm{Im}\left(h_k^{I} h_k^{II*}\right) dk \mp \frac{q^2 A \dot{A}}{2\pi} \, . \tag{4.38}$$

If we now perform the limit $m \to 0$, the first term in Eqs. (4.37) and (4.38) vanishes and we end up with the following anomalous result for the classically conserved currents

$$\partial_\mu \langle J^\mu_{R,L} \rangle_{\rm ren} = \pm \frac{q\dot{A}}{2\pi} \, , \qquad \partial_\mu \langle T^{\mu 1}_{c\,R,L} \rangle_{\rm ren} = \mp \frac{q^2 A \dot{A}}{2\pi} \, , \tag{4.39}$$

in contrast with the classical behaviour (4.30) and (4.32). The anomalies for the (L, R) currents are directly related with the (well-known) two-dimensional chiral anomaly (4.1). The non-vanishing result for the $\nu = 1$ component of the stress-energy tensor shows that, for each chiral sector, there exists an anomaly in the classical translational symmetry. Of course, for a (massless) Dirac field, the anomalies cancel out, restoring the classical (translational and phase) invariance

$$\partial_\mu \big(\langle J^\mu_R \rangle_{\rm ren} + \langle J^\mu_L \rangle_{\rm ren} \big) = 0 \, , \tag{4.40}$$

$$\partial_\mu \big(\langle T^{\mu 1}_{c\,R} \rangle_{\rm ren} + \langle T^{\mu 1}_{c\,L} \rangle_{\rm ren} \big) = 0 \, . \tag{4.41}$$

On the contrary, the axial current for the massless Dirac field remains anomalous

$$\partial_\mu \big(\langle J^\mu_R \rangle_{\rm ren} - \langle J^\mu_L \rangle_{\rm ren} \big) = \frac{q}{\pi} \dot{A} = -\frac{q}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu} \, . \tag{4.42}$$

Relation between particle creation and quantum anomalies

It is very interesting further explore the relationship between quantum anomalies and the underlying process of particle creation, including the breaking of the adiabatic invariance that appears for massless Dirac fields.

First, we note that in two dimensions the electric current is directly related with the axial current via $J^\mu_A = q\epsilon^{\mu\nu}J_\nu$. Therefore, a non-vanishing result for the electric current in the massless limit (4.27) together with the associated non-vanishing particle number (4.25), even in the adiabatic limit, can be interpreted as a necessary consequence required by the chiral anomaly.

We can also look with some detail at the right- and left-moving Weyl sectors. Consider again a pulsed electric field (4.18). For simplicity, we consider $q > 0$, $A_0 < 0$ and therefore $E(t) > 0$. At late times, and using the Bogoliguov transformation method explained in Section 4.1, one obtains the following result:[3]

$$\langle J^0_R \rangle_{\rm ren} = \int_0^{-qA_0} \frac{dk}{2\pi} = -\frac{qA_0}{2\pi} > 0\,, \quad \langle J^0_L \rangle_{\rm ren} = -\int_{qA_0}^{0} \frac{dk}{2\pi} = \frac{qA_0}{2\pi}\,. \quad (4.43)$$

Massless particles with positive charge are created with positive momentum in the interval $(0, -qA_0)$, while antiparticles (with negative charge) are created with negative momentum in the interval $(qA_0, 0)$. The total charge of the system is conserved, however, there is a net creation of chirality $\Delta N_5 = N_5^{\rm out} - N_5^{\rm in} = -\frac{qA_0}{\pi}$, as expected from the anomaly.

We can further explore this situation in terms of the stress-energy tensor. To this end, let us reintroduce the symmetric Belinfante stress-energy tensor,

[3] Remember: $|\beta_k|^2 = 1$ for $k \in (0, -qA_0)$ and zero elsewhere while $|\beta_{-k}|^2 = 1$ for $k \in (qA_0, 0)$ and zero in any other case.

since it is more appropriated to study particle creation. It was already defined in the previous chapter as

$$T^{\mu\nu} = \frac{i}{4}\left(\bar{\psi}\gamma^{\mu}\overleftrightarrow{D}^{\nu}\psi + \bar{\psi}\gamma^{\nu}\overleftrightarrow{D}^{\mu}\psi\right), \tag{4.44}$$

and can be directly related with the canonical stress-energy tensor [40].[4] We can also split this tensor into its left and right Weyl components $T^{\mu\nu}_{R,L}$. The divergence of $\langle T^{\mu 1}_{L,R}\rangle$ can be written in terms of the quantities studied in the first part of this section

$$\partial_{\mu}\langle T^{\mu 1}_{R,L}\rangle = \partial_{\mu}\langle T^{\mu 1}_{c\,R,L}\rangle + q\left(\partial_{\mu}A^{1}\right)\langle J^{\mu}_{R,L}\rangle + qA^{1}\partial_{\mu}\langle J^{\mu}_{R,L}\rangle\,. \tag{4.45}$$

Therefore

$$\partial_{\mu}\langle T^{\mu 1}_{R,L}\rangle_{\rm ren} = \pm\frac{q^{2}A\dot{A}}{2\pi}\,. \tag{4.46}$$

This result is parallel to the one given in (4.39) but with an opposite sign. In terms of the particle spectrum we can write

$$\langle T^{01}_{R}\rangle_{\rm ren} = \int_{0}^{-qA_0}\frac{k}{2\pi}dk = \frac{q^2A_0^2}{4\pi}\,, \quad \langle T^{01}_{L}\rangle_{\rm ren} = \int_{qA_0}^{0}\frac{k}{2\pi}dk = -\frac{q^2A_0^2}{4\pi}\,. \tag{4.47}$$

It is clear that the (R, L) parts of the symmetric stress-energy tensor gives the total momentum of the created pairs with positive (R) and negative (L) momentum. We note that for $E(t) < 0$ (and then $A_0 > 0$), we would have massless antiparticles with negative charge moving to the right and massless particles with positive charge moving to the left. In Figure 4.2 it is represented a pictorial description of the process.

[4] $T^{\mu\nu} = T^{\mu\nu}_{c} + \partial_{\alpha}B^{\alpha\mu\nu} + q\bar{\psi}\gamma^{\mu}\psi A^{\nu}$ where the antisymmetric tensor $B^{\alpha\mu\nu}$ is defined as $B^{\alpha\mu\nu} = \frac{1}{8}\bar{\psi}\left\{\gamma^{\alpha},\sigma^{\mu\nu}\right\}\psi$, and $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu},\gamma^{\nu}\right]$.

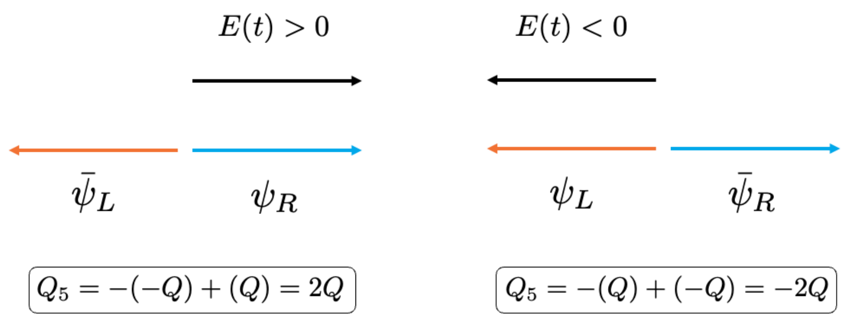

Figure 4.2: Creation of chirality by a time-dependent electric pulse $E(t)$. We have used the relation $Q_5 = \text{sign}(k) \cdot Q$.

Chiral anomaly in 4 dimensions

To finish this chapter, we illustrate how the four-dimensional axial anomaly can also be obtained within our framework and its relation with the breaking of the adiabatic invariance for massless Dirac fields.

For this purpose, let us consider then a four-dimensional, massless Dirac field coupled to an electromagnetic background characterized by the potential vector $A_\mu = (0, 0, Bx^1, -A(t))$. This represents a time-dependent electric pulse $\vec{E} = -\dot{A}\,\hat{z}$ [see Eq. (4.17)] and a constant magnetic field $\vec{B}$ in the z-direction. As in the two-dimensional case, one can split the Dirac spinor in two independent chiral parts $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$. Let us focus on the left sector. The Weyl equation is

$$\partial_0 \psi_L = \vec{\sigma} \vec{D} \psi_L \,, \tag{4.48}$$

and the fourier expansion of the quantized field reads

$$\psi_L(t,\vec{x}) = \sum_n \iint dk_2 dk_3 \left[B_{n,k_2,k_3} u_{n,k_2,k_3}(t,\vec{x}) + D^{\dagger}_{n,k_2,k_3} v_{n,k_2,k_3}(t,\vec{x}) \right], \tag{4.49}$$

with

$$\begin{aligned} u_{n,k_2,k_3}(t,\vec{x}) &= \frac{e^{i(k_2x^2+k_3x^3)}}{2\pi} \begin{pmatrix} h^{I}_{n,k_3}(t)\Phi_{n,k_2}(x^1) \\ -ih^{II}_{n,k_3}(t)\Phi_{n-1,k_2}\left(x^1\right) \end{pmatrix}, \\ v_{n,k_2,k_3}(t,\vec{x}) &= \frac{e^{-i(k_2x^2+k_3x^3)}}{2\pi} \begin{pmatrix} h^{II*}_{n,-k_3}(t)\Phi_{n,-k_2}(x^1) \\ ih^{I*}_{n,-k_3}(t)\Phi_{n-1,-k_2}(x^1) \end{pmatrix}, \end{aligned} \tag{4.50}$$

and where

$$\Phi_{n,k_2}(x^1) = \left(\frac{qB}{\pi}\right)^{1/4} \frac{1}{2^{\frac{n}{2}}\sqrt{n!}} e^{-\xi^2/2} H_n(\xi)\,, \tag{4.51}$$

$\xi = \sqrt{qB}\left(x^1 - k_2/qB\right)$, and $H_n(\xi)$ are the Hermite polynomials with $n = 0, 1, 2, \ldots$[5]. The equations for the time-dependent modes are

$$\begin{aligned} \dot{h}^{I}_{n,k_3} - i\left(k_3 + qA\right) h^{I}_{n,k_3} - i\sqrt{2nqB}\, h^{II}_{n,k_3} &= 0\,, \\ \dot{h}^{II}_{n,k_3} + i\left(k_3 + qA\right) h^{II}_{n,k_3} - i\sqrt{2nqB}\, h^{I}_{n,k_3} &= 0\,. \end{aligned} \tag{4.52}$$

If we compare these equations with the two-dimensional ones (3.7) and (3.8), we see that h^{I}_{n,k_3} and h^{II}_{n,k_3} are coupled through an effective mass $m^2_{\text{eff}} = 2qnB$ that vanish for the mode $n = 0$. For this setup, the most relevant observables are the electric current $\langle J^3 \rangle = -q\langle \bar{\psi}\gamma^3\psi \rangle$ and the chiral charge density $\langle J^0_A \rangle = \langle \bar{\psi}\gamma^0\gamma^5\psi \rangle$.

[5] For $n = -1$, $\Phi_{-1,k_2}(x^1) = 0$.

Let us start with the chiral charge. Repeating the same procedure for the R part and computing its formal vacuum expectation value we find

$$\left\langle J^0_A \right\rangle = \frac{qB}{4\pi^2}\int_{-\infty}^{\infty} dk_3 \left(|h^I_{0,k_3}|^2 - |h^{II}_{0,k_3}|^2\right) . \tag{4.53}$$

We see that only the mode $n = 0$ contributes to this observable. This expression can be renormalized using the two-dimensional adiabatic prescription explained in the previous Chapter [see Eq. (3.29)], giving as a final result

$$\langle J^0_A \rangle_{\rm ren} = \frac{q^2}{2\pi^2} A(t) B \equiv -\frac{q^2}{2\pi^2}\int_{-\infty}^{t} dt' \vec{E}(t')\vec{B}\,, \tag{4.54}$$

which is compatible with the 4-dimensional axial anomaly

$$\partial_\mu \langle J^\mu_A \rangle_{\rm ren} = -\frac{q^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\,. \tag{4.55}$$

On the other hand, for the formal vacuum expectation value of the electric current we obtain

$$\begin{aligned}\left\langle J^3 \right\rangle = \frac{q^2 B}{4\pi^2}\int_{-\infty}^{\infty} dk_3 \left(|h^{II}_{0,k_3}|^2 - |h^I_{0,k_3}|^2\right) \\ + \frac{q^2 B}{2\pi^2}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} dk_3 \left(|h^{II}_{n,k_3}|^2 - |h^I_{n,k_3}|^2\right) .\end{aligned} \tag{4.56}$$

We have split the result into two parts to emphasize the special role of the modes with $n = 0$. The breaking of the adiabatic invariance could be understood as follows. Imagine that the variation of the electric is very slow (adiabatic). Therefore, the modes with $n > 0$ do not contribute to the electric current (after renormalization). As we have seen, the renormalized electric current is directly related with the particle number density $\sim |\beta_{n,k_3}|^2$. This quantity can be obtained from its two-dimensional analog $|\beta_k|^2$ with the replacement $m \to m_{\rm eff}$ (4.21), and vanishes in the adiabatic limit for

$m_{\text{eff}} \neq 0$ [see Eq. (4.23)]. However, the modes $n = 0$ are different. Their effective mass is zero, and therefore $|\beta_{0,k_3}|^2 = 1$ for $k_3 \in (-qA_0, 0)$, even in the adiabatic limit. Consequently, the adiabatic invariance of the particle number $\langle N \rangle \sim \sum_n \int dk_3 (|\beta_{n,k_3}|^2 + |\beta_{n,-k_3}|^2)$ is broken, and the $n = 0$ contribution turns out to be a lower bound for $\langle J^3 \rangle_{\text{ren}}$. It is direct to see that this lower bound is

$$\langle J^3 \rangle_{\text{ren}}^{\text{min}} = -q \langle J_A^0 \rangle_{\text{ren}} \,. \tag{4.57}$$

In terms of particle creation, it means the axial anomaly forces a minimum amount of particles to be created, regardless of the form of the external field. This statement extends to curved backgrounds, where there is a gravitational contribution to the axial anomaly, and also to massless spin-1 fields in curved spacetimes [92, 93, 24].

Chapter 5

The backreaction problem

So far, we have studied the propagation of quantum fields in time-dependent classical backgrounds, and we have learned how to properly renormalize physical quantities in the context of two-dimensional semiclassical electrodynamics. The adiabatic prescription turned out to be a very efficient and systematic method, compatible with general covariance, that is able to reproduce the expected quantum anomalies correctly. On the other hand, the semiclassical approach allowed us to derive some fascinating non-perturbative quantum phenomena. In particular, we have characterized the spontaneous process of particle creation, and we have learned some interesting properties about it, such as its relation with the axial anomaly or its behavior in the adiabatic limit.

In this situation, a natural question arises: *what is the effect of the created particles on the background field?* We devote the first part of this chapter to answering this question in the context of time-dependent electric background fields. This problem can be studied via the semiclassical

Maxwell equations

$$\partial_\mu F^{\mu\nu} = J^\mu_C \quad \rightarrow \quad \partial_\mu F^{\mu\nu} = J^\mu_C + \langle J^\mu_Q \rangle_{\rm ren} \,. \tag{5.1}$$

The semiclassical approach is commonly considered as a truncated, effective version of a full quantized theory with a restricted range of validity. However, it becomes extremely useful to understand some of the main features of the process. This problem was previously studied in Refs. [68, 66, 72], first for a massive scalar field coupled to a time-dependent electric background in two spacetime dimensions and then, also for spin-$\frac{1}{2}$ fields and for four spacetime dimensions. More recently it was also analyzed in Refs. [94, 95]. The backreaction problem in electrodynamics was also studied from other perspectives, for example, using lattice simulations in Ref. [96, 97], solving the Vlasov equation with a source term in Refs. [68, 66, 72, 98] and with classical and statistical field theory techniques in Ref. [95]. Here, we revisit this problem for spin-$\frac{1}{2}$ fields in two dimensions, using, for the first time, the improved adiabatic subtraction scheme proposed in Chapter 3. We also analyze the energy transfer between the created particles and the background field and study in detail the time-dependent particle number. We note that from the exponential factor in Eq. (1.1), it is immediate to see that the order of the critical value for pair production should be

$$E_{\rm crit} \equiv m^2/q \,. \tag{5.2}$$

This characteristic value is used for some of the numerical work presented in this chapter.

