On large orbits of supersoluble subgroups of linear groups

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Abstract

We prove that if G is a finite soluble group, V is a finite faithful completely reducible G-module, and H is a supersoluble subgroup of G, then H has at least one regular orbit on $V \oplus V$.

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1 Introduction

Let G be a finite group acting on a finite set Ω. An element ω of Ω is in a *regular* orbit if $\mathrm{C}_G(\omega) = \{g \in G \mid \omega g = \omega\} = 1$, i.e., the orbit of ω is as large as possible and it has size $|G|$. Regular orbits of actions of linear groups acting on finite vector spaces arise in a variety of contexts, including the study of soluble groups, representation theory of finite groups and finite permutation groups, and it is a lively area of current research.

One of the most important questions in this context is to determine conditions which force a given subgroup of a finite linear group to have a regular orbit. This problem has been extensively investigated with a lot of results available (see [3, 4, 5, 15, 16]). In [11, Theorem A], a common extension of the main results of these papers has been showed.

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Theorem 1 ([11, Theorem A]). *If G is a finite soluble group, V is a faithful completely reducible G-module (possibly of mixed characteristic) and H is a subgroup of G such that the semidirect product VH is S_4-free, then H has at least two regular orbits on $V \oplus V$. Furthermore, if H is $\Gamma(2^3)$-free and* $\mathrm{SL}(2,3)$*-free, then H has at least three regular orbits on $V \oplus V$.*

Halasi and Maróti also proved in [7] that if V is a finite vector space over a finite field of order $q \geq 5$ and of characteristic p and $G \leqslant \mathrm{GL}(V)$ is a p-soluble completely reducible linear group, then there exists a base for G on V of size at most 2. As a consequence, under this hypothesis G possesses a regular orbit over $V \oplus V$. On the other hand, Wolf [12, Theorem A] showed that a finite supersoluble and completely reducible subgroup G of $\mathrm{GL}(V)$, for a finite vector space $0 \neq V$, has at least one regular orbit on $V \oplus V$.

The results just mentioned suggest that the answer to the question of whether or not Wolf's theorem holds for every supersoluble subgroup of a finite completely reducible soluble subgroup G of $\mathrm{GL}(V)$, even if the supersoluble subgroup is not completely reducible, is a natural next objective.

The main aim of this paper is to give a complete answer to this question.

Theorem A. *Let G be a finite soluble group and V be a finite faithful completely reducible G-module (possibly of mixed characteristic). Suppose that H is a supersoluble subgroup of G. Then H has at least one regular orbit on $V \oplus V$.*

By [11, Corollary 3], the answer is affirmative if V is of odd order. Therefore it will be enough to prove Theorem A for a module V over a field of characteristic 2.

The following two examples will show that in Theorem A the subgroup H is not completely reducible on V in general.

Example 1. Let $G = \mathrm{GL}(2,3)$ and $V = \mathrm{GF}(3) \oplus \mathrm{GF}(3)$ the natural faithful module of G over $\mathrm{GF}(3)$. Let $H = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$. Observe that $H \cong S_3$ is supersoluble and V is a non completely reducible H-module. In fact, $V_1 = \{(0,x) \mid x \in \mathrm{GF}(3)\}$ is an H-submodule of V and no complement of V_1 in V is H-invariant.

Example 2. Let $K = \mathrm{GL}(2,2)$ and $W = \mathrm{GF}(2) \oplus \mathrm{GF}(2)$ the natural faithful module of K over $\mathrm{GF}(2)$. Let $S \cong S_2$ be the symmetric group on $\Omega = \{1,2\}$. Write $G = K \wr S$ and $V = W^{\Omega}$. Then V is a faithful, irreducible G-module (see Section 2). Set

$$H = \{(f,\sigma) \in G \mid f \in K, \sigma \in S, f(1) = f(2)\} \cong S_3 \times C_2.$$

Then H is a supersoluble subgroup of G. Let $V_1 = \{v \in V = W^\Omega \mid v(1) = v(2)\}$. Then V_1 is an H-submodule of V. Suppose that there exists an H-submodule V_2 such that $V = V_1 \oplus V_2$ and take $0 \neq u \in V_2$. We have that $u(1) \neq u(2)$, and $u(1) + u(2) \neq 0$. Then $u^{(1,\sigma)} + u \in V_1 \cap V_2 = 0$. This contradiction shows that V is not a completely reducible H-module. Note that G does not have regular orbits on $V \oplus V$ by Wolf's formula, but H does.

We bring the introduction to a close with an application of Theorem A. Denote by $\mathfrak{U}$ the class of all finite supersoluble groups, which is a subgroup-closed saturated formation. Denote by $G^{\mathfrak{U}}$ the $\mathfrak{U}$-residual of group G, Clearly, $G^{\mathfrak{U}} \subseteq G^{\mathfrak{N}}$, where $G^{\mathfrak{N}}$ denotes the nilpotent residual. The following corollary generalises a result of Keller and Yang [9, Theorem 1.2] by replacing the nilpotent residual by the supersoluble residual.

Corollary 2. *Let G be a finite soluble group and V a finite faithful completely reducible G-module, possibly of mixed characteristic. Let M be the largest orbit size in the action of G on V. Then*

$$|G : G^{\mathfrak{U}}| \leqslant M^2.$$

Proof. Since $\mathfrak{U}$ is a saturated formation, by [8, Theorem 3.9] we can take a subgroup H such that $G^{\mathfrak{U}}H = G$ and $H \in \mathfrak{U}$. Then H is supersoluble. By Theorem A, H has a regular orbit on $V \oplus V$. It implies that $|\mathrm{C}_H(v)| \leqslant |H|^{1/2}$ for some $v \in V$. Let M_H be the largest orbit size of H on V. Then it follows that $|H| \leqslant |H : \mathrm{C}_H(v)|^2 \leqslant M_H^2$. Hence clearly $|G/G^{\mathfrak{U}}| \leqslant |H| \leqslant M_H^2 \leqslant M^2$, as desired. $\square$

2 Background results

All groups considered in the sequel will be finite.

The following elementary lemma appears in [11, Lemma 8].

Lemma 3. *Suppose that a group G acts on a non-empty finite set Ω. Then:*

1. *If $|\Omega| - \left|\bigcup_{1 \neq g \in G} \mathrm{C}_\Omega(g)\right| > k|G|$ for some non-negative integer k, then G has at least $k+1$ regular orbits on Ω. In particular, if $k = 0$, then G has at least one regular orbit on Ω.*

2. *If G has k regular orbits on Ω, then a subgroup H of G has at least $|G : H|k$ regular orbits on Ω.*

The following notation and arguments appear in [11, Section 3]. We summarise them here for the benefit of the reader.

Recall that an irreducible G-module V is called *imprimitive* if there is a non-trivial decomposition of V as a direct sum of subspaces $V = V_1 \oplus \cdots \oplus V_n$ $(n > 1)$ such that G permutes the set $\{V_1, \dots V_n\}$. The irreducible G-module V is *primitive* if V is not imprimitive. A linear group $G \leqslant \mathrm{GL}(d, p^k)$, p a prime, is said to be *primitive* if the natural G-module is primitive.

Let G be a group and let V be a faithful G-module. Let $V = \widehat{W_1} \oplus \cdots \oplus \widehat{W_m}$, with $m \geqslant 2$, be a decomposition of V as a direct sum of subspaces such that $\widehat{\Omega} = \{\widehat{W_1}, \dots, \widehat{W_m}\}$ is permuted transitively by G. The action of G on $\widehat{\Omega}$ induces a homomorphism $\sigma\colon G \longrightarrow S_\Omega$, where $\Omega = \{1, \dots, m\}$. Write $W = \widehat{W_1}$ and $H = \mathrm{N}_G(W)/\mathrm{C}_G(W)$ and $S = \sigma(G)$. Let

$$\widehat{G} = H \wr S = \{(f, \sigma) \mid f\colon \Omega \longrightarrow H, \sigma \in S\}$$

with the product $(f_1, \sigma_1)(f_2, \sigma_2) = (g, \sigma_1\sigma_2)$, where $g(\omega) = f_1(\omega)f_2(\omega^{\sigma_1})$ for all $\omega \in \Omega$ be the permutational wreath product of K with S (see [8, Kapitel I, Satz 15.3]). Let

$$W^\Omega = \{f \mid f\colon \Omega \longrightarrow W \text{ is a map}\}. \tag{1}$$

If Y is a subgroup of H, we set $Y^\natural = \{(f, 1) \in H \wr S \mid f(w) \in Y \text{ for all } \omega \in \Omega\}$. In particular, $B = H^\natural$ is called the *base group* of $H \wr S$. If W is a H-module, then $W^\natural$, considered as a subgroup of $([W]H) \wr S$, becomes a $H \wr S$-module with the action given by $g^{(f,\sigma)}(\omega) = g(\omega^{\sigma^{-1}})^{f(\omega^{\sigma^{-1}})}$.

