On finite p-groups of supersoluble type

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Abstract

A finite p-group S is said to be of supersoluble type if every fusion system over S is supersoluble. The main aim of this paper is to characterise the finite p-groups of supersoluble type. Abelian and metacyclic p-groups of supersoluble type are completely described. Furthermore, we show that the Sylow p-subgroups of supersoluble type of a finite simple group must be cyclic.

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1 Introduction

All groups considered in this paper will be finite.

A saturated fusion system $\mathcal{F}$ over a p-group S, p a prime, is a category whose objects are the subgroups of S, whose morphisms are monomorphisms between subgroups of S, and whose morphism sets satisfy certain axioms motivated by properties of conjugacy relations between p-subgroups of a group. If S is a Sylow p-subgroup of a group G, we can associate the saturated fusion system $\mathcal{F}_S(G)$ over S, called the *fusion system* of G, whose morphisms are those homomorphisms induced by conjugation in G. We refer to [1] for

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a detailed introduction to the theory of saturated fusion systems: this book will be our reference for the notation, terminology and results.

In this paper, we continue the study, started in [14], of supersoluble saturated fusion systems.

Definition 1 ([14, Definition 1.2])**.** Let S be a p-group and let $\mathcal{F}$ be a saturated fusion system over S. We say that $\mathcal{F}$ is *supersoluble* if there exists a series $1 = S_0 \leq \cdots \leq S_m = S$ of subgroups of S such that S_i is strongly closed in S with respect to $\mathcal{F}$ and S_{i+1}/S_i is cyclic for each $i \in \{0, \ldots, m-1\}$.

It has been shown that supersoluble fusion systems are precisely the fusion systems of supersoluble groups (see [14, Proposition 1.3(b)]).

It is clear that every nilpotent saturated fusion system in the sense of [11] is supersoluble and every supersoluble saturated fusion system is soluble in the sense of [1, Part II, Definition 12.1].

Definition 2. A p-group S is said to be *of supersoluble type* if for every saturated fusion system $\mathcal{F}$ over S, $\mathcal{F}$ is supersoluble.

We characterise the p-groups of supersoluble type in the following theorem.

Theorem A. *Let S be a p-group. Then S is of supersoluble type if and only if S is resistant and every p'-subgroup of* $\mathrm{Aut}(S)$ *is abelian of exponent dividing $p-1$.*

We will give several applications of the above characterisation theorem. One of the consequences is the following.

Corollary 3. *If S is a p-group of supersoluble type, then* $\mathrm{Aut}(S)$ *is soluble.*

Theorem A and Corollary 3 will be proved in the next section.

We then apply this characterisation to describe the abelian and metacyclic p-groups of supersoluble type in Theorems B and C. With all these results at hand, we can show that the Sylow p-subgroups of supersoluble type of a simple group must be cyclic (Theorem D), and that the structure of metacyclic Sylow p-subgrups of a simple group is quite limited (Theorem 12).

2 Proof of Theorem A

Recall that a p-group S is called *resistant* if S is normal in every saturated fusion system over S (see [13]).

Proof of Theorem A. We prove first the necessity of the condition. Assume that S is of supersoluble type. Let $\mathcal{F}$ be a saturated fusion system over S. By [14, Proposition 1.3(b)], there exists a supersoluble group K such that S is a Sylow p-subgroup of K and $\mathcal{F} = \mathcal{F}_S(K)$. Without loss of generality, we may assume that $\mathrm{O}_{p'}(K) = 1$. Since K is supersoluble with $\mathrm{O}_{p'}(K) = 1$, we have $S \trianglelefteq K$ and thus $S \trianglelefteq \mathcal{F}_S(K) = \mathcal{F}$. Hence S is resistant.

Let H be a p'-subgroup of $\mathrm{Aut}(S)$. We will show that H is abelian of exponent dividing $p-1$. Set $G = S \rtimes H$, the natural semidirect product of S and H. Clearly $\mathrm{C}_G(S) \leq S$ since $\mathrm{C}_H(S) = 1$. Write $\mathcal{F} = \mathcal{F}_S(G)$. As $\mathcal{F}$ is a saturated fusion system over S, it follows that $\mathcal{F}$ is supersoluble. By [14, Proposition 1.3(d)], $\mathrm{Aut}_{\mathcal{F}}(S)$ is p-closed and a Hall p'-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$ is abelian of exponent dividing $p-1$. Note that $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_G(S) \cong \mathrm{N}_G(S)/\mathrm{C}_G(S) = HS/\mathrm{C}_G(S)$. Then $H\mathrm{C}_G(S)/\mathrm{C}_G(S)$ is a Hall p'-subgroup of $G/\mathrm{C}_G(S)$ and so $H \cong H\mathrm{C}_G(S)/\mathrm{C}_G(S)$ is abelian of exponent dividing $p-1$.