In the second part of the chapter, we go deeper into our analysis, focusing on the validity of the semiclassical approximation to study the backreaction problem. Our proposal to deal with this complicated problem is a modified version of the analysis made in Ref. [41] for semiclassical gravity and later adapted for scalar backgrounds in Ref. [42] in the context of chaotic inflation. This approach is based on the behavior of the solutions to the linear response equation. However, as was done in Ref. [42], we do not solve this equation directly. Instead, we build approximate solutions from solutions to the backreaction equations with very close initial conditions. We compute the relative difference between these solutions and analyze their time evolution. If their difference increases significantly over time or rapidly in a short period, we state that quantum fluctuations are large, so the semiclassical approximation breaks down. We pay special attention to the massless limit and its relation with the axial anomaly. The content of this chapter is based on Ref. [28].[1]

5.1 The model 2.0

For our proposals, we consider a model similar to the one described in Section 3.1: a quantized Dirac field interacting with a classical and time-dependent electric field $E(t)$, this time initially generated by a prescribed classical source $J_C(t)$, in two-dimensional Minkowski spacetime. The quantized spin-$\frac{1}{2}$ field ψ obeys the Dirac equation

$$(i\gamma^\mu D_\mu - m)\psi = 0\,, \tag{5.3}$$

where $D_\mu\psi = (\partial_\mu - iqA_\mu)\psi$ and $A_\mu = (0, -A(t))$. Because of the characteristics of the system, we can perform a mode expansion of the Dirac field as in (3.4), namely $\psi = \int_{-\infty}^{\infty} dk[B_k u_k(t,x) + D^\dagger v_k(x,t)]$, where the modes u_k and v_k can be expanded in terms of two complex functions h_k^I and h_k^{II} as in Eqs. (3.5) and (3.6). In terms of h_k^I and h_k^{II} the Dirac equation transforms

[1]In Ref. [28] we also study the scalar field version of this problem. Here, for simplicity, we only focus on the spin-$\frac{1}{2}$ case.

into the following system of equations

$$\dot{h}_k^I - i\,(k+qA)\,h_k^I - imh_k^{II} = 0\,, \tag{5.4}$$

$$\dot{h}_k^{II} + i\,(k+qA)\,h_k^{II} - imh_k^{I} = 0\,. \tag{5.5}$$

On the other hand, the dynamics of the background field are characterized by the semiclassical Maxwell equations

$$\partial_\mu F^{\mu\nu} = J_C^\mu + \langle J_Q^\mu \rangle_{\rm ren}\,, \tag{5.6}$$

where J_C^μ represents an external (and conserved) classical source and $\langle J_Q \rangle_{\rm ren}$ is the renormalized vacuum expectation value of the Dirac current $J_Q^\mu = -q\bar{\psi}\gamma^\mu\psi$, that encapsulates the potential effect of the created particles on the background. In our particular setup, the semiclassical Maxwell equations turn out to be a single equation

$$\ddot{A} = -\dot{E} = J_C + \langle J_Q \rangle_{\rm ren}\,, \tag{5.7}$$

where $J_C(t) = J_C^1$, $J_C^0 = 0$, and

$$\begin{aligned} \langle J_Q \rangle_{\rm ren} = \left\langle J_Q^1 \right\rangle_{\rm ren} &= q\int_{-\infty}^{\infty} \frac{dk}{2\pi}\left(|h_k^{II}|^2 - |h_k^I|^2 - \frac{k}{\omega} - \frac{m^2 qA}{\omega^3}\right) \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left(|h_k^{II}|^2 - |h_k^I|^2 - \frac{k}{\omega}\right) - \frac{q^2}{\pi}A\,. \end{aligned} \tag{5.8}$$

We point out that $\langle J_Q^0 \rangle_{\rm ren} = 0$, which means that no net charge is created. The renormalized vacuum expectation value of the electric current $\langle J_Q \rangle_{\rm ren}$ is obtained using the adiabatic regularization prescription with $A(t)$ of adiabatic order one, as explained in Section 3.1. We note also that in the massless limit $m \to 0$, the induced electric current $\langle J_Q \rangle_{\rm ren}$ takes a particularly simple form

$$\langle J_Q \rangle_{\rm ren} = -\frac{q^2}{\pi}A\,. \tag{5.9}$$

We remember that in two spacetime dimensions, the axial and the electric current are directly related $J_A^\mu = q\epsilon^{\mu\nu}J_\nu$. As we argued in Chapter 4, this non-vanishing result for the electric current can be interpreted as a necessary consequence required by the chiral anomaly (4.1).

Equation (5.7) together with the mode equations (5.4) and (5.5) form a coupled system of non-linear ordinary differential equations that can be solved numerically once the initial conditions $E(t_0)$, $A(t_0)$, $h_k^I(t_0)$ and $h_k^{II}(t_0)$ and the external current $J_C(t)$ are specified.

Particle creation

To better understand the process of particle creation at a given time t, it is useful to define a time-dependent particle number $\langle N(t)\rangle$. As explained in Section 4.1, we recall that the particle number is only unambiguously defined during intervals of time when the potential $A(t)$ is time-independent, that is, if $E(t)$ and its first time derivative are zero. For this reason, and as a first step, let us assume for a moment that the electric field is zero for times $t > t_f$. Defining

$$w(t) \equiv \sqrt{p(t)^2 + m^2}\ , \quad p^2 = (k + qA)^2 + m^2, \tag{5.10}$$

the exact modes for a massive spin-$\frac{1}{2}$ field are for $t \geq t_f$ [see Eq. (4.13)]

$$h_k^I(t) = \alpha_k \sqrt{\tfrac{w(t_f)-p(t_f)}{2w(t_f)}}\, e^{-i\int_{t_0}^t w(t_f)dt_1} + \beta_k \sqrt{\tfrac{w(t_f)+p(t_f)}{2w(t_f)}}\, e^{+i\int_{t_0}^t w(t_f)dt_1}, \tag{5.11}$$

$$h_k^{II}(t) = -\alpha_k \sqrt{\tfrac{w(t_f)+p(t_f)}{2w(t_f)}}\, e^{-i\int_{t_0}^t w(t_f)dt_1} - \beta_k \sqrt{\tfrac{w(t_f)-p(t_f)}{2w(t_f)}}\, e^{+i\int_{t_0}^t w(t_f)dt_1}, \tag{5.12}$$

where t_0 is an arbitrary constant and α_k and β_k are the (time-independent) Bogolubov coefficients, obeying $|\alpha_k|^2 + |\beta_k|^2 = 1$. The positive quantity $|\beta_k|^2$

determines univocally the (late-times) number density of created particles

$$\langle N \rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(|\beta_k|^2 + |\beta_{-k}|^2 \right) = 2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\beta_k|^2 \,, \tag{5.13}$$

where the factor of 2 accounts for antiparticles.

Having this idea on mind, one can define a time-dependent particle number $\langle N(t) \rangle$ based on the adiabatic expansion of the spin-$\frac{1}{2}$ modes of the quantum field when considering $A(t)$ a function of adiabatic order zero [see Eqs. (3.22) and (3.23)]. For scalar fields in electric backgrounds this approach has been considered in Refs. [99, 100, 101] (see also [10, 102] for $\langle N(t) \rangle$ in cosmological backgrounds). Using the leading order of the adiabatic expansion for the Dirac modes,[2] we can define an adiabatic notion of particle at any time t as follows

$$g_k^I \equiv \sqrt{\frac{w-p}{2w}} e^{-i \int_{t_0}^{t} w(t_1) dt_1}, \quad g_k^{II} \equiv -\sqrt{\frac{w+p}{2w}} e^{-i \int_{t_0}^{t} w(t_1) dt_1} \,, \tag{5.14}$$

and then, expand the exact solutions as

$$h_k^I(t) = \alpha_k(t) g_k^I(t) - \beta_k(t) g_k^{II*}(t) \,, \tag{5.15}$$

$$h_k^{II}(t) = \alpha_k(t) g_k^{II}(t) + \beta_k(t) g_k^{I*}(t) \,, \tag{5.16}$$

where now the coefficients $\alpha_k(t)$ and $\beta_k(t)$ depend on time and satisfy the normalization condition $|\alpha_k(t)|^2 + |\beta_k(t)|^2 = 1$. With this motivation we can define the time dependent particle number as

$$\langle N(t) \rangle = 2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\beta_k(t)|^2 \,, \tag{5.17}$$

where $\beta_k(t)$ can be obtained inverting Eqs. (5.15) and (5.16)

$$\beta_k(t) = \left[g_k^I(t) h_k^{II}(t) - g_k^{II}(t) h_k^I(t) \right] . \tag{5.18}$$

[2] with $A(t)$ of adiabatic order zero.

5.2 Numerical solutions to the backreaction equations

We can now proceed to analyze some numerical solutions to the backreaction equations (5.4),(5.5) and (5.6). To this end, we consider the classical source

$$J_C = -E_0 \delta(t) \,. \tag{5.19}$$

If we ignore the effect of the created particles, the solution to the classical Maxwell equation would be $E_C = E_0 \theta(t)$, where $\theta(x)$ is the Heaviside step function. That is, a constant electric field for $t > 0$. We want to study how this classical picture changes when including the effect of the created particles. Because of the form of the external source, we choose as initial conditions for the spin-$\frac{1}{2}$ field the Minkowski vacuum

$$h_k^I(t=0) = \sqrt{\frac{\omega - k}{2\omega}} \,, \qquad h_k^{II}(t=0) = -\sqrt{\frac{\omega + k}{2\omega}} \,. \tag{5.20}$$

We focus on the time evolution of the electric field $E(t)$ and the induced electric current $\langle J_Q \rangle_{\rm ren}$. We also study the particle production via the time-dependent particle number $\langle N(t) \rangle$ defined in Eq. (5.17). Because of the form of the classical source, the total energy of the system is conserved for $t > 0$ [see Eq. (3.36)],

$$\langle \rho(t) \rangle_{\rm ren} + \frac{1}{2} E(t)^2 = cte = \frac{1}{2} E_0^2 \,, \tag{5.21}$$

where

$$\langle \rho \rangle_{\rm ren} = \langle T_{00} \rangle_{\rm ren} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left(i[h_k^{II} \dot{h}_k^{II*} + h_k^I \dot{h}_k^{I*}] + \omega + \frac{kqA}{\omega} + \frac{m^2 q^2 A^2}{2\omega^3} \right). \tag{5.22}$$

Therefore, we can also study the energy transfer process between the electric field and the created particles. As we have argued in the introduction, for the particle production to be significant $E_0 \sim E_{\text{crit}} = m^2/q$, so we restrict our analysis to this region of values for the initial electric field.

For numerical convenience, we use dimensionless variables to solve the backreaction equations in terms of the electric charge, i.e. $k \to k/q$, $m \to m/q$, $\omega \to \omega/q$ and $t \to qt$. We also rescale the physical quantities in terms of the critical electric field, namely

$$\tilde{E} \equiv \frac{E}{E_{\text{crit}}}\,, \qquad \tilde{J} \equiv \frac{J}{qE_{\text{crit}}}\,, \qquad \tilde{\rho} = \frac{\rho}{E_{\text{crit}}^2}\,, \qquad \langle \tilde{N} \rangle = \frac{\langle N \rangle}{E_{\text{crit}}}\,. \tag{5.23}$$

In Figures 5.1 and 5.2 numerical solutions to the backreaction equations are shown for two different values of the initial electric field $E_0 = E_{\text{crit}}$ and $E_0 = 5E_{\text{crit}}$, and for $\frac{m^2}{q^2} = 10$. We immediately see that, as soon as the particle production starts to happen, the initial electric field decays, and the electric current increases due to the created particles. When the electric field has been reduced significantly, the current reaches a plateau, and the particle creation saturates.

For $E_0 = 5E_{\text{crit}}$ we see that after the first creation event, the particle number does not increase anymore. This happens that early because of the Pauli exclusion principle. However, for $E_0 = E_{\text{crit}}$, we still have some creation events after each oscillation, which means that the particle number is not saturated after the first creation burst. We also see that the electric field amplitude is always lower than the initial value E_0, which indicates that part of the energy of the electric field has been permanently transferred to the created particles.

At later times, both the electric field and the electric current oscillate, and the created particles behave like an oscillating plasma fluid. The oscillation of this plasma generates an oscillating electric current that generates oscillations in the electric field.

We can also look at the energy density of the system. We recall that for $t > 0$, the total energy of the system is conserved. Again, if we analyze the energy density envelope of both the electric and the Dirac field, we see that part of the initial energy of the electric field is permanently transferred to the created particles.

Particle Creation Events

In Figure 5.3 we show the time evolution of $|\beta_k(t)|^2$ for two different values of k: $\frac{k}{q} = 30$ and $\frac{k}{q} = 50$, and for the initial condition $E_0 = E_{\text{crit}}$. We see that for a given mode k we can have various creation events but also destruction events. If we analyze the solution for the potential vector $A(t)$ we see that these events (creation and destruction) happen around $|k + qA| \sim m$. In Refs. [99, 100, 101] it was already shown that for a external and approximately constant background field, single particle creation events happen for $|k + qA| \sim m$. Here, we have upgraded the analysis by considering backreaction effects, and we have observed that the resulting plasma oscillations lead to multiple creation and destruction events.

Massless limit

The previous analysis can be easily extended to the massless limit. In this case, the mode equations (5.4) and (5.5) decouple. For the massless version of the initial conditions given in (5.20), the exact solution for the Dirac modes is given by

$$h_k^{I,II}(t) = \pm\theta(\mp k) e^{\pm i \int_{t_0}^{t} (k+qA(t'))dt'} , \tag{5.24}$$

and the electric current $\langle J_Q \rangle_{\rm ren}$ reduces to $\langle J_Q \rangle_{\rm ren} = -\frac{q^2}{\pi} A(t)$. Therefore, the semiclassical Maxwell equation transforms into the equation of an harmonic oscillator

$$\ddot{A} + \frac{q^2}{\pi} A = 0 \,, \qquad (5.25)$$

with frequency $q/\sqrt{\pi}$. This result is consistent with the well-known fact that radiative corrections to the Schwinger model induce a mass for the "photon", with value $m_\gamma^2 = q^2/\pi$ [103].

For the initial conditions $E(0) = E_0$ and $A(0) = 0$ the exact solution of (5.25) is given by $E(t) = \cos(\frac{q}{\sqrt{\pi}} t)$. We can also obtain exact expressions for the energy density and for the particle number

$$\langle \rho \rangle_{\rm ren} = \frac{q^2 A^2}{2\pi} \,, \qquad \langle N \rangle = \frac{|qA|}{\pi} \,. \qquad (5.26)$$

In Figure 5.4 we explicitly show the time evolution of the previous observables for the initial value $\frac{E_0}{q} = 2$. They have to be compared with the massive solutions represented in Figures 5.1 and 5.2. We also observe plasma oscillations. However, there is a crucial difference with the massive case. In this case, the maximum amplitude of the electric field oscillations never decrease, and after each oscillation, the electric field returns to the initial point E_0.