Lemma 4 ([11, Lemma 9])**.** *There exists a monomorphism $\tau\colon G \longrightarrow \widehat{G}$ such that:*

1. *The actions of G on V and $\tau(G)$ on W^Ω are equivalent.*
2. *$\widehat{G} = H^\natural \tau(G)$.*
3. *Write $W_i = \{f \in W^\Omega \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$. Then*

$$\mathrm{N}_{\tau(G)}(W_i)/\,\mathrm{C}_{\tau(G)}(W_i) \cong H, \forall i \in \Omega.$$

Therefore if we are interested in regular orbits of the action of G on V and V is not primitive, we may assume, by Lemma 4, that G is a supersoluble subgroup of a wreath product $\widehat{G} = K \wr S$, where K is a group, W is a faithful K-module, and S is a non-trivial primitive permutation group on a finite set Ω such that $\widehat{G} = K^\natural G$ and $V = W^\Omega$. Since this situation will appear several times in our arguments, we will use some abbreviations to refer to it.

Notation 5. We say that $(\widehat{G}, G, H, S, \Omega)$ satisfies **Condition A** if

- H is a group;
- S is a primitive group on the finite set Ω;
- $\widehat{G} = H \wr S$;
- G is a supersoluble subgroup of $\widehat{G}$ such that $H^{\natural}G = \widehat{G}$.

Notation 6. We say that $(\widehat{G}, G, H, S, \Omega, V, W)$ satisfies **Condition B** if

- $(\widehat{G}, G, H, S, \Omega)$ satisfies **Condition A**;
- W is a faithful H-module over $\mathrm{GF}(2)$;
- $V = W^{\Omega}$ (see Equation (1)), naturally is a faithful $\widehat{G}$-module;

Write $W_i = \{f \in V \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$.

- $\mathrm{N}_G(W_i)/\,\mathrm{C}_G(W_i) \cong H$ for each $i \in \Omega$.

As in [11, Section 3], we are interested here in regular orbits of a group G on completely reducible G-modules V over finite fields and so, in looking for regular orbits of G on V, we can assume without loss of generality that the field is a prime field.

In this context, a result of Wolf [13] that provides a formula to count the exact number of regular orbits $\widehat{G}$ on W^{Ω} is extremely useful. Let S be a transitive permutation group on a finite set Ω and denote by $\Pi_l(\Omega, S)$ the set of all partitions $\{\Delta_1, \dots, \Delta_l\}$ of length l of Ω having the property that the subgroup $\{s \in S \mid \Delta_i^s = \Delta_i \text{ for all } i\}$ of S is trivial.

Theorem 7 (Wolf's formula, [13])**.** *Suppose that $(\widehat{G}, G, H, S, \Omega, V, W)$ satisfies **Condition B**. Let k be the number of regular orbits of H on W. Then the number of regular orbits of $\widehat{G}$(also G) on $V = W^{\Omega}$ is at least*

$$\frac{1}{|S|} \sum_{2 \leqslant l \leqslant m} P(k, l) |\Pi_l(\Omega, S)|,$$

where $P(k, l) = k!/(k-l)!$ if $k \geqslant l$ and $P(k, l) = 0$ otherwise.

The following result is useful to obtain regular orbits in a direct sum of G-modules starting from regular orbits of its terms.

Lemma 8. *Let G be a group and V be a faithful G-module such that $V = W_1 \oplus \cdots \oplus W_s$, where W_i is a G-module, $1 \leqslant i \leqslant s$. If $G/\mathrm{C}_G(W_i)$ has t_i regular orbits on $W_i \oplus W_i$, then G has at least $\prod_{i=1}^{s} t_i$ regular orbits on $V \oplus V$.*

The following result about supersoluble primitive permutation groups is crucial in our inductive arguments.

Lemma 9. *Let S be a supersoluble primitive permutation group on a finite set $\Omega = \{1, \dots, n\}$ with $n \geqslant 2$. Then* $\mathrm{Stab}_S(1) \cap \mathrm{Stab}_S(2) = 1$.

Proof. Since S is supersoluble and primitive, we have that $|\Omega|$ is a prime. Hence S is a transitive permutation group of prime degree. The conclusion follows from [8, Theorem 3.6 (d)]. $\square$

Lemma 10. *Assume that $(\widehat{G}, G, H, S, \Omega)$ satisfies **Condition A**. Write $N = H^{\natural} \cap G$ and assume that $\mathrm{O}_p(N) = 1$ for some prime p. If f is a p-element of $H^{\natural}$ such that $(f, 1) \in N$ and $f(i_0) = 1$ for some $i_0 \in \Omega$, then $f = 1$.*

Proof. Observe that $S \cong \widehat{G}/H^{\natural} \cong G/N$ is supersoluble. Since S is a primitive permutation group, we conclude that S has a unique minimal normal subgroup X such that $|X| = |\Omega| = q$ for some prime q.

Let $P \in \mathrm{Syl}_p(N)$ such that $(f, 1) \in P$. By the Frattini Argument, $G = N\mathrm{N}_G(P)$ and, consequently, $\widehat{G} = H^{\natural}\mathrm{N}_G(P)$. Let $\rho \in X$, $\rho \neq 1$. Then $\rho^q = 1$. Since $\widehat{G} = H^{\natural}\mathrm{N}_G(P)$, there exists $u \in H^{\natural}$ such that $(u, \rho) \in \mathrm{N}_G(P)$ whose projection onto S is ρ. Assume that $\mathrm{o}((u, \rho)) = q^{\alpha}m$ with $\gcd(q, m) = 1$ and $\alpha \in \mathbb{N}$. Then there exist λ, $\mu \in \mathbb{Z}$ such that $\lambda q + \mu m = 1$, and so $(u, \rho)^{1-\lambda q} = (u, \rho)^{\mu m}$ is a q-element of the form $(g, \rho^{1-\lambda q}) = (g, \rho) \in \mathrm{N}_G(P)$. Let $T = P\langle (g, \rho)\rangle$. Note that $T' \leqslant P$ is a p-group and observe that $T' \leqslant G' \leqslant \mathrm{F}(G)$ since G is supersoluble. Thus $T' \leqslant \mathrm{O}_p(G)$. Then $[(f, 1), (g, \rho)] \in T' \cap N \leqslant \mathrm{O}_p(G) \cap N = \mathrm{O}_p(N) = 1$. Thus we have $(f, 1)(g, \rho) = (g, \rho)(f, 1)$, that is, $f(i)g(i) = g(i)f(i^{\rho})$ for all $i \in \Omega$. Therefore $f(i) = 1$ if and only if $f(i^{\rho}) = 1$.

Recall that X acts transitively on Ω. For each $i \in \Omega$, there exists ρ_i (depending on i) in S such that $i_0^{\rho_i} = i$. Since $f(i_0) = 1$, we have that $f(i) = f(i_0^{\rho_i}) = 1$. Thus $f(i) = 1$ for each $i \in \Omega$ and the statement is proved. $\square$

3 Lemmas

In order to prove Theorem A, we will argue by induction by decomposing V as a direct sum of subspaces permuted transitively by G. Therefore our first step will be the study of the case in which there is no such a proper decomposition, that is, V is primitive. In attaining this aim, the following two lemmas are crucial. The first one concerns primitive soluble linear groups over a field of characteristic two.

Let V be the Galois field $\mathrm{GF}(p^n)$ for some prime p and integer n. Then V can be regarded as a vector space over $\mathrm{GF}(p)$ of dimension n. Recall that the semi-linear group of V is

$$\Gamma(V) = \Gamma(p^n) = \{x \longmapsto ax^\tau \mid a \in \mathrm{GF}(p^n)^*, \tau \in \mathrm{Gal}(\mathrm{GF}(p^n)/\,\mathrm{GF}(p))\}.$$

Lemma 11. *Let G be a supersoluble group and V be a faithful primitive G-module over* $\mathrm{GF}(2)$. *Then G has at least four regular orbits on $V \oplus V$ unless $G = \Gamma(V)$ and $|V| = 2^n$, $2 \leqslant n \leqslant 4$. In these cases, G has exactly $n-1$ regular orbits on $V \oplus V$.*

Proof. Let A be a maximal abelian normal subgroup of G. Clearly $A \leqslant \mathrm{C}_G(A) \trianglelefteq G$. Suppose that $A < \mathrm{C}_G(A)$. Then we can take a chief factor T/A of G such that $T \leqslant \mathrm{C}_G(A)$. Since G is supersoluble, T/A is cyclic and $T = \langle A, x\rangle$ for some $x \in \mathrm{C}_G(A)$. Then T is an abelian normal subgroup of G, contrary to the choice of A. Thus $A = \mathrm{C}_G(A)$. Since V is a primitive G-module, V_A is homogeneous by Clifford's theorem [2, Chapter B, Theorem 7.3]. By [10, Lemma 2.2], V_A is irreducible. It follows from [10, Theorem 2.1] that $G \leqslant \Gamma(V)$. Write $|V| = 2^n$ where $n \geqslant 1$ is an integer.