We prove now the sufficiency of the condition. Assume that S is resistant and every p'-subgroup of $\mathrm{Aut}(S)$ is abelian of exponent dividing $p-1$. Let $\mathcal{F}$ be a saturated fusion system over S. We shall show that $\mathcal{F}$ is supersoluble. As S is resistant, $S \trianglelefteq \mathcal{F}$ and $\mathcal{F}$ is constrained. Since $S \trianglelefteq \mathcal{F}$, it is clear that $\mathcal{F}$ is the fusion system of a finite group $G = S \rtimes H$ for some p'-subgroup H of $\mathrm{Aut}(S)$. It then follows from the assumption that H is abelian of exponent dividing $p-1$. Thus G is soluble and every p-chief factor of G is cyclic by [5, Chapter B, Theorem 9.8] and every p'-chief factor is central. Consequently, G is supersoluble. By [14, Proposition 1.3(b)], $\mathcal{F} = \mathcal{F}_S(G)$ is supersoluble. We conclude then that S is of supersoluble type. □

Corollary 4. *Let S be a 2-group. Then S is of supersoluble type if and only if S is resistant and $\mathrm{Aut}(S)$ is a 2-group.*

We can then obtain Corollary 3 by combining Theorem A and the following result.

Theorem 5. *If every p'-subgroup of a group G is abelian of exponent dividing $p-1$, then G is soluble.*

The proof of Theorem 5 requires the following lemma.

Lemma 6. *Let $G = \mathrm{PSL}_2(r^f)$, where r is a prime and $f \geq 1$. Set $u = (r^f-1)/k$ and $s = (r^f+1)/k$, where $k = (r^f-1, 2)$. Then G has two cyclic subgroups U and S of orders u and s, respectively. Moreover $\mathrm{N}_G(U)$ is dihedral of order $2u$ and $\mathrm{N}_G(S)$ is dihedral of order $2s$.*

Proof. It is a consequence of [9, Kapitel II, Satz 8.3 and 8.4]. □

Proof of Theorem 5. Let G be a counterexample of minimal order. Then p is the largest prime dividing the order of G. By [5, Chapter I, Section 2], $p \neq 2$,and $p \neq 3$. The minimal choice of G implies that every proper subgroup and every nontrivial epimorphic image of G are soluble. Hence G is a minimal simple group.

By a result of Thompson (see [9, Kapitel II, Bemerkung 7.5]), G is isomorphic to one of the following groups:

1. $\mathrm{PSL}_2(q)$, where $q > 3$ is a prime and $5 \nmid q^2 - 1$;
2. $\mathrm{PSL}_2(2^q)$, where q is a prime;
3. $\mathrm{PSL}_2(3^q)$, where q is an odd prime;
4. $\mathrm{PSL}_3(3)$;
5. the Suzuki group $\mathrm{Sz}(2^q)$, where q is an odd prime.

If $G \cong \mathrm{PSL}_3(3)$, then $|G| = 13 \cdot 3^3 \cdot 2^4$ and $p = 13$. Observe that $\mathrm{PSL}_3(3) \cong \mathrm{SL}_3(3)$ has a subgroup isomorphic to $\mathrm{SL}_2(3)$, which is a nonabelian $13'$-group, contrary to assumption. Hence G cannot be isomorphic to $\mathrm{PSL}_3(3)$. If $G \cong \mathrm{Sz}(2^q)$, where q is an odd prime, we can apply [10, Chapter XI, Lemma 3.1(a) and Theorem 3.3] to conclude that G has a nonabelian Sylow 2-subgroup. This contradiction shows that G is not isomorphic to $\mathrm{Sz}(2^q)$ for any odd prime q.