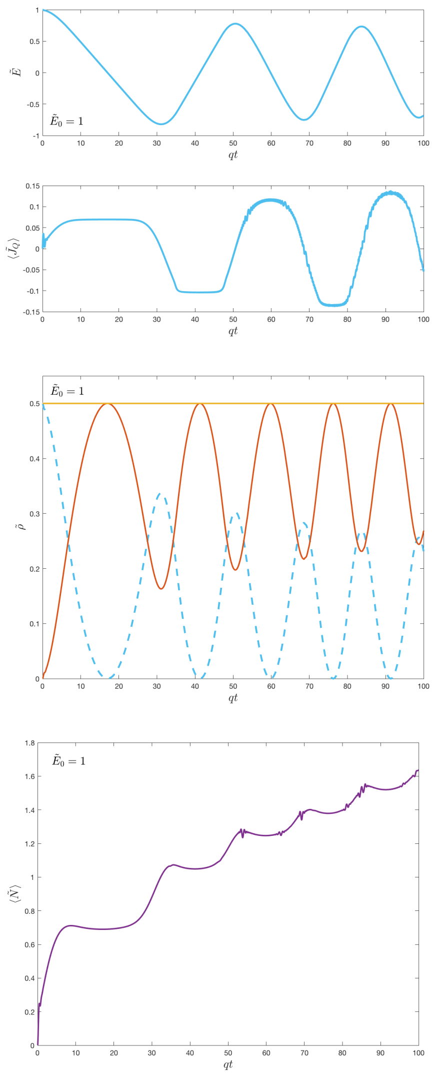

Figure 5.1: Numerical solutions to the backreaction equations for $E_0 = E_{\text{crit}}$ and $\frac{m^2}{q^2} = 10$. In the upper panel, we show the time evolution of the electric field $E(t)$ and the induced electric current $\langle J_Q \rangle_{\text{ren}}$. In the middle panel, we show the energy transfer between the electric field ρ_{elec} (dashed blue line) and the created particles $\langle \rho \rangle_{\text{ren}}$ (orange line). The yellow line represents the conserved quantity $\rho_0 = \rho_{\text{elec}} + \langle \rho \rangle_{\text{ren}}$. In the lower panel we show the time evolution of the time dependent particle number $\langle N(t) \rangle$.

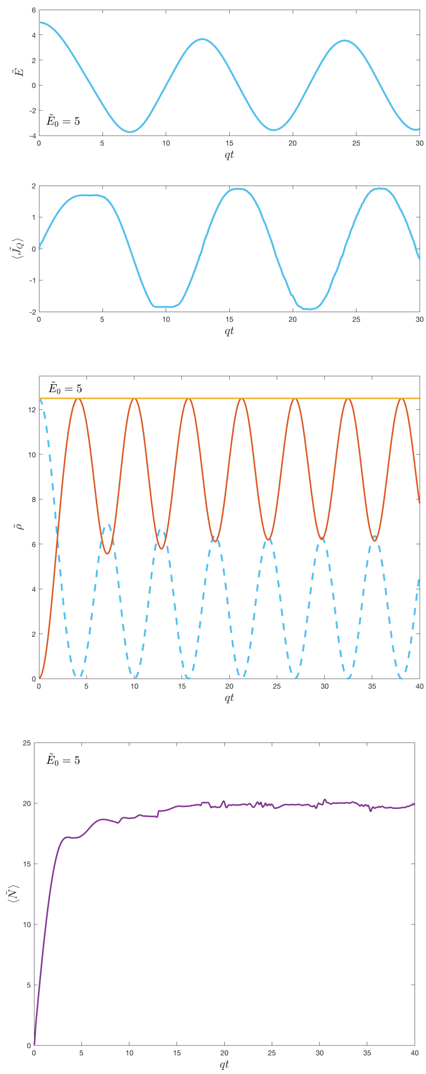

Figure 5.2: Numerical solutions to the backreaction equations for $E_0 = 5E_{\text{crit}}$ and $\frac{m^2}{q^2} = 10$. In the upper panel, we show the time evolution of the electric field $E(t)$ and the induced electric current $\langle J_Q \rangle_{\text{ren}}$. In the middle panel, we show the energy transfer between the electric field ρ_{elec} (dashed blue line) and the created particles $\langle \rho \rangle_{\text{ren}}$ (orange line). The yellow line represents the conserved quantity $\rho_0 = \rho_{\text{elec}} + \langle \rho \rangle_{\text{ren}}$. In the lower panel we show the time evolution of the time dependent particle number $\langle N(t) \rangle$.

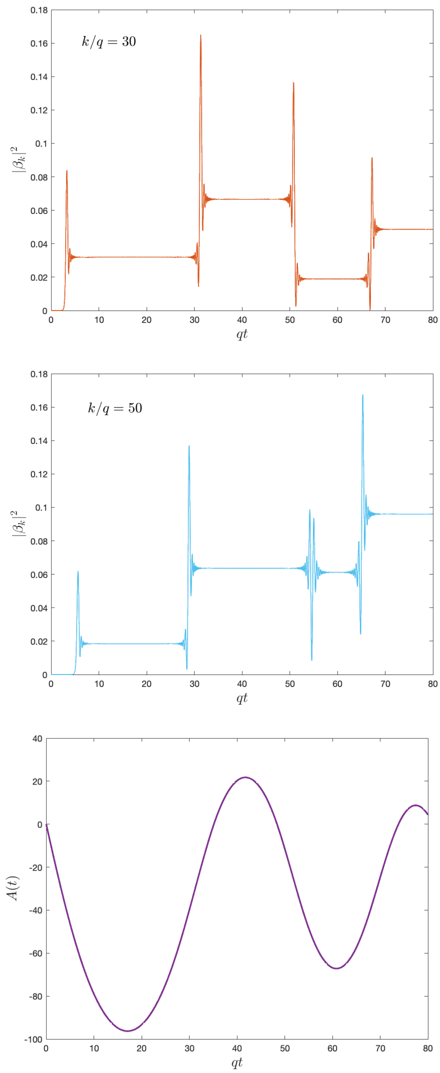

Figure 5.3: In the upper and middle panels we see the spectrum of the time dependent particle number $|\beta_k|^2$ for two individual modes $\frac{k}{q} = 30$ and $\frac{k}{q} = 50$ for the initial condition $E_0 = E_{\text{crit}}$. In the lower panel we see the time evolution of the potential vector $A(t)$.

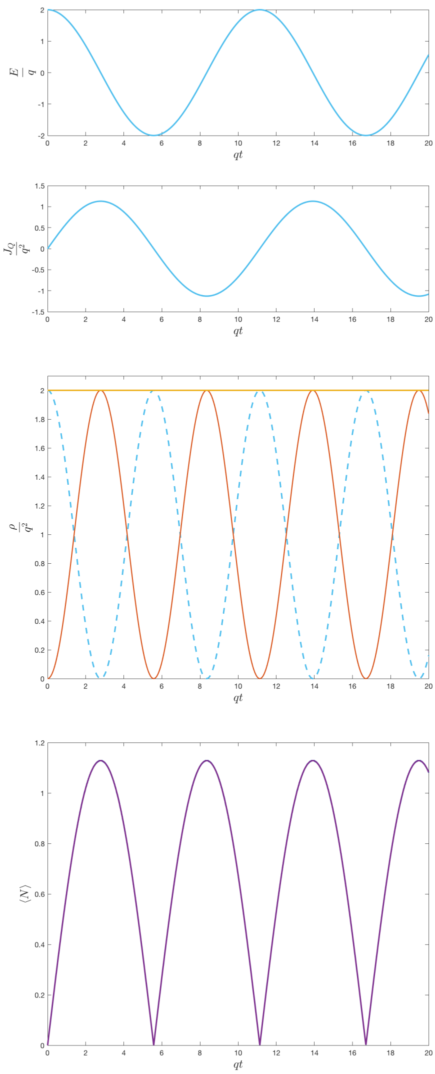

Figure 5.4: Solutions to the massless backreaction equations for $E_0 = 2q$. In the upper panel, we show the time evolution of the electric field $E(t)$ and the induced electric current $\langle J_Q \rangle_{\text{ren}}$. In the middle panel, we show the energy transfer between the electric field ρ_{elec} (dashed blue line) and the created particles $\langle \rho \rangle_{\text{ren}}$ (orange line). The yellow line represents the conserved quantity $\rho_0 = \rho_{\text{elec}} + \langle \rho \rangle_{\text{ren}}$. In the lower panel we show the time evolution of the time dependent particle number $\langle N(t) \rangle$.

5.3 Validity of the semiclassical approximation

In the previous section, we studied the effect of the created particles on the classical electric field background via the semiclassical Maxwell equations. Now, we analyze the validity of this approach, i.e., if it correctly describes the interaction between the created particles and the dynamical background. The problem of the validity of the semiclassical approximation is, in general, a very complex one. However, we can take some advantage because of the particularities of this system.

In two-dimensional dilaton gravity with conformal quantized fields, it is possible to describe with high accuracy the semiclassical evolution of two-dimensional evaporating black holes up to the end of the process [104, 105, 106, 107, 17]. In analogy, the semiclassical approximation in electrodynamics describes the decay of an electric field because of particle production. The main difference between these two situations is that in the decay of a black hole, the creation of particles increases with time. On the contrary, in the electric field decay, the production of particles occurs in bursts and saturates after a specific time, leaving the well-known plasma oscillations. It means that, while one expects the breakdown of the semiclassical picture for black holes to happen at the end of the process, in the electric field decay, we can expect the breakdown to happen at the beginning of the process, since the particle production is more abundant. This argument becomes stronger for spin-$\frac{1}{2}$ fields because of the Pauli exclusion principle.

A second argument to reinforce this idea is that in our approach we are neglecting the interactions between the created particles, which are expected to become more significant at later times. Therefore, our approach is not a good description of the problem at later times anyways. For these two reasons, we restrict our validity analysis to the early times region, at the first cycle of the decay process, ignoring then the plasma oscillations.

In this situation, a natural way to study the validity of the semiclassical approximation is to look at the linear response equation, which can be obtained by perturbing the semiclassical backreaction equation (5.7) around a given solution. In Ref. [41] it was given, in the context of semiclassical gravity, a quantitative test for the validity of the semiclassical approximation in terms of metric fluctuations and a linear response analysis. Later, in Ref. [42] this criterion was adapted and applied to the process of preheating in models of chaotic inflation, focusing on homogeneous solutions to the linear response equation. The validity criterion that we borrow states that "*the semiclassical approximation will break down if any linearized gauge-invariant quantity constructed from solutions to the linear response equation, with finite non-singular data, grows rapidly for some period of time*".

In the case of homogeneous perturbations, the solutions to the linear response equation δE can be approximated by using the solutions of the backreaction equations for two sets of initial conditions that are very close to each other ΔE. As explained in [42, 28], as long as the difference between these solutions does not grow significantly, the difference between these two solutions is an approximate solution to the linear response equation. In this section, we proceed as follows. We present the linear response equation and describe how to obtain approximate solutions to this equation. We also introduce some relevant quantities to study the breakdown of the semiclassical approximation as the (modified) relative difference R. Then, we give the initial conditions that we have used in our analysis, and finally, we present our numerical results. The content of this section is based on the analysis made in Ref. [28].

Approximate solutions to the linear response equation

In semiclassical electrodynamics, the linear response equation for homogeneous perturbations can be obtained by perturbing (5.7) around a background solution

$$\frac{d^2}{dt^2}\delta A(t) = -\frac{d}{dt}\delta E = \delta J_C + \delta \left\langle J_Q \right\rangle \, , \tag{5.27}$$

where

$$\delta \left\langle J_Q \right\rangle_{\rm ren} = -\frac{q^2}{\pi}\delta A(t) + i\int_{-\infty}^{\infty} dx' \int_{-\infty}^{t} dt' \left\langle \left[J_Q(t,x), J_Q\left(t',x'\right)\right]\right\rangle \delta A\left(t'\right) \, , \tag{5.28}$$

with

$$\begin{aligned}\int_{-\infty}^{\infty} & dx' \langle [J_Q(t,x), J_Q(t',x')]\rangle \\ &= \frac{4iq^2}{\pi}\int_{-\infty}^{\infty} dk \, \mathrm{Im}\left\{ h_k^I(t) h_k^{II}(t) h_k^{I*}(t') h_k^{II*}(t')\right\} .\end{aligned} \tag{5.29}$$

The details of this derivation can be found in [28]. We note that in the massless limit, the second term in (5.28) vanish identically, and $\delta\langle J_Q\rangle_{\rm ren}$ reduces to $\delta\langle J_Q\rangle_{\rm ren} = -\frac{q^2}{\pi}\delta A(t)$. For our proposals, it is very convenient to split the background solution $E(t)$ in two parts

$$E = E_C + E_Q \, , \qquad E_C = -\int_{t_0}^{t} J_C(t')dt', \tag{5.30}$$

and therefore, the solutions to the linear response equation, can be splitted in the same way $\delta E = \delta E_C + \delta E_Q$. Because of the linear structure of (5.27), it is clear that our validity criterion can be modified in terms of δE_Q, that is, we can say that if δE_Q grows significantly in a time period, the semiclassical approximation is broken.

As stressed before, the solutions to the linear response equation δE for homogeneous perturbations can be approximated by the difference between two solutions to the backreaction equation with very close initial conditions ΔE as long as the difference between these solutions remains small. To understand this statement, let us consider a classical current of the form

$$J_C = -E_0 \dot{f}(t)\,, \tag{5.31}$$

where $f(t)$ is a well-behaved, time-dependent function. The solution to the classical Maxwell equation is then $E_C = E_0 g(t)$. Let us consider now two solutions to the semiclassical Maxwell equation (5.7) $E_1(t)$ and $E_2(t)$ with initial conditions $E_0 = E_{01}$ and $E_0 = E_{02}$ respectively. It is straightforward to see that the difference $\Delta E = E_2 - E_1$ is an exact solution to the equation

$$-\Delta\dot{E} = \Delta J_C + \Delta\langle J_C\rangle_{\rm ren}\,, \tag{5.32}$$

where $\Delta J_C = J_{C1} - J_{C2} = \Delta E_0 \dot{f}$, and $\Delta\langle J_Q\rangle_{\rm ren} = \langle J_{Q1}\rangle_{\rm ren} - \langle J_{Q2}\rangle_{\rm ren}$. For a small initial difference, ΔE_0 can be considered a perturbation so that $\Delta E_0 = \delta E_0$, Then, it is clear that $\Delta J_C = \delta J_C$ and $\Delta E_C = \delta E_C$.

Therefore, we can say that ΔE is an approximate solution to the linear response equation $\Delta E \approx \delta E$ for some time interval if during this period $\Delta\langle J_Q\rangle_{\rm ren} \approx \delta\langle J_Q\rangle_{\rm ren}$. However, since we are not solving the exact linear response equation, we can rephrase this statement by saying that, as long as ΔE is small, ΔE should be an approximate solution to the linear response equation. For convenience and to have a clear criterion about what *small* means, we work with the relative differences:

$$R = \frac{|\Delta E|}{|E_1| + |E_2|}\,, \quad R_C = \frac{|\Delta E_0|}{|E_{01}| + |E_{02}|}\,, \quad R_Q = \frac{|\Delta E_Q|}{|E_{Q1}| + |E_{Q2}|}\,. \tag{5.33}$$

we recall here that the quantity R can be written in terms of covariant quantities as

$$R = \frac{\left|(F_{\mu\nu})_2\, u^\nu n^\mu - (F_{\mu\nu})_1\, u^\nu n^\mu\right|}{\left|(F_{\mu\nu})_2\, u^\nu n^\mu\right| + \left|(F_{\mu\nu})_1\, u^\nu n^\mu\right|}\,. \tag{5.34}$$

where $u^\mu = (1,0)$ is the covariant velocity associated to the inertial observer for which $E^\mu = F^{\mu\nu}u_\nu = (0, E(t))$ and $n^\mu = (0,1)$ is the tangent vector to the one-dimensional spatial surface. We compare the classical relative difference R_C with the induced quantum difference R_Q.

The analysis of the validity of the semiclassical approximation can be done as follows:

1. If for times $t_0 \leq t \leq t_1$ for some t_1, $R_Q \lesssim R_C$ the validity criterion is satisfied by the approximated homogeneous linear response solutions during this time interval.

2. If for any time between $t_0 \leq t \leq t_1$, $R_Q \gg R_C$, then it means that δE has been grown significantly, and therefore the criterion is not satisfied, and the semiclassical approximation has down. We note that although, at later times $R_Q \lesssim R_C$ the semiclassical approximation is not restored, and the approximated solutions ΔE cannot be used anymore to model the perturbed solutions.

3. If $R_Q > R_C$ but they are still of the same order; the criterion is ambiguous, so we can only say that the accuracy of the semiclassical approximation is lower.