First we assume that $G = \Gamma(V)$. Equivalently, if suffices to consider the regular orbits of $\Gamma(2^n)$ acting on the additive group of the field $\mathrm{GF}(2^n)$. Take the field automorphism $\sigma\colon\ \mathrm{GF}(2^n) \longrightarrow \mathrm{GF}(2^n)$ given by $u \longmapsto u^2$. The Galois group $C = \mathrm{Gal}(\mathrm{GF}(2^n)/\mathrm{GF}(2)) = \langle\sigma\rangle$ is of order n.

For each prime p dividing n, $\langle\sigma^{n/p}\rangle$ is the unique subgroup of C with order p since C is cyclic. Then we have that

$$\mathrm{C}_{\mathrm{GF}(2^n)}(\sigma^{n/p}) = \{u \in \mathrm{GF}(2^n) \mid u^{2^{n/p}} = u\}$$

is a subfield of $\mathrm{GF}(2^n)$ isomorphic to $\mathrm{GF}(2^{n/p})$. Thus $|\mathrm{C}_{\mathrm{GF}(2^n)}(\sigma^{n/p})| = 2^{n/p}$.

In order to prove that C has at least four regular orbits on $\mathrm{GF}(2^n)$ when $n \geqslant 5$, by Lemma 3, it suffices to show that

$$2^n - \sum_{p|n} 2^{n/p} > 3n$$

holds for $n \geqslant 5$. Observe that $\sum_{p|n} 2^{n/p} \leqslant \log_2 n \cdot 2^{n/2}$. It is not difficult to check that $2^n - \sum_{p|n} 2^{n/p} \geqslant 2^n - \log_2 n \cdot 2^{n/2} > 3n$ for $n \geqslant 8$ and it is easy to find that the inequality also holds for $n = 5$, 6, 7.

Thus we have proved that $G \leqslant \Gamma(V)$ has at least four regular orbits on $V \oplus V$ when $n \geqslant 5$.

Assume that $n = 1$. Then $|V| = 2$ and $G = 1$. Hence G has exactly four regular orbits on $V \oplus V$.

Assume that $n = 2$. Then $|V| = 2^2$ and $G \leqslant \Gamma(V) \cong S_3$. If $G < \Gamma(V)$, then G has a regular orbit on V. In this case, G has at least $|V| = 4$ regular orbits on $V \oplus V$. If $G = \Gamma(V)$, we can check that G has exactly one regular orbit on $V \oplus V$.

Assume that $n = 3$. Then $|V| = 2^3$ and $G \leqslant \Gamma(V) \cong [C_7]C_3$. If $G = \Gamma(V)$, then G has exactly two regular orbits on $V \oplus V$. Thus, if $G < \Gamma(V)$, G has at least four regular orbits on $V \oplus V$.

Assume that $n = 4$. Then $|V| = 2^4$ and $G \leqslant \Gamma(V) \cong [C_{15}]C_4$. If $G = \Gamma(V)$, then G has exactly three regular orbits on $V \oplus V$. Thus, if $G < \Gamma(V)$, G has at least six regular orbits on $V \oplus V$.

Thus the lemma is completely proved. □

Lemma 12. *Let G be a soluble primitive group of $\mathrm{GL}(d, 2)$, and let V be the natural G-module. Assume that H is a supersoluble subgroup of G. Then H has at least three regular orbits on $V \oplus V$ unless one of the following two cases occurs:*

1. *$d = 2$ and $H = \Gamma(V) \cong S_3$, then H has just one regular orbit on $V \oplus V$.*
2. *$d = 3$ and $H = \Gamma(V) \cong \Gamma(2^3)$, then H has just two regular orbits on $V \oplus V$.*

Furthermore if H is of odd order, then H has four regular orbits on $V \oplus V$ unless the case 2 occurs.

Proof. Assume first that $H = G$. Then G is supersoluble. It follows from Lemma 11 that the hypothesis of the lemma is satisfied. Now we may assume that $H < G$. By [3, Theorem 3.4], H has at least four regular orbits on $V \oplus V$ provided that G is not isomorphic to $\mathrm{GL}(2, 2)$, $3^{1+2}.\mathrm{SL}(2, 3)$ or $3^{1+2}.\mathrm{GL}(2, 3)$.

If H is a proper subgroup of $G = \mathrm{GL}(2, 2) \cong S_3$, then H is of prime order and there exists $v \in V$ such that $\mathrm{C}_H(v) = 1$. Hence H has at least $|V| = 4$ regular orbits on $V \oplus V$.

Suppose that G is isomorphic to $3^{1+2}.\mathrm{SL}(2, 3)$ or $3^{1+2}.\mathrm{GL}(2, 3)$ (as a subgroup of $\mathrm{GL}(6, 2)$). In this case, one checks with GAP [6] that H has at least three (four if $|H|$ is odd) regular orbits on $V \oplus V$. □

The next definitions reflect what happens in the exceptional cases of Lemma 12.

Definition 13. Let G be a group and let V be a faithful G-module. We say that the G-module V satisfies **Property I** if the following conditions hold:

1. G is an odd order group and $\mathrm{O}_3(G) = 1$.

2. There exists $0 \neq x \in V$ such that $\mathrm{C}_G(x)$ has at least four different orbits on V with representatives y_1, y_2, z_1, z_2 satisfying that $\mathrm{C}_G(x) \cap \mathrm{C}_G(y_i) = 1$ and $\mathrm{C}_G(x) \cap \mathrm{C}_G(z_i)$ is a 3-group for each i.

Definition 14. Let G be a group and let V be a faithful G-module. We say that the G-module V satisfies **Property II** if the following conditions hold:

1. G is an even order group with $\mathrm{O}_2(G) = 1$.

2. There exists $0 \neq x \in V$ such that $\mathrm{C}_G(x)$ at least three different orbits on V with representatives y, z_1, z_2 satisfying that $\mathrm{C}_G(x) \cap \mathrm{C}_G(y) = 1$ and $\mathrm{C}_G(x) \cap \mathrm{C}_G(z_i)$ is a 2-group for each $1 \leqslant i \leqslant 2$.

Note that if the faithful G-module V satisfies either **Property I** or **Property II**, then G has at least one regular orbit on $V \oplus V$. Our strategy will consist in showing by induction that G has at least three regular orbits on $V \oplus V$ or G satisfies either **Property I** or **Property II**. As we will see in Lemmas 15 and 16 below, the existence of regular orbits on $V \oplus V$ in the situation of **Condition B** will depend on the existence of some special orbits of H on $W_1 \oplus W_1$ allowing us to apply Lemma 10. This situation is guaranteed when **Property I** or **Property II** holds.

Let G be a group and Ω be a transitive G-set. Recall that a subset $\Delta \subseteq \Omega$ is said to be a *block* if for every $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \varnothing$. Clearly every transitive G-set Ω has a block Δ such that $1 \leqslant |\Delta| < |\Omega|$ if $|\Omega| \geqslant 2$. If we take such a block Δ of maximal size, then $\mathrm{Stab}_G(\Delta)$ is maximal in G. (see [1, Definition 1.1.1 and Proposition 1.1.2]).

Lemma 15. *Assume that $(\widehat{G}, G, H, S, \Omega, V, W)$ satisfies **Condition B**.*

1. *If $\mathrm{O}_p(H) = 1$ for some prime p and write $N = H^\natural \cap G$, then $\mathrm{O}_p(N) = 1$.*

Let $x \in W$. Suppose that $v \in V = W^\Omega$ is defined by $v(\omega) = x$ for all $\omega \in \Omega$.