Assume that $G = \mathrm{PSL}_2(r^f)$ for some prime r and integer f such that $r^f \geq 4$. Then $|G| = r^f(r^f - 1)(r^f + 1)k^{-1}$, where $k = (2, r^f - 1)$. Observe that $(r^f + 1, r^f - 1) = 1$ or 2. As $p \neq 2$, we can conclude that $r^f - 1$ or $r^f + 1$ is a p'-number. Suppose that $r^f = 4$ or 5. By [9, Kapitel II, Satz 6.14], $G \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$ is isomorphic to the alternating group of degree 5 which has a nonabelian p'-subgroup isomorphic to the alternating group of degree 4. This contradiction yields $r^f \geq 6$. Then $2(r^f + 1/k) > 2(r^f - 1/k) > 4$; by Lemma 6, G has dihedral p'-subgroups. This final contradiction proves the theorem. □

3 Abelian and metacyclic p-groups of supersoluble type

Our aim in this section is to characterise the abelian and metacyclic p-groups of supersoluble type. Such characterizations will be used later in Section 4 to investigate the structure of Sylow p-subgroups of supersoluble type of finite simple groups and the structure of metacyclic Sylow p-subgroups of finite simple groups.

We need two preliminary lemmas. The first one is elementary.

Lemma 7. *Let P be a group isomorphic to $C_{p^n} \times C_{p^n}$ for some positive integer n. Then* $\mathrm{Aut}(P)$ *has a quotient isomorphic to* $\mathrm{GL}_2(p)$.

The second lemma is a consequence of Theorem A.

Lemma 8. *Let S be a resistant p-group. Assume that S has a series of characteristic subgroups $\Phi(S) = D_0 \leq D_1 \leq \cdots \leq D_n = S$ such that D_i/D_{i-1} is cyclic for each $0 < i \leq n$. Then S is of supersoluble type.*

Proof. Without loss of generality, we may assume that D_i/D_{i-1} is of order p for $i = 1, \ldots, n$. Let H be a p'-subgroup of $\mathrm{Aut}(S)$, and let H^* be the smallest normal subgroup of H such that H/H^* is an abelian group of exponent dividing $p-1$. We want to show that $H^* = 1$. Let $1 \leq i \leq n$. Then $H/\mathrm{C}_H(D_i/D_{i-1})$ is isomorphic to a subgroup of $\mathrm{Aut}(D_i/D_{i-1}) \cong \mathrm{Aut}(C_p)$ and so $H/\mathrm{C}_H(D_i/D_{i-1})$ is abelian of exponent dividing $p-1$. Thus $H^* \leq \mathrm{C}_H(D_i/D_{i-1})$ for all i. Therefore H^* stabilises the chain $S = D_n \geq D_{n-1} \geq \cdots \geq D_0 = \Phi(S)$. By [5, Chapter I, Lemma 1.5], H^* acts trivially on $D = S/\Phi(S)$. It follows from [5, Chapter I, Proposition 1.7] that $H^* = 1$. Consequently, S is of supersoluble type by Theorem A. □

Theorem B. *Let S be an abelian p-group of type $(m_1, \ldots, m_t)$. Then S is of supersoluble type if and only if $m_1, \ldots, m_t$ are all distinct.*

Proof. We can assume, arguing by contradiction, that S is of supersoluble type and $m_1, \ldots, m_t$ are not all distinct. Without loss of generality we may suppose that $m_1 = m_2 = n$. Then $S = P \times H$, where $P, H \leq S$ and $P \cong C_{p^n} \times C_{p^n}$. By Lemma 7, $\mathrm{Aut}(P)$ has a quotient isomorphic to $\mathrm{GL}_2(p)$. Observe that $\mathrm{Aut}(P) \times \mathrm{Aut}(H)$ is a subgroup of $\mathrm{Aut}(S)$. Thus $\mathrm{Aut}(S)$ has a section isomorphic to $\mathrm{GL}_2(p)$. Suppose that $p = 2$. Since $\mathrm{GL}_2(2) \cong S_3$, it follows that $\mathrm{Aut}(S)$ is not a 2-group. Hence $p > 2$ by Corollary 4. If p is odd, the Sylow 2-subgroups of $\mathrm{SL}_2(p)$ are nonabelian by [9, Kapitel II, Hauptsatz 8.27]. This contradicts Theorem A. Consequently, $m_1, \ldots, m_t$ are distinct.