Numerical analysis

So far, we have all the ingredients to study the validity on the semiclassical approximation from the solutions to the semiclassical Maxwell equation

(5.7) by using the relative difference R_Q. For this analysis we work with the classical source

$$J_C = -\frac{qE_0}{(1+qt)^2}\,, \tag{5.35}$$

for $t \geq 0$ and $J_C = 0$ for $t < 0$. The classical electric field associated with this source is the asymptotically constant profile

$$E_C(t) = E_0\left(\frac{qt}{1+qt}\right). \tag{5.36}$$

In Figure 5.5 are represented these classical solutions. The initial conditions at $t_0 = 0$ for the electric field and the vector potential are $E(0) = A(0) = 0$. For the spin-$\frac{1}{2}$ field we choose the same initial conditions as for the delta profile (5.20).

Before starting with our analysis, it is particularly interesting to study the massless limit. In this case, the exact linear response equation reduces to

$$\delta\ddot{A} + \frac{q^2}{\pi}\delta A = \delta J_C. \tag{5.37}$$

Therefore, for a setup such that $\delta J_C = \Delta J_C$ it is clear that $\delta A = \Delta A$ and $\delta E = \Delta E$ for all t. In other words

$$R_Q = R_C \tag{5.38}$$

and the validity criterion is always satisfied.

To study the validity of the semiclassical approximation for the massive case, we have solved numerically the backreaction equations for two different sets of initial conditions, determined by the difference $\frac{\Delta E_0}{q} = 10^{-3}$. As stressed before, we focus on the early time solutions. We examine the quantity R_Q [eq. (5.33)] and study its dependence on the mass. As complementary material, we also represent other quantities as the time-dependent

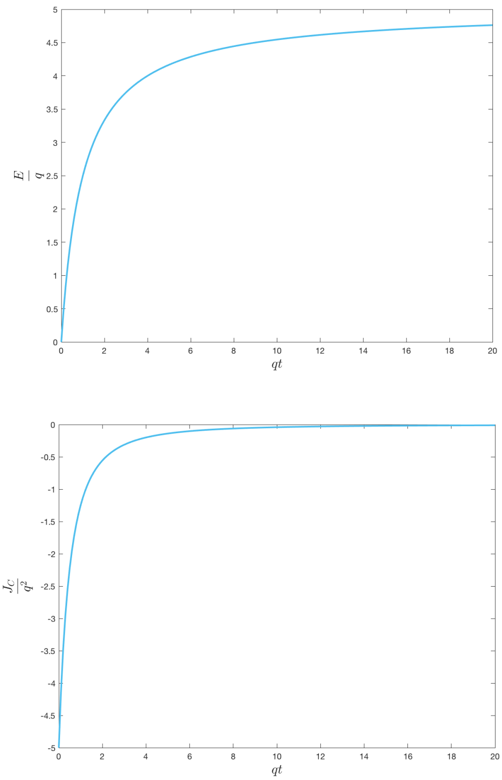

Figure 5.5: Asymptotically constant classical profile for the initial value $E_0 = 5q$. In the upper panel it is shown the electric profile and in the lower panel we show the classical source.

particle number $\langle N(t)\rangle$, the dimensionless electric field E/q and electric current J/q^2.

Consider a fixed value of E_0. For small values of the mass $m \to 0$ the induced electric current should approach to $\langle J_Q\rangle_{\text{ren}} \to \frac{q^2}{\pi}A$ and we expect the semiclassical approximation to be better, that is $R_Q \to R_C$, according to Eq. (5.38). The same is true for the very massive limit $m \to \infty$. In this case, the electric background is not strong enough to create particles, and we should recover the classical limit $\langle J_Q\rangle_{\text{ren}} \to 0$ and $E(t) \to E_C(t)$, in agreement with the decoupling theorem in perturbative quantum field theory [108]. Therefore, we expect a potential break down of the semiclassical approximation to occur for values of the mass such that $E_{\text{crit}} = m^2/q \sim E_0$.

In Figures 5.6 and 5.7 we represent the quantity R_Q for the initial conditions $\frac{E_{01}}{q} = 1$ and $\frac{E_{02}}{q} = 1 + 10^{-3}$ and for different values of the mass $\frac{m^2}{q^2}$. Note that for these values $R_C = 5 \times 10^{-4}$. We also represent the electric field $E(t)$, the induced electric current $\langle J_Q\rangle$ and the time-dependent particle number $\langle N\rangle$ for E_{01}. In Figure 5.7 we focus on values of the mass that make E_0 to be of the same order of E_{crit}. On the contrary, in Figure 5.6 we look at small values of the mass so that $E_0 \gg E_{\text{crit}}$. From Figure 5.6 we see that the closer we are to the massless limit, the better our criterion is satisfied. We also see that the smaller the mass, the faster is the particle production and the damping of the electric field due to backreaction effects. For $\frac{m^2}{q^2} = 0.01$ the growth of R_Q is small and soft. However, for $\frac{m^2}{q^2} = 0.1$ the value of R_Q grows substantially once the particle production has occurred. In Figure 5.7, we see that for the cases with $E_0 \sim E_{\text{crit}}$ there is a significant amount of particle creation and also that, once enough particle production has taken place, the value of R_Q increases rapidly. The (possibly) exponential growth of R_Q continues up to the point where the effect of the created particles is strong enough that the electric field starts to decrease.

In Figure 5.8 we represent R_Q [as well as E, $\langle J_Q \rangle_{\rm ren}$ and $\langle N \rangle$] for the initial conditions $\frac{E_{01}}{q} = 10$ and $\frac{E_{02}}{q} = 10 + 10^{-3}$, and for different values of the mass $\frac{m^2}{q^2}$ that go from the small mass limit to the critical limit $E_0 \sim E_{\rm crit}$. In this case $R_C = 5 \times 10^{-5}$. We see the same behaviour than in the previous case. For very small values of the mass, the growth of R_Q is low and soft. However, as the mass of the created particles increases such that E_0 approaches $E_{\rm crit}$, the growth of R_Q becomes rapid and strong and extends in time until the damping of the electric field is significant enough so that $E(t) \ll E_{\rm crit}$.

From our analysis, we can extract some important conclusions. First, we have seen that the relevant quantity for the validity analysis is $\frac{E_0}{E_{\rm crit}} = \frac{qE_0}{m^2}$, that is, a quotient between the external electric field parameter E_0 and the mass of the created particles. For similar values of $\frac{qE_0}{m^2}$, the behavior of the solutions to the backreaction equations is very similar. Second, as expected, the most problematic region is the critical region $E_0 \sim E_{\rm crit}$. In this case, our criterion is violated, and the semiclassical approximation seems to break down after the first burst of particles is created (this situation is similar to the semiclassical breakdown found in [42] in an inflationary context). For the small mass regime, the growth of R_Q is smaller and smoother, and we can say that our criterion is not explicitly violated. However, we recall that the proposed statement is a necessary but not sufficient condition, which means that if it is violated, we can say that the semiclassical approximation breaks down, but if it is not, we cannot say anything about the validity of the semiclassical approximation. For the large mass limit, we have not computed explicitly the solutions to the backreaction equations, however, one should expect that particle production does not occur, and therefore, the electric field behavior can be predicted by classical electrodynamics.

We expect the first experimental verification of the Schwinger effect to happen in the regime $E_0 \sim E_{\rm crit}$ (a very strong field strength is required, $E_{\rm crit} \sim 10^{18} V/m$). However, the focus of the detection will be in the first particle creation events more than in the backreaction effects. In this context, the semiclassical approximation provides a clear and simple description of the early time quantum phenomena of particle creation.

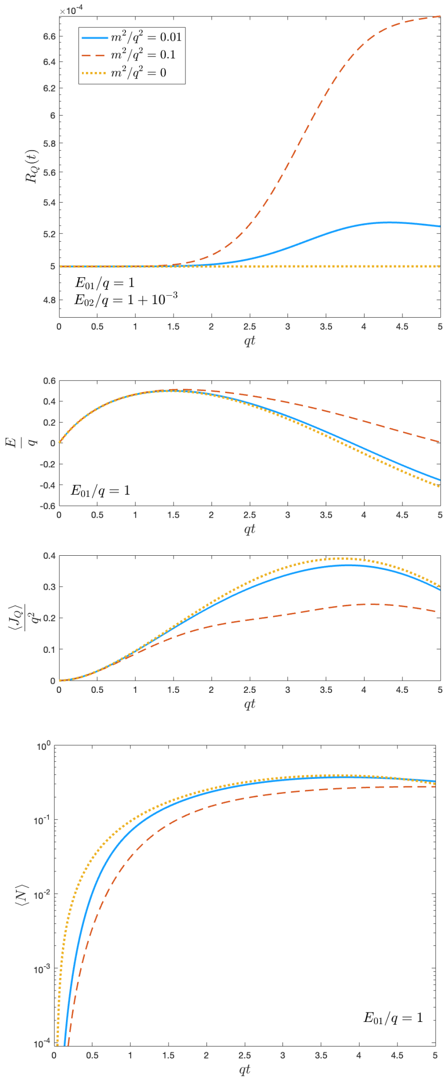

Figure 5.6: Validity analysis for the initial values $\frac{E_{01}}{q} = 1$ and $\frac{E_{02}}{q} = 1 + 10^{-3}$ and for different values of $\frac{m^2}{q^2}$. The masses are chosen such that $E_0 \gg E_{\text{crit}}$. In the upper panel we show the time evolution of the quantity R_Q in a semi-logarithmic plot compared with the optimal value R_C. In the middle panel we show the time evolution of the electric field E and the electric current $\langle J_Q \rangle_{\text{ren}}$. In the lower panel we show the time dependent particle number $\langle N \rangle$ in a semi-logharitmic plot.

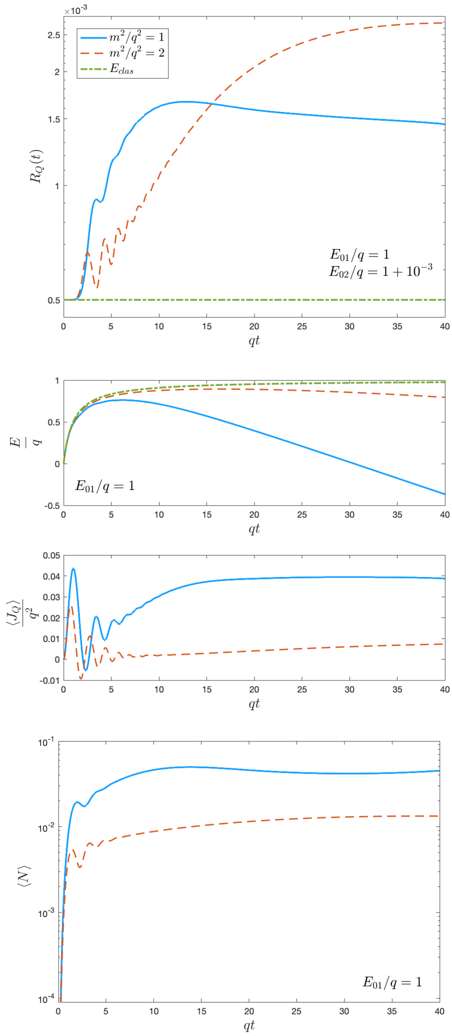

Figure 5.7: Validity analysis for the initial values $\frac{E_{01}}{q} = 1$ and $\frac{E_{02}}{q} = 1 + 10^{-3}$ and for different values of $\frac{m^2}{q^2}$. The masses are chosen such that $E_0 \sim E_{\text{crit}}$. In the upper panel we show the time evolution of the quantity R_Q in a semi-logarithmic plot compared with the optimal value R_C. In the middle panel we show the time evolution of the electric field E and the electric current $\langle J_Q \rangle_{\text{ren}}$. In the lower panel we show the time dependent particle number $\langle N \rangle$ in a semi-logharitmic plot.

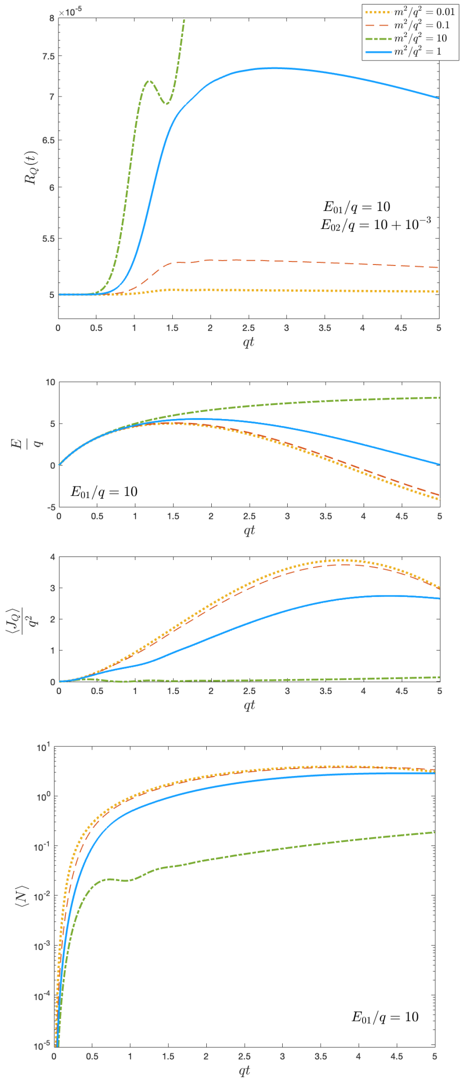

Figure 5.8: Validity analysis for the initial values $\frac{E_{01}}{q} = 10$ and $\frac{E_{02}}{q} = 10 + 10^{-3}$ and for different values of $\frac{m^2}{q^2}$. The masses are chosen from $E_0 \gg E_{\rm crit}$ (massless limit) to $E_0 \sim E_{\rm crit}$. In the upper panel we show the time evolution of the quantity R_Q in a semi-logarithmic plot compared with the optimal value R_C. In the middle panel we show the time evolution of the electric field E and the electric current $\langle J_Q \rangle_{\rm ren}$. In the lower panel we show the time dependent particle number $\langle N \rangle$ in a semi-logharitmic plot.

Chapter 6

Asymptotic expansions

In the first two parts of this thesis, we have investigated some of the main issues in quantum field theory under external conditions, always focusing on homogeneous and time-dependent backgrounds. This setup allowed us to understand the central features of this framework and was very useful for studying fundamental quantum processes such as the spontaneous creation of particles or the backreaction problem. Our methods, particularly the adiabatic renormalization prescription, appear to be consistent with general covariance and are able to capture relevant non-perturbative phenomena. However, although the adiabatic expansion turned out to be very efficient for numerical purposes, its range of applicability is very limited. For this reason, we devote this chapter to studying another type of techniques commonly used in quantum field theory in curved spacetimes (and under other external conditions) that have a more comprehensive range of application.

In the first part of the chapter, we introduce the DeWitt-Schwinger asymptotic expansion of the Feynman Green's function, which is carried out in the Schwinger proper-time formalism and that is commonly used as a point-splitting renormalization technique. In this context, we analyze two essential features. First, we show that, although their apparent difference, the DeWitt-Schwinger expansion is equivalent to the adiabatic expansion when applied to FLRW universes. It was firstly shown in Ref. [43, 32] for the stress-energy tensor and then further studied and generalized in Refs. [44, 45]. The understanding of this feature is crucial for subsequent computations. Second, we study a fascinating non-perturbative property of the DeWitt-Schwinger proper-time expansion: it can be partially summed in all terms containing the curvature scalar $R(x)$ in such a way that the new, resummed expansion does not contain any term that vanishes when the scalar curvature is replaced by zero. This factorization was first conjectured in Ref. [46] and then proved in Ref. [47]. Our contribution is to show that this special factorization has an analog in the adiabatic framework [48].

In the second part of the chapter, we argue that it is also possible to find a non-perturbative factorization for the proper-time asymptotic expansion of the one-loop effective Lagrangian in quantum electrodynamics. This factorization captures all terms proportional to the spacetime-dependent electromagnetic invariants $\mathcal{F}(x) = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\mathcal{G}(x) = \frac{1}{2}\tilde{F}_{\mu\nu}F^{\mu\nu}$. In this case, the non-perturbative global prefactor is the Euler-Heisenberg Lagrangian. As in the gravitational case, the new, partially summed, asymptotic expansion has the advantage of not containing terms that vanish when the electromagnetic invariants are replaced by zero. We study this factorization for both scalars and spin-$\frac{1}{2}$ fields. The starting point for this computation is the the proper-time expansion of the one-loop effective action derived in Ref. [49].