2. *If $(f, \sigma) \in \mathrm{C}_G(v)$, then $f(\omega) \in \mathrm{C}_H(x)$ for all $\omega \in \Omega$.*

3. *Assume that $\{\Delta_1, \dots, \Delta_s\}$ is a partition of Ω such that $\bigcap_i \mathrm{Stab}_S(\Delta_i) = 1$. Assume also that $\mathrm{C}_H(x)$ has different orbits on W with representatives $y_1, \dots,$ y_s such that $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_i)$ is a p-group for $1 \leqslant i \leqslant s$. Construct the elements $v \in V = W^\Omega$ as $v(\omega) = x$ for $\omega \in \Omega$ and $u \in V = W^\Omega$ by $u(\omega) = y_i$ if $\omega \in \Delta_i$, where $1 \leqslant i \leqslant s$, for $\omega \in \Omega$. Then $\mathrm{C}_G(v) \cap \mathrm{C}_G(u)$ is a p-group. Furthermore, if $\mathrm{O}_p(H) = 1$ and $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_k) = 1$ for some $1 \leqslant k \leqslant s$, then $\mathrm{C}_G(v) \cap \mathrm{C}_G(u) = 1$, in particular, (v, u) generates a regular orbit in $V \oplus V$.*

4. *If $\Omega = \{1, 2, 3\}$, y, $z \in W$ belong to different orbits of $\mathrm{C}_H(x)$ on W, $\mathrm{C}_H(x) \cap \mathrm{C}_H(y) = 1$, $\mathrm{C}_H(x) \cap \mathrm{C}_H(z)$ is a 2-group, and $u \in V$ is defined by $u(1) = u(2) = y$, $u(3) = z$, then $\mathrm{C}_G(u) \cap \mathrm{C}_G(v)$ is a 2-group.*

5. *If $\Omega = \{1, 2\}$, $y \in W$ satisfies that $\mathrm{C}_H(x) \cap \mathrm{C}_H(y) = 1$, and $u \in V$ is defined by $u(1) = u(2) = y$, then $\mathrm{C}_G(u) \cap \mathrm{C}_G(v)$ is a 2-group.*

Assume that $\Omega = \{1, 2\}$ and $0 \neq x \in W$. Suppose that $\mathrm{O}_2(H) = 1$ and that $v' \in V = W^\Omega$ is defined by $v'(1) = 0$, $v'(2) = x$.

6. *If $\mathrm{O}_2(G) \neq 1$, y_1 and y_2 lie in different orbits of $\mathrm{C}_H(x)$ on W, $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_2) = 1$ and $u \in V = W^\Omega$ is defined by $u(1) = y_1$, $u(2) = y_2$, then $\mathrm{C}_G(v') \cap \mathrm{C}_G(u) = 1$.*

Proof. 1. Write $W_i = \{f \in V \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$ and note that $N = \bigcap_j \mathrm{N}_G(W_j) \trianglelefteq G$. Consequently, N is a normal subgroup of $\mathrm{N}_G(W_j)$ for each j. $N/(N \cap \mathrm{C}_G(W_j)) \cong N\mathrm{C}_G(W_j)/\mathrm{C}_G(W_j) \leqslant \mathrm{N}_G(W_j)/\mathrm{C}_G(W_j)$, which is isomorphic to H. Since $\mathrm{O}_p(H) = 1$, we conclude that $\mathrm{O}_p(N) \leqslant \mathrm{C}_G(W_j)$ for each j. Therefore

$$\mathrm{O}_p(N) \leqslant \bigcap_j \mathrm{C}_G(W_j) = \mathrm{C}_G(V) = 1,$$

because G acts faithfully on V.

2. Suppose that $(f, \sigma) \in \mathrm{C}_G(v)$. Given $\omega \in \Omega$, $v(\omega)^{f(\omega)} = v(\omega^\sigma)$, which implies that $x^{f(\omega)} = x$ and so $f(\omega) \in \mathrm{C}_H(x)$ for all $\omega \in \Omega$.

3. Let $(f, \sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u)$. Given $\omega \in \Omega$, $v(\omega)^{f(\omega)} = v(\omega^\sigma)$, which implies that $x^{f(\omega)} = x$ and so $f(\omega) \in \mathrm{C}_H(x)$ for $\omega \in \Omega$. Moreover, $u(\omega)^{f(\omega)} = u(\omega^\sigma)$ for $\omega \in \Omega$. If $\omega \in \Delta_i$, since the y_i belong to different orbits under the action of $\mathrm{C}_H(x)$, we conclude that $\omega^\sigma \in \Delta_i$. It follows that $\sigma \in \bigcap_i \mathrm{Stab}_S(\Delta_i) = 1$ and, if $\omega \in \Delta_i$, $y_i^{f(\omega)} = y_i$, that is, $f(\omega) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_i)$, which is a p-group for all i. Therefore $(f, \sigma) = (f, 1)$ is a p-element. It follows that $\mathrm{C}_G(v) \cap \mathrm{C}_G(u)$ is a p-group.

   Suppose that, in addition, $\mathrm{O}_p(H) = 1$ and that $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_1) = 1$. In this case, for $\omega \in \Delta_1$, we obtain that $y_1^{f(\omega)} = y_1$, and hence $f(\omega) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_1) = 1$. Consequently $f(\omega) = 1$ for $\omega \in \Delta_1$. Furthermore, $(f, \sigma) = (f, 1) \in H^\natural \cap G = N$ is a p-element and $f(\omega) = 1$ for $\omega \in \Delta_1$. Since $\mathrm{O}_p(H) = 1$, we obtain that $\mathrm{O}_p(N) = 1$ by the statement 1. By Lemma 10, we conclude that $f = 1$.

4. Let $(f,\sigma) \in \mathrm{C}_G(v) \cap \mathrm{C}_G(u)$. Then $v(i)^{f(i)} = v(i^\sigma)$. It follows that $x^{f(i)} = x$, that is, $f(i) \in \mathrm{C}_H(x)$ for all $i \in \Omega$. Moreover, $u(i)^{f(i)} = u(i^\sigma)$. Since y and z belong to different orbits of $\mathrm{C}_G(x)$ in W_1, we conclude that $\sigma \in \langle(12)\rangle$. Moreover, $u(i)^{f(i)} = u(i^\sigma)$ for $i \in \{1,2\}$ implies that $y^{f(i)} = y$, that is, $f(1)$, $f(2) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y) = 1$, and $u(3)^{f(3)} = u(3^\sigma) = u(3)$ implies that $z^{f(3)} = z$, that is, $f(3) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(z)$, a 2-group. Therefore (f,σ) is a 2-element.

5. The proof of this statement is similar to the proof of the previous statement.

6. Since $\mathrm{O}_2(H) = 1$, we have that $\mathrm{O}_2(N) = 1$ by Statement 1. Since $G/N \cong S \cong S_2$, we have that N is a maximal subgroup of G. Moreover, $N \cap \mathrm{O}_2(G) \leqslant \mathrm{O}_2(N) = 1$. As $\mathrm{O}_2(G) \neq 1$, consequently, $G = N\mathrm{O}_2(G) = H^\natural \mathrm{O}_2(G)$ and $[N, \mathrm{O}_2(G)] = 1$.

   Let $(f,\sigma) \in \mathrm{C}_G(v') \cap \mathrm{C}_G(u)$, with $f \in H^\natural$, $\sigma \in S$. Since $v'(2^\sigma) = v'(2)^{f(2)} = x^{f(2)} \neq 0$, we conclude that $\sigma = 1$. Furthermore, $u(2) = u(2)^{f(2)}$, which implies that $f(2) \in \mathrm{C}_H(x) \cap \mathrm{C}_H(y_2) = 1$. Note that $(f,\sigma) = (f,1) \in H^\natural \cap G = N$.

   Let $\rho = (12) \in S$. Since $G = H^\natural \mathrm{O}_2(G)$, there exists $g \in H^\natural$ such that $(g,\rho) \in \mathrm{O}_2(G)$. Since $[N, \mathrm{O}_2(G)] = 1$, $(f,1)(g,\rho) = (g,\rho)(f,1)$. It follows that $f(1) = f(2^\rho) = f(2)^{g(2)} = 1$, and so $f(1) = 1$. Consequently, $(f,\sigma) = (1,1)$. We conclude that $\mathrm{C}_G(v') \cap \mathrm{C}_G(u) = 1$. □

The arguments needed for the induction step are collected in the following lemma.