Assume that $m_1, \ldots, m_t$ are all distinct and $m_1 < m_2 < \cdots < m_t$. We shall show that S is of supersoluble type. By [1, Part I, Corollary 4.7], S is resistant. Let $D_i = \Omega_i(S)\Phi(S)$, where $\Omega_i(S)$ is the subgroup generated by all elements of S of order dividing p^i. Then there exists a positive integer n such that

$$\Phi(S) = D_0 \leq D_1 \leq \cdots \leq D_n = S. \quad (1)$$

Then (1) is a characteristic series of S such that D_i/D_{i-1} is cyclic for each $0 < i \leq n$. By Lemma 8, S is of supersoluble type. $\square$

Theorem C. *Let S be a metacyclic p-group. Then S is of supersoluble type if and only if S is none of the following groups:*

1. *the abelian group $C_{p^n} \times C_{p^n}$ for some positive integer n,*
2. *dihedral, semidihedral or generalised quaternion if $p = 2$.*

Proof. First assume that $p = 2$ and let S be a metacyclic 2-group. Applying [4, Theorems 1.1] and Corollary 4, we have that S is of supersoluble type if and only if S is none of the groups listed in the statement of the theorem.

Now assume that p is odd. If S is an abelian p-group, then by Theorem B, S is of supersoluble type if and only if S is not isomorphic to $C_{p^n} \times C_{p^n}$ for any positive integer n.

Suppose that S is a nonabelian metacyclic p-group. We prove that S is of supersolvable type. By [13, Proposition 5.4], S is resistant. Since S' is a nontrivial cyclic subgroup of S, we can apply [9, Kapitel III, Satz 10.2(c)] to conclude that S is regular.

Assume that the exponent of S is p^m. Since S is regular, we can apply [9, Kapitel III, Hauptsatz 10.5(b)] to conclude that

$$\mho_{m-1}(S) = \langle x^{p^{m-1}} : x \in S \rangle = \{x^{p^{m-1}} : x \in S\}.$$

Moreover, by [9, Kapitel III, Satz 10.6], $\mho_{m-1}(S)$ is elementary abelian. Since $\mho_{m-1}(S)$ is metacyclic, we have that $\mho_{m-1}(S) \cong C_p$ or $\mho_{m-1}(S) \cong C_p \times C_p$.

Suppose that $\mho_{m-1}(S) \cong C_p$. By [9, Kapitel III, Satz 10.7 (a)], we have that $|S/\Omega_{m-1}(S)| = |\mho_{m-1}(S)| = p$. Since S is 2-generated, $|S/\Phi(S)| = p^2$. Hence $\Phi(S) \leq \Omega_{m-1}(S) \leq S$ is a characteristic series of S with cyclic factors. Applying Lemma 8, we conclude that S is of supersoluble type.

Suppose that $\mho_{m-1}(S) \cong C_p \times C_p$. Since S' is a nontrivial cyclic subgroup of S, we have $\mho_{m-1}(S)$ is not contained in S'. Moreover $\mho_{m-1}(S) \cap S' \neq 1$, because otherwise $\mho_{m-1}(S)S' = \mho_{m-1}(S) \times S' \cong (C_p \times C_p) \times C_{p^t}$, $t > 0$, would not be metacyclic. Hence $|\mho_{m-1}(S) \cap S'| = p$.

Let D be the subgroup of S such that $\Omega_{m-1}(S/S') = D/S'$. Clearly D is a characteristic subgroup of G, and

$$\begin{aligned} |S : D| &= |S/S' : D/S'| = |S/S' : \Omega_{m-1}(S/S')| = |\mho_{m-1}(S/S')| \\ &= |\mho_{m-1}(S)S'/S'| = |\mho_{m-1}(S) : \mho_{m-1}(S) \cap S'| = p. \end{aligned}$$

It follows that $\Phi(S) \leq D \leq S$ is a characteristic series of S with $|S/D| = |D/\Phi(S)| = p$. By Lemma 8, S is of supersoluble type. $\square$

4 Simple groups with Sylow p-subgroups of supersoluble type

The aim of this section is to determine the Sylow p-subgroups of simple groups that are of supersoluble type. As an application we also determine the metacyclic Sylow p-subgroups of simple groups. This is achieved in the last two theorems in the section and requires some preliminary results. The first lemma is well known.