6.1 Equivalence between adiabatic and DeWitt Schwinger asymptotic expansions in FLRW universes

There are several strategies to renormalize physical observables in curved spacetimes. Some of the most important approaches use *point splitting techniques* [74, 75, 76, 109, 110]. In this context, the divergences of vacuum expectation values of composite field operators (such as the two-point function or the stress-energy tensor) arise when the split spacetime points, x, and x', are brought together. It is easy to see that these divergences are purely geometric. In other words, they do not depend on the quantum state. In this approach, renormalized quantities are computed by identifying and then subtracting the (geometric) divergent terms and then taking the coincident limit $x' \to x$. The main object in this framework is the Feynman Green's Function $iG_F(x, x')$, as defined in [111]. It is constructed as in flat spacetime by giving an infinitesimal negative imaginary part to the mass parameter ($m^2 \to m^2 - i\epsilon$). The (short-distance) divergences of more involved quantities (e.g., $\langle T_{\mu\nu} \rangle$) can be directly determined from the divergences of $G_F(x, x')$.[1]

On the other hand, the adiabatic regularization prescription, introduced in Chapter 2, is fundamentally based on the mode expansion of the quantized fields. This approach takes advantage of the isometries of the FLRW spacetime and deals with the high-frequency behavior of the field modes $h_k(t)$ through its adiabatic expansion.

[1] Different renormalization methods can give different answers for the stress-energy tensor. However these differences are not arbitrary: a consistent renormalized stress-energy tensor has to satisfy the Wald axioms [112, 113].

This section aims to show that, despite their conceptually different formulations, the DeWitt-Schwinger (point-splitting) renormalization method and the adiabatic regularization procedure are equivalent when evaluated in FLRW universes. To be able to make this comparison we start by upgrading the adiabatic method with a spatial point splitting $\langle\phi^2\rangle \to \langle\phi(x)\phi(x')\rangle$ ($t = t'$ but $\vec{x} \neq \vec{x}'$). For simplicity we restrict ourselves to a neutral scalar field in a four-dimensional flat FLRW spacetime. Then, we introduce the key ingredients of the DeWitt-Schwinger formalism, and finally, we illustrate the mentioned equivalence by means of the two-point function.

Adiabatic expansion with spatial point splitting

Using the results given in Section 2.2, we can easily compute the adiabatic expansion of the two-point-function

$$^{(2n)}\langle\phi^2\rangle_{\rm Ad} = \int \frac{d^3k}{2(2\pi)^3 a^3} \sum_{j=0}^{n} (W_k^{-1})^{(2j)} \,. \tag{6.1}$$

The first terms of the expansion are given in (2.23).[2] As we have argued, this expansion captures in its leading terms the UV divergences of the two-point function. Using the mode expansion of the scalar field (2.7), it is straightforward to extend this result and compute the Feynman Green's function at separated spatial points

$$\begin{aligned} i\,^{(2n)}G_F(x,x')_{\rm Ad} &\equiv \, ^{(2n)}\langle\phi(t,\vec{x})\phi(t,\vec{x}')\rangle_{\rm Ad} \qquad (6.2) \\ &= \int \frac{d^3k}{2(2\pi)^3 a^3} e^{i\vec{k}\Delta\vec{x}} \sum_{j=0}^{n} (W_k^{-1})^{(2j)} \,. \end{aligned}$$

[2]Remember: for this background configuration, all functions with odd adiabatic order $(2n+1)$ are zero.

The main difference of this result with respect to the usual (coincident) adiabatic expansion (6.1) is that the extra exponential factor $e^{i\vec{k}\Delta\vec{x}}$ acts as a regulator, making the momentum integral convergent. Therefore, Eq. (6.2) can be systematically integrated, allowing us to translate the momentum representation of the adiabatic expansion to its spacetime form. The integrals can be easily computed using the Mathematica software and are given, for example, in [44].

De Witt-Schwinger expansion

As explained in Section 3.2, the DeWitt-Schwinger expansion is an asymptotic expansion of the Feynman Green's function introduced by DeWitt in terms of the Schwinger proper-time formalism [111]. This expansion encodes, in its lowest orders, the short-distance divergent behaviour of the Green's function and it is of utmost importance in quantum field theory in curved spacetimes. Let us see how it works.

Consider a scalar field ϕ propagating on a general smooth four-dimensional spacetime. It satisfies the Klein-Gordon equation. Its associated Feynman's Green function satisfies the equation

$$\left(\Box_x + m^2 + \xi R\right) G_F(x,x') = -|g(x)|^{-1/2}\delta(x-x')\,. \tag{6.3}$$

The DeWitt-Schwinger representation of the Feynman Green's function is given by (m^2 is understood to have an infinitesimal negative imaginary part $-i\epsilon$) [111]

$$G_F(x,x') = -i\int_0^\infty ds\; e^{-im^2 s}\langle x,s|x',0\rangle\,, \tag{6.4}$$

where the kernel $\langle x,s|x',0\rangle$ satisfies (for $s>0$)

$$i\frac{\partial}{\partial s}\langle x,s|x',0\rangle = \left(\Box_x + \xi R\right)\langle x,s|x',0\rangle\,, \tag{6.5}$$

with boundary condition $|g(x)|^{-1/2}\delta\,(x-x')$ for $s\to 0$. The kernel $\langle x,s|x',0\rangle$ admits, for $s\to 0$ and x close to x', the following asymptotic expansion

$$\langle x,s|x',0\rangle = i\frac{\Delta^{1/2}\,(x,x')}{(4\pi)^2(is)^2}e^{(\sigma(x,x')+i\epsilon)/(2is)}F(x,x';is)\,, \tag{6.6}$$

where[3]

$$F(x,x';is)=\sum_0^\infty a_n(x,x')(is)^n\,. \tag{6.7}$$

The interval $\sigma(x,x')$ is one-half of the square of the geodesic distance between x and x' and obeys the defining equation $2\sigma=-g_{\mu\nu}\sigma^{;\mu}\sigma^{;\nu}$. Note that for x' close to x (in a normal neighborhood) there is only one geodesic connecting the two points. The bi-scalar $\Delta^{1/2}$ is the Van Vleck-Morette determinant and satisfies the boundary condition $\Delta^{1/2}(x,x)\to 1$. The coefficients a_n are defined by a recurrence relation derived from Eq. (6.3) and the boundary condition $a_0=1$. They are symmetric in the exchange of x and x', regular when $x'\to x$, and do not depend on the spacetime dimension. In practice, it is not always possible to find a closed form for the coefficients $a_n(x,x')$ and it becomes necessary to perform a covariant Taylor series expansion for x'

$$a_n(x,x')=a_n(x)+a_{n\mu}(x)\sigma^{;\mu}++a_{n\mu\nu}(x)\sigma^{;\mu}\sigma^{;\nu}+... \tag{6.8}$$

as well as for other geometric quantities such as $\Delta^{1/2}$. The first coincident ($x'\to x$) DeWitt coefficients are ($\bar\xi=\xi-\frac{1}{6}$) [111, 78, 79]

$$\begin{aligned}&a_0(x)=1,\quad a_1(x)=-\bar\xi R\,,\\&a_2(x)=\frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-\frac{1}{180}R^{\alpha\beta}R_{\alpha\beta}-\frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box R+\frac{1}{2}\bar\xi^2R^2\,.\end{aligned} \tag{6.9}$$

[3]Note: In spacetimes with boundaries and singularities there are additional terms (see Ref. [79]).

For higher orders see, for example, Refs. [78, 79, 36]. Using all previous results, the proper-time expansion of the Feynman Green's function in terms of the DeWitt coefficients reads

$$G_F(x,x')_{DS} = \frac{\Delta^{1/2}(x,x')}{(4\pi)^2} \int_0^\infty \frac{ds}{(is)^2} e^{-im^2 s} e^{\frac{\sigma(x,x')}{2is}} \sum_{j=0}^{\infty} a_j(x,x')(is)^j \,. \tag{6.10}$$

We note that, at coincidence $x' \to x$, this expansion can be also understood as an expansion in number of derivatives of the metric with a fixed leading term.[4]

Equivalence of the expansions

The equivalence between these two expansions in FLRW universes, and therefore between their associated renormalization prescriptions was proved in Ref. [44] and further analyzed in Ref. [45] in terms of the isometries of the spacetime. At coincidence, this equivalence can be expressed as

$$^{(2n)}G_F(x,x)_{DS} = {}^{(2n)}G_F(x,x)_{\text{Ad}} \,, \tag{6.11}$$

meaning that it holds at any adiabatic order. Here we are not going to give the strict proof but to give a nice intuition to the reader by showing that both expansions are equivalent at second adiabatic order and for $x' \to x$. Integrating (6.2) in the momentum space for $2n = 2$ (that is, at second adiabatic order) we find

$$\begin{aligned} i\,{}^{(2)}G_F(x,x')_{\text{Ad}} =& \frac{m}{4\pi^2 a|\Delta\vec{x}|} K_1(am|\Delta\vec{x}|) + \frac{R}{288\pi^2}(ma|\Delta\vec{x}|)K_1(am|\Delta\vec{x}|) \\ & - \frac{\bar{\xi}R}{8\pi^2} K_0(am|\Delta\vec{x}|) - \frac{\dot{a}^2}{96\pi^2 a^2}(am|\Delta\vec{x}|)^2 K_0(am|\Delta\vec{x}|) \,. \end{aligned} \tag{6.12}$$

[4] In FLRW spacetimes an "expansion in the number of derivatives" is equivalent to an adiabatic expansion.

where $K_0(z)$ and $K_1(z)$ are the modified Bessel functions of the second kind [91].

On the other hand, the integral of (6.10) in the proper time, considering only the first two terms of the expansion gives

$$i\,G_F(x,x')_{DS} = \frac{\Delta^{1/2}}{4\pi^2}\left[\frac{m}{\sqrt{-2\sigma}}K_1(m\sqrt{-2\sigma}) + \frac{a_1\,(x,x')}{2}K_0(m\sqrt{-2\sigma})\right]. \tag{6.13}$$

In the cosmological setup, for short distances and at time coincidence $\Delta t = 0$, the geometric quantities reduce to [114, 115] $(-2\sigma) \to a^2|\Delta\vec{x}|^2$, $a_1(x,x') \to a_1(x)$ and

$$\frac{1}{\sqrt{-2\sigma}} \to \frac{1}{a|\Delta\vec{x}|} - \frac{\dot{a}^2}{24a}|\Delta\vec{x}|\,, \tag{6.14}$$

$$\Delta^{1/2} \to 1 + \frac{1}{12}\Big(2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\Big)|\Delta\vec{x}|^2\,. \tag{6.15}$$

We note that to obtain the short-distance expansions for a spatial point splitting, we must first compute the full covariant expansions assuming $\Delta t \neq 0$. Only at the end of the computation we can take $\Delta t = 0$. Using now the Taylor expansions of the Bessel functions around $z = 0$

$$K_0(z) \sim -\gamma_E - \ln(\tfrac{z}{2}) + \frac{z^2}{4}\big(1 - \gamma_E - \ln(\tfrac{z}{2})\big) + ... \tag{6.16}$$

$$K_1(z) \sim \frac{1}{z} + \frac{z}{2}\big(-\frac{1}{2} + \gamma_E + \ln(\tfrac{z}{2})\big) + ... \tag{6.17}$$

we easily arrive to

$$^{(2)}G_F(x,x)_{DS} = {}^{(2)}G_F(x,x)_{\text{Ad}}\,. \tag{6.18}$$

This equivalence can be further extended to all adiabatic orders (6.11) but also to all orders in the short-distance expansion. In Ref. [44] it is explicitly shown that the mentioned equivalence holds at 4th adiabatic order and at second order in the short distance expansion, namely

$$^{(4)}G_F(x,x')_{DS} = {}^{(4)}G_F(x,x')_{\rm Ad} + O(|\Delta\vec{x}|^4)\,, \tag{6.19}$$

which ensures the equivalence of the renormalized stress-energy tensor.

We conclude that the adiabatic expansion can be regarded as an upgraded version of a more involved (and manifestly covariant) expansion that becomes especially useful in homogeneous and time-dependent backgrounds. This equivalence has an immediate and powerful consequence: if we find a property of the DeWitt-Schwinger expansion, it should be possible to directly find an adiabatic version of this property.

6.2 R-summed form of the adiabatic expansion in cosmological backgrounds

In 1985 it was first conjectured by L. Parker, and D. Toms [46] and then proved by I. Jack and L. Parker [47] that the DeWitt-Schwinger proper-time expansion can be partially summed in all terms containing the scalar curvature R. This partial resummation is encapsulated in an exponential factor $\exp(-is\bar{\xi}R)$, with $\bar{\xi} = \xi - \frac{1}{6}$, in such a way that the $\bar{a}_n$ coefficients of the new (resummed) expansion do not contain any term that vanishes when R is replaced by zero. This result had major physical significance to account for the effective dynamics of the Universe, and the observed cosmological acceleration [116, 117, 118, 119, 120, 121, 122, 123]. This section aims to explore this property of the DeWitt-Schwinger expansion in terms of the adiabatic expansion. We first introduce the R-summed DeWitt-Schwinger expansion and then propose a R-summed form of the adiabatic expansion. Finally, we test the equivalence between these two expansions making use of Eq. (6.11). Our results are based on Ref. [48].

In Refs. [46, 47] it was shown that the heat-kernel (6.5) adimits a expansion of the form

$$\langle x, s|x', 0\rangle = i\frac{\Delta^{1/2}(x,x')}{(4\pi)^2(is)^2}e^{\frac{\sigma(x,x')}{2is}}e^{-i\bar{\xi}R(x')s}\bar{F}(x,x';is)\,, \tag{6.20}$$

where, the function $\bar{F}(x,x';is)$ accepts the proper-time expansion

$$\bar{F}(x,x';is) = \sum_j \bar{a}_j(x,x')(is)^j. \tag{6.21}$$

At coincidence $x' \to x$, the first coefficients of the expansion are [7]

$$\begin{aligned} &\bar{a}_0(x) = 1, \quad \bar{a}_1(x) = 0, \\ &\bar{a}_2(x) = \frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180}R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box R\,. \end{aligned} \tag{6.22}$$

The main advantage of this proposal is that the new coefficients $\bar{a}_n$ do not depend on R (although they can depend on its derivatives): all the dependence with the scalar curvature is captured in the exponential prefactor. More specifically, it has been proven for general spacetimes in arbitrary dimensions, that (6.20) depends on R only by the overall exponential factor. For simplicity and without loss of generality, from now on we restrict ourselves to the coincident limit. From Eqs. (6.6) and (6.20) we can directly find the relation between the standard and the new coefficients,

$$\bar{a}_n(x) = \sum_{k=0}^{n} a_{n-k}(x)\frac{(\bar{\xi}R)^k}{k!}\,. \tag{6.23}$$

The R-summed form of the DeWitt-Schwinger expansion of the Feynman Green's function at coincidence $x' \to x$ takes the form

$$^{(2n)}\bar{G}_F(x,x)_{DS} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{(is)^2}e^{-i(m^2+\bar{\xi}R)s}\sum_{j=0}^{n}\bar{a}_j(x)(is)^j\,, \tag{6.24}$$

where the coefficients $\bar{a}_n$ are of adiabatic order $2n$.