Lemma 16. *Let G be a supersoluble group and V be a faithful G-module over $\mathrm{GF}(2)$. Assume that there is a decomposition $V = V_1 \oplus \cdots \oplus V_m$ ($m \geqslant 1$) as a direct sum of subspaces which are permuted transitively by G. Let $K = \mathrm{N}_G(V_1)/\mathrm{C}_G(V_1)$, then V_1 can be regarded as a faithful K-module. Then we have:*

1. *If K has at least four regular orbits on $V_1 \oplus V_1$, then G has at least four regular orbits on $V \oplus V$.*

2. *If K is of even order and K has at least three regular orbits on $V_1 \oplus V_1$, then G has at least three regular orbits on $V \oplus V$.*

3. *If the K-module V_1 satisfies* ***Property I*** *and G is of odd order, then G has at least four regular orbits on $V \oplus V$ or the G-module V satisfies* ***Property I****.*

4. *If the K-module V_1 satisfies **Property II**, then either G has three regular orbits on $V \oplus V$ or the G-module V satisfies **Property II**.*

5. *If the K-module V_1 satisfies **Property I**, then either G has three regular orbits on $V \oplus V$ or the G-module V satisfies **Property I** or **Property II**.*

Proof. We argue by induction on m. Clearly Statements 1–5 hold when $m = 1$. Now we assume that $m \geqslant 2$. Since G acts transitively on $\{V_1, \dots, V_m\}$, we can take a block Δ of $\{V_1, \dots, V_m\}$ such that $\mathrm{Stab}_G(\Delta)$ is maximal in G. Without loss of generality, we may assume that $\Delta = \{V_1, \dots, V_s\}$ with $s \geqslant 1$.

Let $W = \sum_{i=1}^{s} V_i$ and $L = \mathrm{N}_G(W)$. Then $L = \mathrm{Stab}_G(\Delta)$ is maximal in G. Assume that $\{g_1, g_2, \dots, g_t\}$, where $g_1 = 1$, is a right transversal of L in G with $t = |G : L| \geqslant 2$. Note that $V = Wg_1 \oplus \cdots \oplus Wg_t$ and the action of G on $\{Wg_1, \dots, Wg_t\}$ induces a homomorphism $\sigma\colon G \longrightarrow S_\Omega$ such that $Wg_ig = Wg_{i\sigma(g)}$, where $\Omega = \{1, \dots, m\}$. Write $S = \sigma(G)$ and S acts faithfully and primitively on Ω.

Let $H = L/\mathrm{C}_G(W)$, $\widehat{G} = H \wr S$. By Lemma 4, there exists a monomorphism $\tau\colon G \longrightarrow \widehat{G}$ such that:

1. The actions of G on V and $\tau(G)$ on W^Ω are equivalent.

2. $\widehat{G} = H^\natural \tau(G)$.

3. Write $W_i = \{f \in W^\Omega \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$. Then

$$\mathrm{N}_{\tau(G)}(W_i)/\,\mathrm{C}_{\tau(G)}(W_i) \cong H, \forall i \in \Omega.$$

It is easy to check that $(\widehat{G}, \tau(G), H, S, \Omega, W^\Omega, W)$ satisfies **Condition B**. Since the action of G on V and the action of $\tau(G)$ on W^Ω are equivalent, without loss of generality, we may assume that $G = \tau(G)$, $V = W^\Omega$ and $(\widehat{G}, G, H, S, \Omega, V, W)$ satisfies **Condition B**.

Write $N = H^\natural \cap G$ and $W_i = \{f \in V \mid f(j) = 0, \forall j \neq i\}$ for each $i \in \Omega$. It is easy to see that $N = \bigcap_i \mathrm{N}_G(W_i)$, moreover, $S \cong \widehat{G}/H^\natural \cong G/N$ is supersoluble. Thus t is a prime.

Recall that $W = V_1 \oplus \cdots \oplus V_s$ is a faithful H-module and $\Delta = \{V_1, \dots, V_s\}$ is a block of the action of G on $\{V_1, ..., V_m\}$. It follows from [1, Theorem 1.13] that L (and also H) acts transitively on $\Delta = \{V_1, \dots, V_s\}$. Write $J = \mathrm{N}_H(V_1)/\mathrm{C}_H(V_1)$ and $J_0 = \mathrm{N}_L(V_1)\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1) \leqslant K$. It is not difficult to see that the action of J on V_1 is equivalent to the action of J_0 on V_1.

Now we will prove Statements 1–5. Our strategy is first to apply induction on (W, H, V_1, J) and then to calculate the number of regular orbits by Theorem 7.

1. By hypothesis, $J_0 \leqslant K$ has at least four regular orbits on $V_1 \oplus V_1$. Thus J has at least four regular orbits on $V_1 \oplus V_1$. Since $s = m/t < m$, by induction, H has at least four regular orbits on $W \oplus W$.

   Suppose that S has a regular orbit on the power set of Ω. Then $|\Pi_2(\Omega, S)| \geqslant |S|/2$. Consequently, in this case, $\widehat{G} = H \wr S$ has at least four regular orbits on $V \oplus V$ by Theorem 7 and so does G. Therefore we may assume that S has no regular orbit on $\mathcal{P}(\Omega)$ and so S is one of the exceptional cases of [10, Theorem 5.6] and $3 \leqslant t \leqslant 9$. By [13, Theorem 3.1 (iii)], we have that $|\Pi_3(\Omega, S)| \geqslant |S|$ for $5 \leqslant t \leqslant 9$, which implies that $G \leqslant H \wr S$ has at least four regular orbits on $V \oplus V$ by Theorem 7. Thus we may assume that $t = 3$ since t is a prime. In this case, $S \cong S_3$. It is not difficult to calculate that $|\Pi_2(\Omega, S)| = 0$ and $|\Pi_3(\Omega, S)| = 1$. Thus G, as a subgroup of $\widehat{G}$, has at least four regular orbits on $V \oplus V$. Thus Statement 1 is proved.

2. If J is of odd order, then so is J_0. Since K is of even order, $|K : J_0| \geqslant 2$. Thus J_0 (and also J) has at least six regular orbits on $V_1 \oplus V_1$. Applying Statement 1 on (W, H, V_1, J), we conclude that H has at least four regular orbits on $W_1 \oplus W_1$. Applying Statement 1 on (V, G, W, H) again, we obtain that G has at least four regular orbits on $V \oplus V$, as desired.

   Now we assume that J is of even order. By induction, H has at least three regular orbits on $W \oplus W$. By [14, Proposition 3.2 (2)] and Theorem 7, we may assume that $t \leqslant 4$ and S has no regular orbit on $\mathcal{P}(\Omega)$. Note that t is a prime. Thus, by [10, Theorem 5.6], we conclude that $|\Omega| = 3$ and $S \cong S_3$. In this case, $|\Pi_2(\Omega, S)| = 0$ and $|\Pi_3(\Omega, S)| = 1$. In particular, $\widehat{G}$ has at least one regular orbit on $V \oplus V$.

   Observe that H is of even order since J is of even order. Then $\widehat{G}$ has a subgroup isomorphic to $C_2 \wr S_3$ and so $\widehat{G}$ is not supersoluble. Thus we have that G is a proper subgroup of $\widehat{G}$. Suppose that $|\widehat{G} : G| = 2$. Then $G \triangleleft \widehat{G}$ and $B = H^\natural$ is not contained in G. Recall that $N = B \cap G$. Then N is normal in $\widehat{G}$ and $|B : N| = 2$. In particular, there exists a direct factor $H_1 \cong H$ of B which is not contained in N. Then $B = H_1 N$ and $|H_1 : H_1 \cap N| = 2$. Note that $C = (H_1 \cap N)^\natural$ is a normal subgroup of $\widehat{G}$ contained in B such that $\widehat{G}/C \cong C_2 \wr S_3$. Thus there exists a normal subgroup X of $\widehat{G}$ contained in B such that $\widehat{G}/X \cong S_4$ and clearly $|B : X| = 2^2$. If $X \leqslant G$, we have that $X \leqslant N$ and $|N : X| = 2$. It implies that N/X is a normal subgroup with order 2 of $G/X \cong S_4$, which is impossible. Therefore $\widehat{G} = XG$ and $G/G \cap X \cong \widehat{G}/X \cong S_4$, contrary to assumption. Consequently, $|\widehat{G} : G| \geqslant 3$ and so G has at least three regular orbits on $V \oplus V$. Thus the conclusion 2 is proved.

3. Since the K-module V_1 satisfies **Property I**, K has at least two regular orbits on $V_1 \oplus V_1$. If J_0 is a proper subgroup of K, then J_0 has at least four regular orbits on $V_1 \oplus V_1$ and so does J. Applying Statement 1 twice, we obtain that G has at least four regular orbits on $V \oplus V$.

   Then we may assume $J_0 = K$. Consequently the J_0-module V_1 (and also V_1 as a J-module) satisfies **Property I**. By induction, H has at least four regular orbits on $W \oplus W$ or the H-module W satisfies **Property I**. If H has at least four regular orbits on $W \oplus W$, then, by Statement 1, G has at least four regular orbits on $V \oplus V$, as desired.