Lemma 9. *If p is an odd prime, then the group $\mathrm{SL}_2(p)$ has an element of order $p+1$.*

Lemma 10. *If S is an extraspecial group of order p^3 and exponent p, with p an odd prime, then S is not of supersoluble type.*

Proof. It is enough to prove the existence of a p'-automorphism of S with order not dividing $p-1$. Since every p'-automorphism of the elementary abelian quotient of S lifts to S, Lemma 9 yields that $\mathrm{Aut}(S)$ has an element of order divisible by $p+1$. Since $p+1$ is not a divisor of $p-1$, the result follows as a consequence of Theorem A.. □

Lemma 11. *The Sylow p-subgroups of $G = \mathrm{PSU}_3(q)$ for q a power of the prime p are not of supersoluble type.*

Proof. As in [9, Kapitel II, Satz 10.12], we consider $\mathrm{GU}_3(q)$ as the group of matrices $\mathsf{M} \in \mathrm{GL}_3(q^2)$ such that $\mathsf{M}^{\varphi}\mathsf{J}\mathsf{M} = \mathsf{J}$, where

$$\mathsf{J} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and φ acts on each entry of the matrix as the field automorphism $x \longmapsto x^q$. Then the Sylow p-subgroups of $\mathrm{PSU}_3(q)$ are isomorhpic to the Sylow p-subgroups of $\mathrm{GU}_3(q)$, and there is a Sylow p-subgroup U of $\mathrm{GU}_3(q)$ composed of matrices of the form

$$\mathsf{M}(c,d) = \begin{bmatrix} 1 & c & d \\ 0 & 1 & -c^q \\ 0 & 0 & 1 \end{bmatrix},$$

where $d \in \mathrm{GF}(q^2)$ and $c \in \mathrm{GF}(q^2)$ satisfy $cc^q = -(d + d^q)$ and multiplication given by $\mathsf{M}(c,d)\mathsf{M}(e,f) = \mathsf{M}(c+e, d+f-ce^q)$. Let U be the set composed of all these matrices with c, $d \in \mathrm{GF}(q^2)$ and $cc^q = -(d+d^q)$.

Let ζ be a generator of the multiplicative group of $\mathrm{GF}(q) \subseteq \mathrm{GF}(q^2)$. Then the matrix

$$\mathsf{D} = \begin{bmatrix} \zeta^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{bmatrix}$$

is an element of $\mathrm{GU}_3(q)$ that induces by conjugation an automorphism δ of $\mathrm{GU}_3(q)$ such that $\mathsf{D}^{-1}\mathsf{M}(c,d)\mathsf{D} = \mathsf{M}(\zeta c, \zeta^2 d)$, and since $\zeta^q = \zeta$, $(\zeta c)(\zeta c)^q = \zeta^2 cc^q = -\zeta^2(d+d^q) = -(\zeta^2 d + (\zeta^2 d)^q)$ and $\mathsf{M}(\zeta c, \zeta^2 d) \in U$. We note that this automorphism has order $q-1$, because if ξ is a generator of the multiplicative group of $\mathrm{GF}(q^2)$, then $\mathsf{A} = \mathsf{M}(1, -\xi/(\xi+\xi^q)) \in U$ and $\mathsf{A}^{\delta^t} = \mathsf{M}(\zeta^r, -\zeta^{2r}\xi/(\xi+\xi^q))$. Hence $\mathsf{A}^{\delta^t} = \mathsf{A}$ if and only if $q-1 \mid t$, and so the order of δ is divisible by $q-1$. If U is of supersoluble type, then by Theorem A we have that $q = p$, a prime number.

Suppose that $G = \mathrm{PSU}_3(p)$, with p prime, then $p > 2$ since $\mathrm{PSU}_3(2)$ is soluble. Let $\mathsf{A} = \mathsf{M}(1, -\xi/(\xi+\xi^p))$, $\mathsf{B} = \mathsf{M}(\xi, -\xi^2\xi^p/(\xi+\xi^p))$, where ξ is a generator of $\mathrm{GF}(p^2)^\times$, Since $\xi - \xi^p \neq 0$, these elements do not commute, because $\mathsf{AB} = \mathsf{M}(\xi+1, -\xi-\xi^2\xi^q/(\xi+\xi^q))$ and $\mathsf{BA} = \mathsf{M}(\xi+1, -\xi^q-\xi^2\xi^q/(\xi+\xi^q))$. Moreover, the elements of U have order p, since $\mathsf{M}(c,d)^r = \mathsf{M}(rc, rd - (r(r-1)/2)cc^p)$. We conclude that U is an extraspecial group of order p^3 and exponent p. By Lemma 10, this case is also ruled out. □