The question now is simple: how can we translate this factorization to the adiabatic expansion of the field modes? We recall that the adiabatic expansion of the scalar modes $h_k(t)$ is univocally defined once the leading order $\omega_k^{(0)}$ is fixed. Therefore, in order to find a new adiabatic expansion we should change the leading order. If we look at the two first terms of the adiabatic expansion of the two-point function

$$|h_k|^2_{\mathrm{Ad}} \sim \frac{1}{\omega} - \frac{\bar{\xi} R}{2\omega^3}\,, \tag{6.25}$$

we find that they can be regarded as the first orders of the expansion

$$\frac{1}{(\frac{k^2}{a^2} + m^2 + \bar{\xi} R)^{1/2}} \sim \frac{1}{\omega} - \frac{\bar{\xi} R}{2\omega^3} + O(R^2)\cdots \tag{6.26}$$

From this intuition, we propose a new adiabatic expansion with leading order

$$\omega \quad \rightarrow \quad \bar{\omega} = \sqrt{\frac{k^2}{a^2} + m^2 + \bar{\xi} R}\,. \tag{6.27}$$

The new adiabatic expansion can be built as usual. First, we propose the (modified) WKB ansatz with its associated adiabatic expansion

$$h_k(t) = \frac{1}{\sqrt{\bar{W}_k(t)}} e^{-i\int^t \bar{W}_k(t')dt'}\,, \quad \bar{W}_k(t) = \sum_{n=0}^{\infty} \bar{\omega}_k^{(n)}\,. \tag{6.28}$$

Then, inserting the WKB ansatz into the mode equation (2.8) and regrouping the terms to explicitly include the modified frequency (6.27) we find

$$\bar{W}_k^2 = \bar{\omega}^2 + \bar{\sigma} + \frac{3}{4}\frac{\dot{\bar{W}}_k^2}{\bar{W}_k^2} - \frac{1}{2}\frac{\ddot{\bar{W}}_k}{\bar{W}_k}\,, \tag{6.29}$$

where $\bar{\sigma} = \sigma - \bar{\xi} R$ is a function of adiabatic order two. As stated above, we fix $\bar{\omega}_k^{(0)} = \bar{\omega}$. As usual, we can obtain the next-to-leading order terms

$\bar{\omega}_k^{(n)}$ by expanding the function $\bar{W}_k$ adiabatically and grouping terms with the same adiabatic order. However, there is an important difference with respect of the usual adiabatic expansion. We note that although $\bar{\xi}R$ appears at the leading order of the expansion, it should be still considered a function of adiabatic order two. For example, if we look at the time derivative of the modified frequency

$$\dot{\bar{\omega}} = -\frac{2k^2\dot{a}}{2a^3\bar{\omega}} + \frac{\bar{\xi}\dot{R}}{2\bar{\omega}}\,, \tag{6.30}$$

we see that there is a term of adiabatic order one ($\dot{a}$) and a term of adiabatic order three ($\dot{R}$). It means that one should be careful when computing the $\bar{\omega}_k^{(n)}$ coefficients. For practical proposes, the most convenient way to do it is to use the same expressions that we have for the usual adiabatic expressions [see Eqs. (2.23) and (2.24)] with the obvious changes and truncate them after the time derivatives are performed. For example, for the second adiabatic order we get

$$\bar{\omega}_k^{(2)} = \frac{\bar{\sigma}}{2\bar{\omega}} + \frac{3\dot{\bar{\omega}}^2}{8\bar{\omega}^3} - \frac{\ddot{\bar{\omega}}}{4\bar{\omega}^2}\bigg|_{(2)} = +\frac{\bar{\sigma}}{2\bar{\omega}} + \frac{5\dot{a}^2k^4}{8a^6\bar{\omega}^5} - \frac{3\dot{a}^2k^2}{4a^4\bar{\omega}^3} + \frac{k^2\ddot{a}}{4a^3\bar{\omega}^3}\,. \tag{6.31}$$

The next-to-leading orders can be obtained using this (iterative/truncating) procedure, and can be found in Ref. [48]. As in the usual adiabatic expansion $\bar{\omega}_k^{(2n+1)} = 0$. The new adiabatic expansion of the Green's function at coincidence is

$$i\,{}^{(n)}\bar{G}_F(x,x)_{\rm Ad} = \frac{1}{4\pi^2a^3}\int_0^\infty dk\,k^2\sum_{j=0}^{n}(\bar{W}_k^{-1})^{(j)}\,. \tag{6.32}$$

Based on previous results, we claim that the R-summed adiabatic expansion above is equivalent to the DeWitt-Schwinger expansion in FLRW universes at any adiabatic order, i.e.,

$${}^{(2n)}\bar{G}_F(x,x)_{\rm Ad} = {}^{(2n)}\bar{G}_F(x,x)_{DS}\,. \tag{6.33}$$

Let us test if this conjecture holds.

Equivalence between the R-summed expansions

Since we already know that the standard adiabatic and DeWitt-Schwinger expansions are equivalent (6.11), to test our conjecture, we do not need to compare the modified expansions $\bar{G}_F$ explicitly, but only the (finite) differences

$$^{(2n)}\Delta G_{DS} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{(is)^2}\sum_{j=0}^{n}\left[e^{-is(m^2+\bar{\xi}R)}\,\bar{a}_j(x)(is)^j - e^{-ism^2}a_j(x)(is)^j\right], \tag{6.34}$$

and

$$^{(2n)}\Delta G_{\rm Ad} = \frac{(-i)}{4\pi^2 a^3}\int_0^\infty dk\,k^2\sum_{j=0}^{2n}\left[\left(\bar{W}_k^{-1}\right)^{(j)} - \left(W_k^{-1}\right)^{(j)}\right], \tag{6.35}$$

where

$$^{(2n)}\Delta G_h = {}^{(2n)}\bar{G}_F(x,x)_h - {}^{(2n)}G_F(x,x)_h\,, \tag{6.36}$$

with $h = DS, \mathrm{Ad}$. That is, by checking that

$$^{(2n)}\Delta G_{DS} = {}^{(2n)}\Delta G_{\rm Ad} \tag{6.37}$$

holds, we automatically check that our conjecture (6.33) is satisfied. We note that the differences (6.34) and (6.35) are finite by construction, and therefore, the integrals can be easily evaluated.

On one hand, the difference between the DeWitt-Schwinger expansions (6.34) can be directly integrated. For $n \geq 2$ we find

$$^{(2n)}\Delta G_{DS} = \frac{(-i)}{(4\pi)^2}\left[M^2\log\left(\frac{M^2}{m^2}\right) - \bar{\xi}R + \sum_{j=2}^{n}(j-2)!\left(\frac{\bar{a}_j}{M^{2j-2}} - \frac{a_j}{m^{2j-2}}\right)\right], \tag{6.38}$$

where $M^2 = m^2 + \bar{\xi}R$. Note that for $n = 1$ the difference gives the the two first terms of the expression above. On the other hand, the adiabatic difference (6.35) can be also integrated in the momentum space, giving us an expression in terms of the scale factor $a(t)$ and its derivatives. If the conjecture is satisfied, therefore

$$^{(2n)}\Delta G_{\mathrm{Ad}} = \frac{(-i)}{(4\pi)^2}\Big[M^2 \log\Big(\frac{M^2}{m^2}\Big) - \bar{\xi}R + \sum_{j=2}^{n}(j-2)!\Big(\frac{\bar{a}_j}{M^{2j-2}} - \frac{a_j}{m^{2j-2}}\Big)\Big]\,. \tag{6.39}$$

We have checked that this equality is satisfied up to eight adiabatic order ($n = 4$). We believe that this provides a good evidence of our conjecture.

Furthermore, for $n \geq 2$, the momentum integrals in $\bar{G}_{\mathrm{Ad}}$ (as well as the proper time integrals in $\bar{G}_{DS}$) are finite. This allows us to obtain the $\bar{a}_n$ coefficients for FLRW universes directly from the modified adiabatic expansion, namely

$$\frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \left(\bar{W}_k^{-1}\right)^{(2n)} = \frac{i}{(4\pi)^2}\int_0^\infty \frac{ds}{(is)^2} e^{-is\left(m^2+\bar{\xi}R\right)}\bar{a}_n(x)(is)^n$$

$$= \frac{(n-2)!}{(4\pi)^2}\frac{\bar{a}_n}{M^{2n-2}}\,. \tag{6.40}$$

The explicit expressions of the modified DeWitt coefficients $\bar{a}_n$ in FLRW universes can be found, for example, in Ref. [48].

Effective action

One of the advantages of the R-summed form of the DeWitt-Schwinger expansion is that it allows to derive approximate forms of one-loop effective actions easily. We find it useful to show how it works. It will serve to familiarize with some of the techniques used in the next section. The formal one-loop effective action W can be obtained by integrating out the degrees of freedom of the quantized scalar field. It is given by [7]

$$W = \mathcal{S}_{\rm class} - \frac{1}{2}\int d^4x \int_0^\infty \frac{ds}{is} e^{-ism^2} \langle x, s | x, 0\rangle \,. \tag{6.41}$$

where $\mathcal{S}_{\rm class}$ is the classical action that includes the gravitational interaction. According to Eq. (6.20), the quantum contribution can be written as

$$\frac{1}{32\pi^2}\int d^4x \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \bar{F}(x,x;is)\,. \tag{6.42}$$

where $\bar{F}(x,x;is)$ is the R-summed expansion given in Eq. (6.21). The expansion above is ultraviolet divergent and has to be renormalized. In this context, and following the same approach as with the Feynman propagator, we renormalize the effective action by subtracting the usual DeWitt-Schwinger expansion up to and including the second adiabatic order. For the quantum state, we also cut the modified DeWitt expansion at second order. With this prescription, we find

$$W = \mathcal{S}_{\rm class} + \frac{1}{32\pi^2}\int d^4x \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \Big[e^{-is\bar{\xi}R} \left(1 + \bar{a}_1(is) + \bar{a}_2(is)^2\right) - \left(1 + a_1(is) + a_2(is)^2\right) \Big] \,. \tag{6.43}$$

The (finite) quantum contribution can be integrated in the proper time. We obtain the following effective contribution

$$W = \mathcal{S}_{\rm class} + \int d^4x \mathcal{L}_{\rm eff}\,, \tag{6.44}$$

where

$$\mathcal{L}_{\rm eff} = \frac{1}{64\pi^2}\left\{ (\bar{\xi}R)\left(m^2 + \frac{3}{2}\bar{\xi}R\right) - \left(M^4 + 2\bar{a}_2\right)\log\left|\frac{M^2}{m^2}\right| \right\} + \frac{i}{64\pi}\left[M^4 + 2\bar{a}_2\right]\Theta\left(-M^2\right)\,. \tag{6.45}$$

With this easy computation we recover the result given in Ref. [116] obtained by using the ζ-function regularization method. All the results presented in this section can be extended when including also a scalar background field Φ ($\mathcal{L}_{\text{class}} \to \mathcal{L}_{\text{class}} - h\Phi\phi^2$). In this case it is also possible to build a (R, Φ)-summed form of the Feynman propagator with the factor $\exp[-is(\bar{\xi}R + h\Phi)]$. The effective mass of the R-summed adiabatic expansion transforms into $M^2 = \bar{\xi}R + h\Phi$, and both expansions turned out to be equivalent as in the pure gravitational case (for more details, see [48]).

In the following section, we propose a novel factorization for the one-loop QED effective action, analogous to the R-summed form in gravitational backgrounds. This time we restrict ourselves to Minkowski spacetime and work with the proper-time expansion of the effective action for both scalar and spin-$\frac{1}{2}$ quantum fields.

6.3 $(\mathcal{F}, \mathcal{G})$-summed form of the QED effective action

Let us now consider quantized scalar and spin-$\frac{1}{2}$ fields propagating in Minkowski spacetime in the presence of a electromagnetic (background) field. In this context, a very interesting problem is to find corrections to the classical electromagnetic Lagrangian $\mathcal{L}_{\text{class}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ induced by quantum effects. As one can imagine, it is, in general, a very complex task. However, we can still find some answers. If one restricts to a constant electromagnetic background, it is possible to find an exact expression for the one-loop effective Lagrangian for both scalars and Dirac fields. In 1936 Euler and Heisenberg derived the formal expression of the one-loop effective Lagrangian for spinor QED [37]. Some months later, Weisskopf derived a similar expression for scalar QED [124] (for the spin-1 version, see [125]).

The so-called Euler-Heisenberg Lagrangian is usually written in terms of the proper-time parameter s, but, apart from this, it can be expressed in different forms. For convenience, we use the version proposed in [7], namely

$$\mathcal{L}^{(1)}_{\text{scalar}} = \int_0^\infty \frac{ds}{s} e^{-im^2 s} \left[\det\left(\frac{esF}{\sinh(esF)}\right)\right]^{1/2}, \tag{6.46}$$

$$\mathcal{L}^{(1)}_{\text{spinor}} = -\frac{1}{2}\int_0^\infty \frac{ds}{s} e^{-im^2 s} \left[\det\left(\frac{esF}{\sinh(esF)}\right)\right]^{1/2} \operatorname{tr}\left[e^{-\frac{1}{2}esF_{\mu\nu}\sigma^{\mu\nu}}\right], \tag{6.47}$$

where $F = F^{\mu}_{\nu}$ and $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Although it is not explicit from Eqs. (6.46) and (6.47), we recall that the expressions above only depend on the electromagnetic invariants $\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\mathcal{G} = \frac{1}{4}\tilde{F}_{\mu\nu}F^{\mu\nu}$. These one-loop effective corrections to the classical Lagrangian had very important implications associated with their intrinsic non-linear nature, such as light-by-light scattering, vacuum polarization or pair creation from vacuum [18], etc (see Refs. [86, 126]).

For generic backgrounds, a closed expression for the one-loop QED effective Lagrangian is still unknown. Nevertheless, it is possible to build an asymptotic expansion in terms of the proper-time that captures some relevant information for arbitrary external configurations. This task is analogous to the construction of the DeWitt-Schwinger proper-time expansion for the Feynman propagator and the heat-kernel $\langle x, s|x, 0\rangle$ in curved spacetimes, explained at the beginning of Chapter 6. In this context, it was found that the proper-time expansion expansion admits an exact resummation in all terms involving the scalar curvature $R(x)$. This partial resummation generates an alternative asymptotic expansion (6.20) that captures the exact dependence on the Ricci scalar in an an overall factor, that for scalar fields reads $e^{-is\bar{\xi}R(x)}$ (see Refs. [46, 47] and also Section 6.2). The new expansion has clear advantage with respect to the previous one: its coefficients do not contain any term that vanishes when $R(x)$ is replaced by zero. The idea of this factorization came from the exact solution of the heat-kernel for the Static Einstein Universe [127],

$$\langle x, s|x, 0\rangle^{\text{static}} = \frac{i}{(4\pi i s)^2} e^{-i\bar{\xi}Rs}. \tag{6.48}$$

Keeping this idea in mind, we propose the following conjecture in the context of scalar and spinor QED [50]: *The proper-time asymptotic expansion of the QED effective Lagrangian admits an exact resummation in all terms involving the field-strength in variants* $F(x) = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ *and* $\mathcal{G}(x) = \frac{1}{4}\tilde{F}_{\mu\nu}F^{\mu\nu}$*. The form of the factor is just the Heisenberg-Euler Lagrangian for QED, where the electric and magnetic fields depend arbitrarily on space-time coordinates.*

In the rest of the section we study in detail this conjecture, giving strong evidence for its validity. The content of this section is based on Refs. [50, 51]. Our results are built on previous calculations for the proper-time expansion of the effective action obtained in the context of the string-inspired worldline formalism [49, 128, 129, 130, 131]. Most of the computations in this work have been done with the help of the xAct package of the *Mathematica Software* [132]. The traces of products of gamma matrices for spin-$\frac{1}{2}$ fields have been evaluated with the FeynCalc package [133].