   Now we assume that the H-module W satisfies **Property I**. By hypothesis, we have that $\mathrm{O}_3(H) = 1$. Moreover, there exists $0 \neq x \in W$ such that $\mathrm{C}_H(x)$ has at least four different orbits on W with representatives y_1, y_2, z_1, z_2 satisfying that $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_i) = 1$ and $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_i)$ is a 3-group for each i.

   Since G is of odd order, we have that S is of odd order. Consequently t is an odd prime and $t \geqslant 3$. By [10, Theorem 5.6], S has a strongly regular orbit on $\mathcal{P}(\Omega)$. We may assume that $\Delta \subseteq \Omega$ satisfies that $\mathrm{Stab}_S(\Delta) = 1$ and $|\Delta| \neq |\Omega \setminus \Delta|$. Take $v \in V = W_1^{\Omega}$ such that $v(i) = x$ for each $i \in \Omega$ and define u_j, $1 \leqslant j \leqslant 4$, as follows:

$$\begin{array}{llll}
u_1(i) = y_1, & i \in \Delta; & u_1(i) = y_2, & i \in \Omega \setminus \Delta; \\
u_2(i) = y_2, & i \in \Delta; & u_2(i) = y_1, & i \in \Omega \setminus \Delta; \\
u_3(i) = y_1, & i \in \Delta; & u_3(i) = z_1, & i \in \Omega \setminus \Delta; \\
u_4(i) = y_2, & i \in \Delta; & u_4(i) = z_2, & i \in \Omega \setminus \Delta.
\end{array}$$

   It is not difficult to find that u_j, $1 \leqslant j \leqslant 4$, lie in different orbits of $\mathrm{C}_G(v)$ on V. By Lemma 15 (3), (v, u_j), $1 \leqslant j \leqslant 4$, generate four different regular orbits of G on $V \oplus V$. Thus the conclusion 3 is proved.

4. Since the K-module V_1 satisfies **Property II**, we may assume that

   (a) K is an even order group with $\mathrm{O}_2(K) = 1$, and

   (b) there exist $0 \neq x' \in V_1$ and three different $\mathrm{C}_K(x')$-orbits with representatives y', z'_1, z'_2 satisfying that $\mathrm{C}_K(x') \cap \mathrm{C}_K(y') = 1$ and $\mathrm{C}_K(x') \cap \mathrm{C}_K(z'_i)$ is a 2-group for each i.

   If J_0 is of odd order, then J_0 is proper in K. Then J_0 has at least two regular orbits on $V \oplus V$ and $\mathrm{C}_{J_0}(x') \cap \mathrm{C}_{J_0}(z'_i)$ is a 2-group for each i, which implies that J_0 has at least four regular orbits on $V_1 \oplus V_1$ and so does J. Applying Statement 1 twice, we see that G has at least four regular orbits on $V \oplus V$.

Thus we may assume that J_0 is of even order. Suppose that $|K : J_0| \geqslant 3$. Then J_0 (also J) has at least three regular orbits on $V_1 \oplus V_1$. It follows from Statement 2 that H has at least three regular orbits on $W \oplus W$. Observe that $|H|$ is even since $|J|$ is even. Applying Statement 2 again, we conclude that G has at least three regular orbits on $V \oplus V$.

Now we may assume that $|K : J_0| \leqslant 2$. Consequently $J_0 \triangleleft K$ and $\mathrm{O}_2(J_0) \leqslant \mathrm{O}_2(K) = 1$. Then V_1, as a J-module (and so as a J_0-module), satisfies **Property II**.

By induction, H has at least three regular orbits on $W \oplus W$ or the H-module W satisfies **Property II**. Suppose that H has at least three regular orbits on $W \oplus W$. Since $|H|$ is even, G has at least three regular orbits on $V \oplus V$ by Statement 2, as desired.

Now we assume that the H-module W satisfies **Property II**, that is:

(a) H is an even order group with $\mathrm{O}_2(H) = 1$.

(b) There exist $0 \neq x \in W$ and three different $\mathrm{C}_H(x)$-orbits with representatives y, z_1, z_2 satisfying that $\mathrm{C}_H(x) \cap \mathrm{C}_H(y) = 1$ and $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_i)$ is a 2-group for each i.

First we consider the case $|\Omega| = t \geqslant 5$. By Lemma 9, $\mathrm{Stab}_S(1) \cap \mathrm{Stab}_S(2) = 1$. Let us take $v \in V = W^\Omega$ such that $v(i) = x$ for each $i \in \Omega$. Consider the elements $u_j \in V$, with $1 \leqslant j \leqslant 3$, defined by

$$\begin{array}{llll} u_1(1) = y; & u_1(2) = z_2; & u_1(i) = z_1, & i \in \Omega \setminus \{1,2\}; \\ u_2(1) = z_1; & u_2(2) = y; & u_2(i) = z_2, & i \in \Omega \setminus \{1,2\}; \\ u_3(1) = z_2; & u_3(2) = z_1; & u_3(i) = y, & i \in \Omega \setminus \{1,2\}. \end{array}$$

Since y, z_1, z_2 lie in different orbits of $\mathrm{C}_H(x)$ on W_1, it is not difficult to conclude that u_1, u_2 and u_3 lie in different orbits of $\mathrm{C}_G(v)$ on V. By Lemma 15 (3), we have that (v, u_j), $1 \leqslant j \leqslant 3$, generate three different regular orbits of G on $V \oplus V$, as desired.

Recall that $|\Omega| = t$ is a prime. Thus we only have to consider the cases $t = 2$ or $t = 3$.

Assume that $t = 3$. In this case, $S = S_3$ or $S = \langle (123) \rangle$. Take $v \in V = W^\Omega$ such that $v(i) = x$ for each $i \in \Omega$. Consider the elements

$u_j \in V$, where $1 \leqslant j \leqslant 3$, defined by

$$\begin{aligned}
&u_1(1) = y, && u_1(2) = z_1, && u_1(3) = z_2;\\
&u_2(1) = y, && u_2(2) = y, && u_2(3) = z_1;\\
&u_3(1) = y, && u_3(2) = y, && u_3(3) = z_2.
\end{aligned}$$

It is clear that u_1, u_2 and u_3 belong to different orbits of $\mathrm{C}_G(v)$ on V. By Lemma 15 (3), $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_1) = 1$. By Lemma 15 (4), we have that $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ is 2-group for $j \in \{2, 3\}$.

As $\mathrm{O}_2(H) = 1$, by Lemma 15 (1), $\mathrm{O}_p(N) = 1$. Observe that $\mathrm{O}_2(G/N) \cong \mathrm{O}_2(S) = 1$ and consequently $\mathrm{O}_2(G) \leqslant \mathrm{O}_2(N) = 1$. Furthermore, G is of even order since H is of even order. Thus the G-module V satisfies **Property II**, as desired.

Finally we assume that $|\Omega| = 2$ and $S \cong S_2$. Take $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements $u_1, u_2, u_3 \in V$ defined by

$$\begin{aligned}
&u_1(1) = z_1, && u_1(2) = y;\\
&u_2(1) = z_2, && u_2(2) = y;\\
&u_3(1) = z_1, && u_3(2) = z_2.
\end{aligned}$$

We have that u_1, u_2 and u_3 belong to different orbits of $\mathrm{C}_G(v)$ on V and, by Lemma 15 (3), $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$ for $j \in \{1, 2\}$ and $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_3)$ is 2-group.

Assume first that $\mathrm{O}_2(G) = 1$. Then, since G is of even order, we can conclude that the G-module V satisfies **Property II**, as desired. Now we assume that $\mathrm{O}_2(G) \neq 1$. By Lemma 15 (6), if we take $v' \in V$ such that $v'(1) = 0$ and $v'(2) = x$, then $\mathrm{C}_G(v') \cap \mathrm{C}_G(u_1) = 1$. We observe that $(v, u_1), (v, u_2)$ and (v', u_1) lie in different regular orbits of G on $V \oplus V$, as desired. Thus the conclusion 4 is completely proved.

5. Since the K-module V_1 satisfies **Property I**, K has at least two regular orbits on $V_1 \oplus V_1$. If J_0 is proper in K, then J_0 has at least four regular orbits on $V_1 \oplus V_1$ and so does J. By Statement 1, H has at least four regular orbits on $W_1 \oplus W_1$. Applying Statement 1 again, we obtain that G has at least four regular orbits on $V \oplus V$. Thus we may assume $J_0 = K$. Consequently V_1 as a J-module, and so as a J_0-module, satisfies **Property I**.