Theorem D. *Let S be a Sylow p-subgroup of a finite simple group G. If S is of supersoluble type, then S is cyclic.*

Proof. Since S is resistant by Theorem A, we have that $S \trianglelefteq \mathcal{F}_S(G)$. Then G is a p-Goldschmidt group (see [1, Part II, Definition 12.9]). According to results of Foote and Flores and Foote ([7, 6], see also [1, Part II, Theorem 12.10]), G is a p-Goldschmidt group if and only if one of the following conditions holds:

1. S is abelian.
2. G is of Lie type in characteristic p of Lie rank 1.
3. $p = 5$ and $G \cong \mathrm{McL}$.
4. $p = 11$ and $G \cong \mathrm{J}_4$.
5. $p = 3$ and $G \cong \mathrm{J}_2$.
6. $p = 5$ and $G \cong \mathrm{HS}$, Co_2, or Co_3.
7. $p = 3$ and $G \cong \mathrm{G}_2(q)$ for some prime power q prime to 3 such that q is not congruent to ± 1 modulo 9.

8. $p = 3$ and $G \cong \mathrm{J}_3$.

First assume that S is not abelian. In Cases 3–7, according to the Atlas [3], the Sylow p-subgroup is extraspecial of order p^3 and exponent p. These cases are ruled out by Lemma 10. In Case 8, if $p = 3$ and $G = \mathrm{J}_3$, then $|S| = 3^5$ and we can check with GAP [8] that $\mathrm{Aut}(S)$ is a $\{2, 3\}$-group whose Sylow 2-subgroup is isomorphic to a semidihedral group QD_{16} of order 16, therefore the Sylow 3-subgroup of J_3 is not of supersoluble type by Theorem A. Consequently we can assume G is of Lie type in characteristic p and G has Lie rank 1. Since the Sylow p-subgroups of $\mathrm{PSL}_2(q)$ for $q = p^f$ are isomorphic to the multiplicative group of the field $\mathrm{GF}(q)$, that is abelian, we have that G is either isomorphic to $\mathrm{PSU}_3(q)$ for $q = p^f$ a prime power, or to a Suzuki group $\mathrm{Sz}(2^{2m+1})$ for $p = 2$, or to a Ree group ${}^2\mathrm{G}_2(3^{2m+1})$ for $p = 3$. By Lemma 11, the Sylow p-subgroups of $\mathrm{PSU}_3(q)$ are not of supersoluble type. In the Suzuki and Ree cases, the field automorphism $x \mapsto x^p$ induces an automorphism of the Sylow subgroup S of order $2m + 1 \geq 3$, that cannot be a divisor of $p - 1$ and thus S is not of supersoluble type by Theorem A.

Therefore we can suppose that S is abelian. Assume that S is of type $(m_1, \ldots, m_t)$. Now, by Theorem B, we know that $m_1, \ldots, m_t$ are all distinct. Moreover, it is shown in [12] that S is isomorphic to a direct product of copies of a cyclic group. Hence S must be cyclic. This completes the proof of the theorem. □

By combining Theorem C, Theorem D, and [2, Theorem 1], we can determine the structure the metacyclic Sylow p-subgroups of finite simple groups.

Theorem 12. *Let S be a Sylow p-subgroup of a finite simple group G. If S is metacyclic, then S is one of the following:*

1. *$C_{p^n} \times C_{p^n}$ for some positive integer n,*
2. *cyclic if $p \neq 2$,*
3. *dihedral or semidihedral if $p = 2$.*

Remark 13. It is clear that the classes of metacyclic p-group listed in Theorem 12 do occur as Sylow p-subgroups of some finite simple groups. For instance, A_7, the alternating group of degree 7, has dihedral Sylow 2-subgroups, has cyclic Sylow 7-subgroups, and has elementary Sylow 3-subgroups of order 9. And the linear group $\mathrm{PSL}_3(7)$ has semidihedral Sylow 2-subgroups.

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