Scalar fields

Consider a quantized scalar field propagating in an electromagnetic background field. It satisfies the Klein-Gordon equation

$$(D_\mu D^\mu + m^2)\phi = 0\,, \tag{6.49}$$

where $D_\mu\phi = (\partial_\mu + ieA)\phi$. The one-loop effective Lagrangian associated with the scalar field $\mathcal{L}^{(1)}_{\text{scalar}}$ admits an asymptotic expansion in terms of the proper-time parameter that captures some general behavior of the (unknown) formal one-loop effective Lagrangian for arbitrary electromagnetic backgrounds. It consists of an expansion in the number of derivatives and external fields and is related to the heat-kernel (DeWitt-Schwinger) expansion that we have used in a gravitational scenario (6.6). However, for the effective action, we have some extra freedom. In this case, the relevant asymptotic expansion can be defined up to total derivatives. For this reason, we find it extremely convenient to work with the asymptotic expansion proposed in Ref. [49], that has one clear advantage: its the coefficients are written on a minimal basis [134]. This expansion reads

$$\mathcal{L}^{(1)}_{\text{scalar}} = -i\int_0^\infty \frac{ds}{s} e^{-im^2 s} g(x;is)\,, \tag{6.50}$$

where

$$g(x;is) = \frac{i}{(4\pi is)^2}\sum_{n=0}^{\infty}\frac{(-is)^n}{n!}O_n(x)\,. \tag{6.51}$$

We note that

$$g(x;is) = \langle x,s|x,0\rangle + \textit{total derivatives}\,. \tag{6.52}$$

The first coefficients of the expansion are[5]

$$\begin{aligned}
O_1 =&0\,,\\
O_2 =&-\tfrac{e^2}{6}F_{\kappa\lambda}F^{\kappa\lambda}\,,\\
O_3 =&-\tfrac{e^2}{20}\partial_\mu F_{\kappa\lambda}\partial^\mu F^{\kappa\lambda}\,,\\
O_4 =&\tfrac{e^4}{15}F^\mu_\kappa F^{\kappa\lambda}F^\nu_\lambda F_{\mu\nu} + \tfrac{e^4}{12}F_{\kappa\lambda}F^{\kappa\lambda}F_{\mu\nu}F^{\mu\nu}\\
&-\tfrac{e^2}{70}\partial_\nu\partial_\mu F_{\kappa\lambda}\partial^\nu\partial^\mu F^{\kappa\lambda}\,,
\end{aligned} \tag{6.53}$$

[5] They have been obtained up to $n = 6$.

$$
\begin{aligned}
O_5 =& \tfrac{2e^4}{7} F^{\kappa\lambda} F^{\mu\nu} \partial_\lambda F_{\nu\rho} \partial_\mu F_\kappa{}^\rho - \tfrac{4e^4}{63} F_\kappa{}^\mu F^{\kappa\lambda} \partial_\lambda F^{\nu\rho} \partial_\mu F_{\nu\rho} \\
& - \tfrac{e^4}{9} F_\kappa{}^\mu F^{\kappa\lambda} F^{\nu\rho} \partial_\mu \partial_\lambda F_{\nu\rho} - \tfrac{16e^4}{63} F^{\kappa\lambda} F^{\mu\nu} \partial_\mu F_\kappa{}^\rho \partial_\nu F_{\lambda\rho} \\
& + \tfrac{5e^4}{18} F^{\kappa\lambda} F^{\mu\nu} \partial_\rho F_{\mu\nu} \partial^\rho F_{\kappa\lambda} + \tfrac{34e^4}{189} F^{\kappa\lambda} F^{\mu\nu} \partial_\nu F_{\lambda\rho} \partial^\rho F_{\kappa\mu} \\
& + \tfrac{25e^4}{189} F^{\kappa\lambda} F^{\mu\nu} \partial_\rho F_{\lambda\nu} \partial^\rho F_{\kappa\mu} + \tfrac{4e^4}{21} F_\kappa{}^\mu F^{\kappa\lambda} \partial_\rho F_{\mu\nu} \partial^\rho F_\lambda{}^\mu \\
& + \tfrac{e^4}{12} F_{\kappa\lambda} F^{\kappa\lambda} \partial_\rho F_{\mu\nu} \partial^\rho F^{\mu\nu} - \tfrac{e^2}{252} \partial_\rho \partial_\nu \partial_\mu F_{\kappa\lambda} \partial^\rho \partial^\nu \partial^\mu F^{\kappa\lambda} .
\end{aligned}
\tag{6.54}
$$

We recall that this expansion can also be regarded as an "adiabatic" expansion with A_μ considered of adiabatic order one (and therefore $F_{\mu\nu}$ of adiabatic order 2, $\partial_\mu F_{\nu\lambda}$ of adiabatic order three, and so on). As we have stressed, the advantage of the expansion that we are using here is that the coefficients O_n are written on a minimal basis. They have been obtained using the Bianchi identities, the antisymmetry of $F_{\mu\nu}$ and also integration by parts.

From this expansion, we can easily understand our conjecture. We claim that Eq. (6.51) can be partially summed in all terms containing the field strength invariants $\mathcal{F}$ and $\mathcal{G}$. The result of this partial sum is a new asymptotic expansion $\bar{g}(x; is)$ with some new coefficients $\bar{O}_n$ that, if our conjecture is satisfied, do not contain any term that vanishes when $\mathcal{F}$ and $\mathcal{G}$ are replaced by zero. The partial sum is encapsulated in an overall factor with the same form as the (scalar) Euler-Heisenberg effective Lagrangian but with an arbitrary spacetime dependence. That is

$$
\mathcal{L}^{(1)}_{\text{scalar}} = \int_0^\infty \frac{ds}{s} e^{-im^2 s} \left[\det \left(\frac{esF(x)}{\sinh(esF(x))} \right) \right]^{1/2} \bar{g}(x; is), \tag{6.55}
$$

where

$$
\bar{g}(x; is) = \frac{1}{(4\pi i s)^2} \sum_{n=0}^{\infty} \frac{(-is)^n}{n!} \bar{O}_n(x) \,. \tag{6.56}
$$

The two asymptotic expansions are then related by the following equation

$$g(x; is) = \left[\det\left(\frac{esF(x)}{\sinh(esF(x))}\right)\right]^{1/2} \bar{g}(x; is)\,. \tag{6.57}$$

The way to test our conjecture is as follows. From Eq. (6.57) and using the expressions of the coefficients O_n [see Eqs. (6.53) and (6.54)] and also the $s \to 0$ Taylor expansion of the Euler-Heisenberg determinant

$$\left[\det\left(\frac{esF(x)}{\sinh(esF(x))}\right)\right]^{1/2} \sim 1 + U_2(x)(-is)^2 + U_4(x)(-is)^4 + U_6(x)(-is)^6 + \cdots \tag{6.58}$$

where

$$U_2(x) = \frac{e^2}{12}\,\mathrm{Tr}\left(F^2\right), \tag{6.59}$$

$$U_4(x) = \frac{e^4}{288}\,\mathrm{Tr}\left(F^2\right)^2 + \frac{e^4}{360}\,\mathrm{Tr}\left(F^4\right), \tag{6.60}$$

$$U_6(x) = \frac{e^6}{10368}\,\mathrm{Tr}\left(F^2\right)^3 + \frac{e^6}{4320}\,\mathrm{Tr}\left(F^2\right)\mathrm{Tr}\left(F^4\right) + \frac{e^6}{5670}\,\mathrm{Tr}\left(F^6\right), \tag{6.61}$$

we can directly obtain the coefficients of the new expansion $\bar{O}_n$. If these coefficients do not have any term that vainsh when $\mathcal{F}$ and $\mathcal{G}$ are replaced by zero, then, our conjecture is satisfied. More specifically, substituting (6.51), (6.56) and (6.58) into (6.57) we end up with

$$\begin{aligned} O_0 + O_1 + (-is) + \frac{O_2}{2}(-is)^2 \ldots \\ = \left(1 + U_2(-is)^2 + \ldots\right) \cdot \left(\bar{O}_0 + \bar{O}_1(-is) + \frac{\bar{O}_2}{2}(-is)^2 + \ldots\right) \end{aligned} \tag{6.62}$$

and grouping terms with the same order we arrive to $\bar{O}_0 = 1, \bar{O}_1 = 0$,

$$
\begin{aligned}
\bar{O}_2 &= O_2 - 2U_2 = 0\,,\\
\bar{O}_3 &= O_3\,,\\
\bar{O}_4 &= O_4 - 4!U_4\,,\\
\bar{O}_5 &= O_4 - 20U_2O_3\,,\\
\bar{O}_6 &= O_6 - 6!U_6 - 30U_2\bar{O}_4\,.
\end{aligned}
\tag{6.63}
$$

More explicitly

$$
\begin{aligned}
\bar{O}_3 =& -\tfrac{e^2}{20}\partial_\mu F_{\kappa\lambda}\partial^\mu F^{\kappa\lambda}\,,\\
\bar{O}_4 =& -\tfrac{e^2}{70}\partial_\nu\partial_\mu F_{\kappa\lambda}\partial^\nu\partial^\mu F^{\kappa\lambda}\,,\\
\bar{O}_5 =& \tfrac{2e^4}{7}F^{\kappa\lambda}F^{\mu\nu}\partial_\lambda F_{\nu\rho}\partial_\mu F^\rho_\kappa - \tfrac{4e^4}{63}F^\mu_\kappa F^{\kappa\lambda}\partial_\lambda F^{\nu\rho}\partial_\mu F_{\nu\rho}\\
&- \tfrac{e^4}{9}F^\mu_\kappa F^{\kappa\lambda}F^{\nu\rho}\partial_\mu\partial_\lambda F_{\nu\rho} - \tfrac{16e^4}{63}F^{\kappa\lambda}F^{\mu\nu}\partial_\mu F_\kappa{}^\rho\partial_\nu F_{\lambda\rho}\\
&+ \tfrac{5e^4}{18}F^{\kappa\lambda}F^{\mu\nu}\partial_\rho F_{\mu\nu}\partial^\rho F_{\kappa\lambda} + \tfrac{34e^4}{189}F^{\kappa\lambda}F^{\mu\nu}\partial_\nu F_{\lambda\rho}\partial^\rho F_{\kappa\mu}\\
&+ \tfrac{25e^4}{189}F^{\kappa\lambda}F^{\mu\nu}\partial_\rho F_{\lambda\nu}\partial^\rho F_{\kappa\mu} + \tfrac{4e^4}{21}F^\mu_\kappa F^{\kappa\lambda}\partial_\rho F_{\mu\nu}\partial^\rho F^\nu_\lambda\\
&- \tfrac{e^2}{252}\partial_\rho\partial_\nu\partial_\mu F_{\kappa\lambda}\partial^\rho\partial^\nu\partial^\mu F^{\kappa\lambda}\,.
\end{aligned}
\tag{6.64}
$$

The coefficient $\bar{O}_6$ is given in [51] and has 36 terms. The first non-trivial test for our conjecture is in the coefficient $\bar{O}_5$. If our conjecture is true, in O_5 there should be a term going as $U_2O_3 \sim \mathcal{F}O_3$ that has to be completely reabsorbed by the Euler-Heisenberg factor. As we can see, this is the case, and the term proportional to $\mathcal{F}$ in (6.54) has disappeared in $\bar{O}_5$. The same happens with O_6 (which is the last available coefficient in the literature). It has 41 terms, of which 5 are proportional to $\mathcal{F}$ or $\mathcal{G}$. $\bar{O}_6$ has only 36 terms, none of them depending on the electromagnetic invariants.

As a final remark, we would like to stress that the same factorization happens if we include a scalar background field $\Phi(x)$. In this case, the

Klein-Gordon equation becomes $(D^\mu D_\mu + m^2 + \Phi)\phi = 0$. For this situation there also exist in the literature an asymptotic expansion as the one given in (6.51) (see Refs. [128, 49]). We have checked that we can build a double factorization, that is,

$$g(x;is) = \left[\det\left(\frac{esF(x)}{\sinh(esF(x))}\right)\right]^{1/2} e^{-is\Phi(x)}\bar{g}(x;is) \tag{6.65}$$

where the $\bar{O}_n$ coefficients of the new $\bar{g}$ expansion (6.56) do not have any terms that vanish when $\mathcal{F}$, $\mathcal{G}$ or Φ are replaced by zero. As before, we have checked this extended conjecture up to $n = 6$ (the last available coefficient). We recall that O_6 has 97 terms while $\bar{O}_6$ has only 62. The factorization of the scalar field was also found in Ref. [47, 128] without an electromagnetic background.

Spin-$\frac{1}{2}$ fields

Consider now a charged, massive spin-$\frac{1}{2}$ field ψ. It obeys the (modified) Klein-Gordon equation

$$(D_\mu D^\mu + m^2 - \frac{i}{2}eF_{\mu\nu}\sigma^{\mu\nu})\psi = 0, \tag{6.66}$$

As in the scalar case, the induced one-loop effective Lagrangian for an arbitrary electromagnetic background admits a proper-time asymptotic expansion

$$\mathcal{L}^{(1)}_{\text{spinor}} = \frac{i}{2}\int_0^\infty \frac{ds}{s} e^{-im^2 s} h(x;is), \tag{6.67}$$

where

$$\begin{aligned} h(x;is) &= \frac{i}{(4\pi i s)^2}\,\mathrm{tr}\sum_{n=0}^{\infty}\frac{(-is)^n}{n!}O_n(x) \\ &\equiv \frac{i}{(4\pi i s)^2}\sum_{n=0}^{\infty}\frac{(-is)^n}{n!}o_n(x) \end{aligned} \tag{6.68}$$

with $o_n = \operatorname{tr} O_n$, and $O_n = I_{4\times 4}$ so that $o_0 = 4$ (note that the trace "tr" refers to the Dirac trace). This expansion is also obtained using the string-inspired method in the word-line formalism (we have borrowed the coefficients from Ref. [49]), and is directly related with the heat-kernel expansion,

$$h(x;is) = \operatorname{tr}\langle x,s|x,0\rangle + \textit{total derivatives}\,. \tag{6.69}$$

The first coefficients of the expansion are

$$\begin{aligned}
o_1 =&0\,,\\
o_2 =&\tfrac{4e^2}{3}F_{\kappa\lambda}F^{\kappa\lambda}\,,\\
o_3 =&\tfrac{4e^2}{5}\partial_\mu F_{\kappa\lambda}\partial^\mu F^{\kappa\lambda}\,,\\
o_4 =&-\tfrac{56e^4}{15}F_\kappa{}^\mu F^{\kappa\lambda}F_\lambda{}^\nu F_{\mu\nu}+\tfrac{4e^4}{3}F_{\kappa\lambda}F^{\kappa\lambda}F_{\mu\nu}F^{\mu\nu}\\
&+\tfrac{12e^2}{35}\partial_\nu\partial_\mu F_{\kappa\lambda}\partial^\nu\partial^\mu F^{\kappa\lambda}\,,\\
o_5 =&\tfrac{8e^4}{7}F^{\kappa\lambda}F^{\mu\nu}\partial_\lambda F_{\nu\rho}\partial_\mu F_\kappa{}^\rho-\tfrac{16e^4}{63}F_\kappa{}^\mu F^{\kappa\lambda}\partial_\lambda F^{\nu\rho}\partial_\mu F_{\nu\rho}\\
&+\tfrac{8e^4}{9}F_\kappa{}^\mu F^{\kappa\lambda}F^{\nu\rho}\partial_\mu\partial_\lambda F_{\nu\rho}-\tfrac{232e^4}{63}F^{\kappa\lambda}F^{\mu\nu}\partial_\mu F_\kappa{}^\rho\partial_\nu F_{\lambda\rho}\\
&+\tfrac{40e^4}{9}F^{\kappa\lambda}F^{\mu\nu}\partial_\rho F_{\mu\nu}\partial^\rho F_{\kappa\lambda}+\tfrac{136e^4}{189}F^{\kappa\lambda}F^{\mu\nu}\partial_\nu F_{\lambda\rho}\partial^\rho F_{\kappa\mu}\\
&-\tfrac{656e^4}{189}F^{\kappa\lambda}F^{\mu\nu}\partial_\rho F_{\lambda\nu}\partial^\rho F_{\kappa\mu}-\tfrac{320e^4}{21}F_\kappa{}^\mu F^{\kappa\lambda}\partial_\rho F_{\mu\nu}\partial^\rho F_\lambda{}^\nu\\
&+\tfrac{8e^4}{3}F_{\kappa\lambda}F^{\kappa\lambda}\partial_\rho F_{\mu\nu}\partial^\rho F^{\mu\nu}+\tfrac{8e^2}{63}\partial_\rho\partial_\nu\partial_\mu F_{\kappa\lambda}\partial^\rho\partial^\nu\partial^\mu F^{\kappa\lambda}.
\end{aligned} \tag{6.70}$$