   When H is of even order, by induction, H has at least three regular orbits on $W \oplus W$ or the H-module W_1 satisfies **Property I** or **Property II**. Since H is of even order, clearly the H-module W does not satisfy **Property I**. If H has at least three regular orbits on $W \oplus W$, then it follows from Statement 2 that G has at least three regular orbits on $V \oplus V$, as desired. If the H-module W satisfies **Property II**, then we can conclude by Statement 4 that G has at least three regular orbits on $V \oplus V$ or the G-module V satisfies **Property II**, as desired. When H is of odd order, applying Statement 3 on (W, H, V_1, J), we can conclude that the H-module W satisfies **Property I** or H has at least four regular orbits on $W \oplus W$. If the latter case holds, then it follows from Statement 1 that G has at least four regular orbits on $V \oplus V$, as desired.

Thus we can suppose that the H-module W satisfies **Property I**. Then we have:

(a) H is an odd order group and $\mathrm{O}_3(H) = 1$.

(b) There exist $0 \neq x \in W$ and four different $\mathrm{C}_H(x)$-orbits with representatives y_1, y_2, z_1, z_2 satisfying that $\mathrm{C}_H(x) \cap \mathrm{C}_H(y_i) = 1$ and $\mathrm{C}_H(x) \cap \mathrm{C}_H(z_i)$ is a 3-group for each i.

First we consider the case $|\Omega| = t \geqslant 3$. By Lemma 9, $\mathrm{Stab}_S(1) \cap \mathrm{Stab}_S(2) = 1$.

Take $v \in V = W^\Omega$ such that $v(i) = x$ for each $i \in \Omega$. Consider the elements $u_j \in V$, where $1 \leqslant j \leqslant 3$, defined by

$$\begin{array}{llll} u_1(1) = y_1; & u_1(2) = y_2; & u_1(i) = z_1, & i \in \Omega \setminus \{1,2\}; \\ u_2(1) = y_1; & u_2(2) = y_2; & u_2(i) = z_2, & i \in \Omega \setminus \{1,2\}; \\ u_3(1) = y_1; & u_3(2) = z_1; & u_3(i) = z_2, & i \in \Omega \setminus \{1,2\}. \end{array}$$

Since y_1, y_2, z_1 and z_2 lie in different orbits of $\mathrm{C}_H(x)$ on W, it follows that u_1, u_2 and u_3 lie in different orbits of $\mathrm{C}_G(v)$ on V. By Lemma 15 (3), we have that $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j) = 1$ for $1 \leqslant j \leqslant 3$. Thus G has at least three regular orbits on $V \oplus V$, as desired.

Now we assume that $|\Omega| = 2$ and $S \cong S_2$. Let $v \in V$ such that $v(i) = x$ for each $i \in \Omega$ and consider the elements $u_1, u_2, u_3 \in V$ defined by

$$\begin{array}{ll} u_1(1) = y_1, & u_1(2) = y_2; \\ u_2(1) = y_1, & u_2(2) = y_1; \\ u_3(1) = y_2, & u_3(2) = y_2. \end{array}$$

Clearly u_j, $1 \leqslant j \leqslant 3$ lie in different orbits of $\mathrm{C}_G(v)$ on V. By Lemma 15 (3), $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_1) = 1$. By Lemma 15 (5), $\mathrm{C}_G(v) \cap \mathrm{C}_G(u_j)$ are 2-groups for $j \in \{2, 3\}$.

Assume first that $\mathrm{O}_2(G) = 1$. Then, since $G/N \cong S_2$, G has even order and we conclude that the G-module V satisfies **Property II**, as desired.

Now we assume that $\mathrm{O}_2(G) \neq 1$. By Lemma 15 (6), if we take $v' \in V$ such that $v'(1) = x$ and $v'(2) = 0$ and define $u'_j \in V$, $1 \leqslant j \leqslant 2$ as follows:

$$\begin{aligned} u'_1(1) &= y_1, & u'_1(2) &= z_1; \\ u'_2(1) &= y_1, & u'_2(2) &= z_2. \end{aligned}$$

We have that $\mathrm{C}_G(v') \cap \mathrm{C}_G(u'_j) = 1$, $1 \leqslant j \leqslant 2$. We also observe that (v, u_1), (v', u'_1) and (v', u'_2) lie in different regular orbits of G on $V \oplus V$, as desired. Thus the conclusion 5 is completely proved. $\square$

4 Proof of the main theorems

Theorem 17. *Let G be a soluble group and let V be an irreducible and faithful G-module over* $\mathrm{GF}(2)$*. If H is an odd order supersoluble subgroup of G, then H has at least four regular orbits on $V \oplus V$ or the H-module V satisfies **Property I**.*

Proof. We argue by induction on $|G|$. By Lemma 12, if V is primitive, then H has four regular orbits on $V \oplus V$ or $|V| = 2^3, H = \Gamma(V) \cong [C_7]C_3$. In the latter case, **Property I** holds, as desired. Now we may assume that V is an imprimitive G-module. Assume that $V = V_1 \oplus \cdots \oplus V_m$ ($m \geqslant 2$) is a direct sum of subspaces which are permuted transitively by G. If we do this so that m is as small as possible, then we can assume that $L = \mathrm{N}_G(V_1)$ is maximal in G, and we observe also that L acts irreducibly on V_1. Write $U = L/\mathrm{C}_G(V_1)$ and V_1 is a faithful and irreducible U-module.

Assume that $\Omega_1, \ldots, \Omega_s$ ($s \geqslant 1$) are all the H-orbits in $\{V_1, \ldots, V_m\}$. Set $W_j = \sum_{W \in \Omega_j} W$. First we claim that $H/\mathrm{C}_H(W_j)$ has at least four regular orbits on $W_j \oplus W_j$ or the $H/\mathrm{C}_H(W_j)$-module W_j satisfies **Property I** for each j.

We can assume without loss of generality $j = 1$ and $\Omega_1 = \{V_1, \ldots, V_t\}$, where $t = |H : L \cap H|$. Write $W = W_1$, $K = H/\mathrm{C}_H(W_1)$ and $J = \mathrm{N}_K(V_1)/\mathrm{C}_K(V_1)$.

Now we claim that K has at least four regular orbits on $W \oplus W$ or the K-module W satisfies **Property I**. Observe that the action of J on V_1 is equivalent to the action of $A := (L \cap H)\,\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1) \leqslant U$ on V_1. Then the triple (U, A, V_1) satisfies the hypotheses of the theorem. By induction, A

(and so J) has at least four regular orbits on $V_1 \oplus V_1$ or the A-module V_1 (and so the J-module V_1) satisfies **Property I**. If J has at least four regular orbits on $V_1 \oplus V_1$, then it follows from Lemma 16 (1) that K has at least four regular orbits on $W \oplus W$, as claimed. If the J-module V_1 satisfies **Property I**, since $|H|$ is odd, then it follows from Lemma 16 (3) that K has at least four regular orbits on $W \oplus W$ or the K-module W satisfies **Property I**, as claimed.

Thus $H/\mathrm{C}_H(W_j)$ has at least two regular orbits on $W_j \oplus W_j$ for each $1 \leqslant j \leqslant s$. If $s \geqslant 2$, then H has at least four regular orbits on $V \oplus V$ by Lemma 8, as desired. Now we may assume that $s = 1$, that is, H acts transitively on $\{V_1, \dots, V_m\}$. Thus $H = K$ and $W = V$, and consequently H has at least four regular orbits on $V \oplus V$ or the H-module V satisfies **Property I**. The theorem is proved. □

Theorem 18. *Let G be a soluble group and V be an irreducible and faithful G-module over* GF(2)*. If H is a supersoluble subgroup of G, then either H has at least three regular orbits on $V \oplus V$ or V, as an H-module, satisfies* ***Property I*** *or* ***Property II****.*

Proof. Work by induction on $|GV|$. If V is a primitive G-module, it follows from Lemma 11 that either H has at least three regular orbits on $V \oplus V$ or the H-module V satisfies:

1. $|V| = 2^2$ and $H = \Gamma(V) \cong S_3$, or
2. $|V| = 2^3$ and $H = \Gamma(V) \cong [C_7]C_3$.

It is not difficult to find that, in the first case, V satisfies **Property II** and in the second case, V satisfies **Property I**, as desired. Consequently, we assume that V is an imprimitive G-module. Then there $V = V_1 \oplus \cdots \oplus V_m$ ($m \geqslant 2$) is a direct sum of subspaces which are permuted transitively by G. If we do this so that m is as small as possible, then we can assume that $L = \mathrm{N}_G(V_1)$ is maximal in G, and we observe also that L acts irreducibly on V_1. Write $U = L/\mathrm{C}_G(V_1)$ and V_1 is a faithful, irreducible U-module.

Assume that $\Omega_1, \dots, \Omega_s$ ($s \geqslant 1$) are all the H-orbits in $\{V_1, \dots, V_m\}$. Set $W_j = \sum_{W \in \Omega_j} W$.

First we claim that $H/\mathrm{C}_H(W_j)$ has at least three regular orbits on $W_j \oplus W_j$ or the $H/\mathrm{C}_H(W_j)$-module W_j satisfies **Property I** or **Property II** for each j.

Without loss of generality, we may suppose $j = 1$ and $\Omega_1 = \{V_1, \dots, V_t\}$, where $t = |H : L \cap H|$. Write $W = W_1$, $K = H/\mathrm{C}_H(W_1)$ and $J = \mathrm{N}_K(V_1)/\mathrm{C}_K(V_1)$. Then W is a faithful H-module. Observe that the action of J on V_1 is equivalent to the action of $A := (L \cap H)\,\mathrm{C}_G(V_1)/\mathrm{C}_G(V_1) \leqslant U$ on V_1. Then the triple (U, A, V_1) satisfies the hypotheses of the theorem. By induction, either A (and also J) has at least three regular orbits on $V_1 \oplus V_1$ or V_1 regarded as an A-module (and also as a J-module) satisfies **Property I** or **Property II**.

If the J-module V_1 satisfies **Property I**, then our claim follows from Lemma 16 (5). If the J-module V_1 satisfies **Property II**, then our claim follows from Lemma 16 (4). Now we assume that J has at least three regular orbits on $V_1 \oplus V_1$. If J is of even order, then K has at least three regular orbits on $W \oplus W$ by Lemma 16 (2). If J is of odd order, then A is of odd order and the triple (U, A, V_1) satisfies the hypotheses of Theorem 17. Thus A (and also J) has at least four regular orbits on $V_1 \oplus V_1$ or V_1, regarded as an A-module (also as a J-module) satisfies **Property I**. If J has at least four regular orbits on $V_1 \oplus V_1$, then K has at least four regular orbits on $W \oplus W$ by Lemma 16 (1), as claimed. If the J-module V_1 satisfies **Property I**, then, by Lemma 16 (5) again, our claim holds.

Now we have proven that $H/\mathrm{C}_H(W_j)$ has at least three regular orbits on $W_j \oplus W_j$ or the $H/\mathrm{C}_H(W_j)$-module W_j satisfies **Property I** or **Property II** for each $1 \leqslant j \leqslant s$. In particular, $H/\mathrm{C}_H(W_j)$ has at least one regular orbit on $W_j \oplus W_j$ for each $1 \leqslant j \leqslant s$. If there exists some $j \in \{1, \ldots, s\}$ such that $H/\mathrm{C}_H(W_j)$ has at least three regular orbits on $W_j \oplus W_j$, then we can conclude that H has at least three regular orbits on $V \oplus V$ by Lemma 8, as desired.

Now we can assume that the $H/\mathrm{C}_H(W_j)$-module W_j satisfies **Property I** or **Property II** for each $1 \leqslant j \leqslant s$. Thus if $s = 1$, then V, as an H-module, satisfies **Property I** or **Property II**, as desired. Consequently, we can assume $s \geqslant 2$.

Take

$$\mathcal{C} = \{1 \leqslant j \leqslant s \mid \text{the } H/\mathrm{C}_H(W_j)\text{-module } W_j \text{ satisfies } \textbf{Property II}\}.$$

First we assume that $\mathcal{C} = \varnothing$. Then the $H/\mathrm{C}_H(W_j)$-module W_j satisfies **Property I** for each $1 \leqslant j \leqslant s$. It implies that $H/\mathrm{C}_H(W_j)$ has at least two regular orbits on $W_j \oplus W_j$. Since $s \geqslant 2$, then we can conclude that H has at least four regular orbits on $V \oplus V$ by Lemma 8, as desired.

Now we assume that $\mathcal{C} \neq \varnothing$, then, without loss of generality, we may assume that $\mathcal{C} = \{1, \ldots, l\}$ for some $1 \leqslant l \leqslant s$.

Write $K_j = H/\mathrm{C}_H(W_j)$. For $j = 1$, we have

1. K_1 is an even order group and $\mathrm{O}_2(K_1) = 1$.

2. There exists $0 \neq x_1 \in W_1$ such that $\mathrm{C}_{K_1}(x_1)$ has three different orbits on V_1 with representatives y_1, z_1, z_2 such that $\mathrm{C}_{K_1}(x_1) \cap \mathrm{C}_{K_1}(y_1) = 1$ and $\mathrm{C}_{K_1}(x_1) \cap \mathrm{C}_{K_1}(z_i)$ is a 2-group for $i = 1$, 2.

Recall that K_j has at least one regular orbit on $V_j \oplus V_j$ for each $2 \leqslant j \leqslant s$. We can assume that $\mathrm{C}_{K_j}(x_j) \cap \mathrm{C}_{K_j}(y_j) = 1$ for some $x_j, y_j \in V_j$.

Thus we can conclude that $\mathrm{C}_H(x_j) \cap \mathrm{C}_H(y_j) \subseteq \mathrm{C}_H(W_j)$ for each $1 \leqslant j \leqslant s$ and $X_i/\mathrm{C}_H(W_1)$ is a 2-group, where $X_i = \mathrm{C}_H(x_1) \cap \mathrm{C}_H(z_i)$ for $i = 1$, 2.

Write $v = \sum_{i=1}^{s} x_i$, $u = \sum_{i=1}^{s} y_i$, $w_1 = z_1 + \sum_{i=2}^{s} y_i$ and $w_2 = z_1 + \sum_{i=2}^{s} y_i$. It is not difficult to find that u, w_1, w_2 lie in different orbits of $\mathrm{C}_H(v)$ on V. Moreover, we have

$$\mathrm{C}_H(v) \cap \mathrm{C}_H(u) = \bigcap_{j=1}^{s} \big(\mathrm{C}_H(x_j) \cap \mathrm{C}_H(y_j)\big) \subseteq \bigcap_{j=1}^{s} \mathrm{C}_H(W_j) = 1$$

and

$$\mathrm{C}_H(v) \cap \mathrm{C}_H(w_i) \subseteq X_i \cap \bigcap_{j=2}^{s} \mathrm{C}_H(W_j) \cong (X_i \cap \bigcap_{j=2}^{s} \mathrm{C}_H(W_j))\, \mathrm{C}_H(W_1)/\mathrm{C}_H(W_1)$$

is a 2-group for $i = 1$, 2.

On the other hand, H is of even order since $H/\mathrm{C}_H(W_j)$ is of even order. Moreover, for each $j \in \mathcal{C}$, we have that $H/\mathrm{C}_H(W_j)$ is an even order group and $\mathrm{O}_2(H/\mathrm{C}_H(W_j)) = 1$, and for each $j \in \{1, \ldots, s\} \setminus \mathcal{C}$, we have that $H/\mathrm{C}_H(W_j)$ is an odd order group. Thus $\mathrm{O}_2(H) \leqslant \bigcap_{i=1}^{s} \mathrm{C}_H(W_j) = 1$. Hence the H-module V satisfies **Property II**, as desired. Thus the theorem is completely proved. $\square$

Proof of Theorem A. Assume that the theorem is false and let (G, H, V) be the counterexample such that $|G| + |H| + |V|$ minimal. First we claim that V is an irreducible G-module. Assume that this is false. Let $V = V_1 \oplus V_2$, where $0 \neq V_i$ is a G-module for $i \in \{1, 2\}$. Then V_i is a faithful, completely reducible $G/\mathrm{C}_G(V_i)$-module for $i \in \{1, 2\}$. Observe that $H\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ satisfies the hypotheses for $i \in \{1, 2\}$. Hence, by the choice of (G, H, V), $H\mathrm{C}_G(V_i)/\mathrm{C}_G(V_i)$ has at least one regular orbit on $V_i \oplus V_i$ for $i \in \{1, 2\}$. Thus H has at least one regular orbit on $V \oplus V$, against the choice of (G, H, V). This contradiction shows that V is an irreducible G-module over a field of characteristic p for some prime p. Then V is a completely reducible G-module over the field $\mathrm{GF}(p)$ of p elements.

Arguing as in the previous paragraph, we may assume that V is an irreducible, faithful G-module over $\mathrm{GF}(p)$. If p is odd, then it follows from Lemma [11, Corollary 3] that H has at least two regular orbits on $V \oplus V$. Thus we may assume that $p = 2$. It follows from Theorem 18 that H has at least three regular orbits on $V \oplus V$, or the H-module V satisfies **Property I** or **Property II**. In all these cases, we can conclude that H has at least one regular orbit on $V \oplus V$ and the main theorem is completely proved. $\square$

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