These coefficients have been obtained up to order $n = 6$ and are also expressed on the minimal basis. Our conjecture for the spin-$\frac{1}{2}$ can be tested in the same way as for the scalar field. We claim that expansion (6.68) can be partially summed in all terms containing the electromagnetic invariants $\mathcal{F}$ and $\mathcal{G}$ in such a way that the new, resummed expansion does not contain any term that vanishes when the invariants are replaced by zero. We also claim that the non-perturbative factor is precisely the Euler-Heisenberg Lagrangian but with an arbitrary spacetime dependence. More explicitly, we propose a new, partially summed asymptotic expansion

$$\mathcal{L}^{(1)}_{\text{spinor}} = -\frac{1}{2}\int_0^\infty \frac{ds}{s} e^{-im^2 s}\left[\det\left(\frac{esF(x)}{\sinh(esF(x))}\right)\right]^{1/2} \times \operatorname{tr}[e^{-\frac{1}{2}esF_{\mu\nu}(x)\sigma^{\mu\nu}}]\,\bar{h}(x;is)\,, \tag{6.71}$$

where

$$\bar{h}(x;is) = \frac{i}{(4\pi is)^2}\sum_{n=0}^{\infty}\frac{(-is)^n}{n!}\bar{o}_n(x)\,. \tag{6.72}$$

The usual and the new $(\mathcal{F},\mathcal{G})$-summed expansions are directly related by

$$\begin{aligned} h(x;is) &= \left[\det\left(\frac{esF(x)}{\sinh(esF(x))}\right)\right]^{1/2}\operatorname{tr}[e^{-es\frac{1}{2}F_{\mu\nu}(x)\sigma^{\mu\nu}}]\;\bar{h}(x;is) \\ &\equiv W(x;is)\,\bar{h}(x;is)\;. \end{aligned} \tag{6.73}$$

Therefore, the $\bar{o}_n$ coefficients of the new expansion are directly obtained from the standard asymptotic expansion. Expanding $W(x;is)$ around $s\to 0$ we find

$$W(x;is) \sim \operatorname{tr} I + W_2(x)(-is)^2 + W_4(x)(-is)^4 + W_6(x)(-is)^6 + \cdots\,, \tag{6.74}$$

$$W_2(x) = -\tfrac{2e^2}{3}\operatorname{Tr}(F^2), \tag{6.75}$$

$$W_4(x) = \tfrac{e^4}{18}\operatorname{Tr}(F^2)^2 - \tfrac{7e^4}{45}\operatorname{Tr}(F^4), \tag{6.76}$$

$$W_6(x) = -\tfrac{e^6}{324}\operatorname{Tr}(F^2)^3 + \tfrac{7e^6}{270}\operatorname{Tr}(F^2)\operatorname{Tr}(F^4) - \tfrac{124e^6}{2835}\operatorname{Tr}(F^6). \tag{6.77}$$

Inserting this expansion into Eq. (6.73), together with expansions (6.68) and (6.71) and grouping terms with the same order, we finally obtain the $\bar{o}_n$ coefficients. Our conjecture is satisfied if the coefficients of the new expansion do not contain any term that vanish when the electromagnetic invariants are replaced by zero. The first five orders are $\bar{o}_0 = 1$, $\bar{o}_1 = 0$, $\bar{o}_2 = 0$, $\bar{o}_3 = \frac{1}{4}o_3$,

$$\begin{aligned}
4\bar{o}_4 =& \tfrac{12e^2}{35}\partial_\nu\partial_\mu F_{\kappa\lambda}\partial^\nu\partial^\mu F^{\kappa\lambda}\,,\\
4\bar{o}_5 =& \tfrac{8e^4}{7}F^{\kappa\lambda}F^{\mu\nu}\partial_\lambda F_{\nu\rho}\partial_\mu F_\kappa{}^\rho - \tfrac{16e^4}{63}F_\kappa{}^\mu F^{\kappa\lambda}\partial_\lambda F^{\nu\rho}\partial_\mu F_{\nu\rho}\\
&+ \tfrac{8e^4}{9}F_\kappa{}^\mu F^{\kappa\lambda}F^{\nu\rho}\partial_\mu\partial_\lambda F_{\nu\rho} - \tfrac{232e^4}{63}F^{\kappa\lambda}F^{\mu\nu}\partial_\mu F_\kappa{}^\rho\partial_\nu F_{\lambda\rho}\\
&+ \tfrac{40e^4}{9}F^{\kappa\lambda}F^{\mu\nu}\partial_\rho F_{\mu\nu}\partial^\rho F_{\kappa\lambda} + \tfrac{136e^4}{189}F^{\kappa\lambda}F^{\mu\nu}\partial_\nu F_{\lambda\rho}\partial^\rho F_{\kappa\mu}\\
&- \tfrac{656e^4}{189}F^{\kappa\lambda}F^{\mu\nu}\partial_\rho F_{\lambda\nu}\partial^\rho F_{\kappa\mu} - \tfrac{320e^4}{21}F_\kappa{}^\mu F^{\kappa\lambda}\partial_\rho F_{\mu\nu}\partial^\rho F_\lambda{}^\nu\\
&+ \tfrac{8e^2}{63}\partial_\rho\partial_\nu\partial_\mu F_{\kappa\lambda}\partial^\rho\partial^\nu\partial^\mu F^{\kappa\lambda}\,.
\end{aligned} \tag{6.78}$$

If we compare these new coefficients with the ones given in (6.70), we directly see that all terms containing $\mathcal{F}$ and $\mathcal{G}$ have disappeared. As in the scalar case, the first non-trivial test appears at $n = 5$. In o_5 we have a term that goes as $\mathcal{F}o_3 \sim F_{\kappa\lambda}F^{\kappa\lambda}o_3$. The fact that it does not appear in $\bar{o}_5$ means that it has been captured in the non-perturbative factor and that, as we have claimed, the new asymptotic expansion does not contain terms proportional to $\mathcal{F}$ and $\mathcal{G}$. We have also checked that the conjecture is satisfied for $n = 6$ (the last available coefficient).

Physical consequences

The existence of this factorization for both the scalar and the spin-$\frac{1}{2}$ field has immediate physical consequences.

The first result that we would like to point out is that the factorization allows us to find exact expressions of the one-loop effective Lagrangian for some external configurations. It is obvious that for constant electromagnetic backgrounds, we recover the Euler-Heisenberg effective Lagrangian. For an electric and magnetic field pointing in the $\hat{z}$ direction, which depend arbitrarily on the light-cone coordinate $z^+ = (t+z)$ it is also possible to find a closed form for the effective scalar and spin-$\frac{1}{2}$ Lagrangians. For this external configuration we find $\bar{O}_0 = 1 = \bar{o}_0$ and $\bar{O}_n = 0 = \bar{o}_n$ for $n \geq 1$. Therefore, the exact one-loop effective Lagrangian turns out to be the spacetime-dependent version of the Euler-Heisenberg Lagrangian,

$$\mathcal{L}^{(1)}_{\text{scalar}} = \frac{-1}{16\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-im^2 s} \frac{e^2 s^2 E\left(z^+\right) B\left(z^+\right)}{\sinh esE\left(z^+\right) \sin esB\left(z^+\right)}, \quad (6.79)$$

$$\mathcal{L}^{(1)}_{\text{spinor}} = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-im^2 s} \frac{e^2 s^2 E\left(z^+\right) B\left(z^+\right)}{\tanh esE\left(z^+\right) \tan esB\left(z^+\right)}. \quad (6.80)$$

This result includes also the case of a pure electric field $\vec{E} = E(z^+)\hat{z}$ and a pure magnetic field $\vec{B} = B(z^+)\hat{z}$. More explicitly, for the pure electric field case we find

$$\mathcal{L}^{(1)}_{\text{scalar}} = \frac{-1}{16\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-im^2 s} \frac{esE\left(z^+\right)}{\sinh esE\left(z^+\right)}, \quad (6.81)$$

$$\mathcal{L}^{(1)}_{\text{spinor}} = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-im^2 s} \frac{esE\left(z^+\right)\cosh esE\left(z^+\right)}{\sinh esE\left(z^+\right)}. \quad (6.82)$$

The same is true if the spacetime dependence is on $z^- = (t-z)$. The renormalized version of these expressions can be implemented by subtracting the first terms of the usual $g(x;is)$ or $h(x;is)$ proper-time expansions (up to and including $n=2$). These expressions are consistent with the results found in Refs. [135, 136, 137], where they obtained equivalent expressions using different techniques. Finally, for a single electromagnetic plane wave we directly find $\mathcal{L}^{(1)}_{\text{scalar}} = 0 = \mathcal{L}^{(1)}_{\text{spinor}}$. For this profile the electromagnetic invariants are zero $\mathcal{F}(x) = \mathcal{G}(x) = 0$, and therefore the one-loop effective Lagrangian trivially vanishes. This is in agreement with the outcome given in Ref. [20] by other methods.

The factorization (6.71) also allows us to estimate the imaginary part of the one-loop effective action, which is directly related to particle production [18]. Let us focus on the pure electric case. The factorization suggests that the poles of the imaginary part of the one-loop effective action are at the same points as in the constant electric field case $\tau_n = n\pi/e|\vec{E}|$, that is

$$\operatorname{Im} \mathcal{S}^{(1)}_{\text{spinor}} = -2\pi i \int d^4x \sum_{n=1}^{\infty} \operatorname{Res}\left[\frac{e^{-m^2\tau}}{\tau} \frac{e\tau E(x) \cos e\tau E(x)}{\sin e\tau E(x)} \bar{h}(x;\tau), \tau_n\right]. \tag{6.83}$$

If we consider only the leading order of $\bar{h}(x;is)$, we replicate the Schwinger's pair production rate, but now the electric field can (slowly) vary on x. We can take the next-to-leading orders to get perturbative weak-field corrections. However, since the particle-production phenomenon is strongly non-perturbative, a better option is to make a (second) partial resummation, considering, for example, to sum all terms with a given number of derivatives. This is the approach used in Ref. [138], where they give first derivative corrections to the Schwinger formula via a derivative expansion of the one-loop effective Lagrangian.

As a final comment we would like to stress that the factorization seems to be robust in presence of gravity. In particular, in Refs. [139, 140] the Euler-Heisenberg factorization appears in presence of (linearized) gravity for an electromagnetic field satisfying $\nabla_\rho F^{\mu\nu} = 0$. Furthermore, we believe that there should be a double factorization. The R-summed gravitational factorization $\operatorname{Exp}[-is\bar{\xi}R]$ for scalars and $\operatorname{Exp}[-is\frac{1}{12}R]$ for spin-$\frac{1}{2}$ fields is ensured (see Ref. [47]). Therefore, the double factorization should read as

$$g(x;is) = e^{-is\bar{\xi}R}\left[\det\left(\frac{esF}{\sinh(esF)}\right)\right]^{1/2} \tilde{g}(x;is)\,, \tag{6.84}$$

$$h(x;is) = e^{-is\frac{1}{12}R}\left[\det\left(\frac{esF}{\sinh(esF)}\right)\right]^{1/2} \operatorname{tr}[e^{-es\frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}}]\,\tilde{h}(x;is)\,. \tag{6.85}$$

where the expansions $\tilde{g}(x;is)$ and $\tilde{h}(x;is)$ do not contain any term that vanish when $\mathcal{F}(x)$, $\mathcal{G}(x)$, or $R(x)$ are replaced by zero. The leading terms coincide with [141, 142].

Chapter 7

Conclusions and future directions

This thesis summarizes central results of the research carried out by the author in collaboration with her supervisor and other scientific collaborators [specially from Wake Forest University and the University of Sheffield] during the last four and a half years. This research has mainly focused on exploring non-perturbative aspects of Quantum Field Theory within the semiclassical framework.

In Chapter 3, we have contributed to the improvement of the adiabatic regularization/renormalization scheme. We have shown that the renormalization subtractions can be consistently constructed when both gravity and an electromagnetic background are present. Previous analysis in the literature overlooked this point. We have tested the consistency of the proposal given in Ref. [33] using three different arguments. Namely, energy conservation, equivalence with the DeWitt-Schwinger method, and obtention of the expected quantum anomalies [34, 35, 36]. In Chapter 4 we have widely studied the close relationship between chiral quantum anomalies and the underlying particle creation process. In particular, we have found that the conventional adiabatic invariance of the particle number observable is broken in some special external conditions. These conditions are those for which the background induces a chiral anomaly. Therefore, a minimum amount of particles has to be created regardless of the behavior of the external fields (electromagnetic or gravitational). We have also found that the standard chiral anomaly for spin-$\frac{1}{2}$ fields in two spacetime dimensions is accompanied by a new (translational) quantum anomaly for the (L, R) Weyl sectors [39, 40].

The projects above motivated the analysis made in Chapter 5, where we have extensively investigated the backreaction problem (i.e., the effect of created particles on the background field) in the massive Schwinger model in two-dimensional electrodynamics. We have solved the backreaction equations numerically in the mean-field (or semiclassical) approximation. We have found some special limits where the semiclassical viewpoint is accurate: the massless limit, related with the chiral anomaly, and the very massive limit, by virtue of the decoupling theorem [28]. Finally, in Chapter 6 we have worked with different asymptotic expansions for the heat-kernel and for the effective action in curved spacetimes and QED. In Section 6.2 we have found an equivalence between the resumed form of the asymptotic DeWitt-Schwinger series expansion and a new resumed adiabatic (Parker-Fulling) expansion of the scalar field modes in cosmological spacetimes [48]. On the other side, in Section 6.3 we have proposed a new, resumed asymptotic expansion for the one-loop QED effective action, encapsulating all terms containing the field-strength invariants in a non-perturbative factor [50, 51].

The research presented in this thesis can lead to different future projects.

The work developed in Section 6.3 can be generalized in two independent ways. First, extending the $(\mathcal{F}, \mathcal{G})$ factorization to non-Abelian background fields, where we also expect a similar result for the effective action. In parallel, it should be checked if, for the effective action for spin-$\frac{1}{2}$ fields in curved spacetime, it is possible to find a second factorization (apart from the R-summed factor), similar to the one found in QED. Second, exploring the phenomenological consequences of the $(\mathcal{F}, \mathcal{G})$ factorization when gravity also enters into the game. One interesting feature that should be analyzed is the emergence of a logarithmic correction in the QED effective action due to the R-summed gravitational factor and its potential effects on light propagation.

A second research direction is to further explore the issue of preferred vacua states for quantized fields in expanding universes, introduced in Chapter 2. One way to address the problem is to take advantage of the emergent conformal symmetry near the big bang (it can also been further justified by the Weyl curvature hypothesis [52]). It would be interesting to probe this possibility aiming at applying it for a proper radiation-dominated universe or, alternatively, as a preinflationary era. It would also be convenient to investigate the connections between different renormalization methods in curved spacetimes. In particular, between Hadamard renormalization and adiabatic regularization for charged scalar fields. The understanding of these interconnections will allow us to further understand the correspondence between the Hadamard condition and the adiabatic regularity condition.

The work presented in Chapter 5 can also be extended in two complementary ways. First, we recall that to study the validity of the semiclassical approximation, we have used approximated solutions to the linear response equation. A more rigorous analysis should involve exact numerical solutions to this equation [53]. Second, a more realistic analysis of the backreaction problem requires our work to be extended to four spacetime dimensions. For the case with a pure time-dependent background field, we can use the scheme presented in Ref. [36], where the adiabatic regularization method for (four-dimensional) Dirac fields in time-dependent electric backgrounds is proposed. A more intriguing option could be the case in which there also exists a constant magnetic field in the same direction. In this second case, it could be interesting to analyze the situations where the chiral anomaly turns out to be relevant for understanding the dynamics of the system [see, for example, Eq. (4.57)].

The general formalism of quantum field theory in curved spacetime (foundations and physical consequences) and its connection with quantum gravity also becomes a possible research area to explore. In particular, it would be interesting to further study how field theory in curved spacetime emerges as a low-energy approximation in the sense of effective field theory and Wilsonian renormalization. The Schwinger model, which is exactly solvable in the full quantum theory, could be regarded as a toy model for this more ambitious proposal.

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