Triply factorised groups and the structure of skew left braces

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Abstract

The algebraic structure of skew left brace has proved to be useful as a source of set-theoretic solutions of the Yang-Baxter equation. We study in this paper the connections between left and right π-nilpotency and the structure of finite skew left braces. We also study factorisations of skew left braces and their impact on the skew left brace structure. As a consequence of our study, we define a Fitting-like ideal of a left brace. Our approach depends strongly on a description of a skew left brace in terms of a triply factorised group obtained from the action of the multiplicative group of the skew left brace on its additive group.

Keywords: Skew left brace, trifactorised group, triply factorised group, left nilpotent skew left brace, right nilpotent skew left brace, ideal, left Fitting ideal, factorised skew left brace

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1 Introduction

The quantum Yang-Baxter equation is one of the basic equations of mathematical physics [21] that turns out to be an important tool in the theory of quantum groups and related areas [12]. In order to find solutions of this equation, Drinfeld [7] proposed to study the important class of the set-theoretic ones. Drinfeld's paper considerably stimulated research in the area, especially in developing some algebraic tools.

Rump [16] found a connection between set-theoretic solutions and radical rings, and introduced in [16] a new algebraic structure, called *left brace*, that

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generalises radical rings and provides an algebraic way to construct involutive non-degenerate set-theoretic solutions of the Yang-Baxter equation.

Guarnieri and Vendramin [9] introduced skew left braces as a natural generalisation of left braces to the non-abelian setting. This construction was extremely useful to produce and study bijective non-degenerate set-theoretic solutions of the Yang-Baxter equation.

A *skew left brace* $(B, +, \cdot)$ consists of a set B with two operations, denoted by $+$ and $\cdot$, respectively, such that $(B, +)$ and $(B, \cdot)$ are groups and, for every a, b, $c \in B$, it holds that $a \cdot (b + c) = (a \cdot b) - a + (a \cdot c)$, where $-a$ denotes the inverse of a in $(B, +)$ and $a - b$ denotes $a + (-b)$. We will usually write ab instead of $a \cdot b$ and we will follow the usual convention in arithmetic that multiplications are done before additions unless we use parentheses to specify another order for the operations. Hence the previous expression can be written as $a(b + c) = ab - a + ac$.

Skew left braces with abelian additive group are Rump's left braces.

As skew left braces are an interaction of two compatible group structures, it is quite natural to approach them by group theoretical methods. From this point of view, Rump [16] introduced left and right nilpotent skew left braces, which were studied extensively by Cedó, Smoktunowicz and Vendramin in [5]. Left and right p-nilpotency of finite left braces, p a prime number, were studied by Meng and the authors of this paper in [13], whereas left and right p-nilpotency of finite skew left braces were studied by Acri, Lutowski and Vendramin in [1]. Some connections between the additive and the multiplicative group some classes of skew braces have been studied by Nasybullov in [14]. The relation between both groups has also been the key in the papers of Guarnieri and Vendramin [9] and Bardakov, Neshchadim, and Yadav [3] to classify skew left braces of small orders.

On the other hand, Sysak asked about extending the results of factorisation of groups to skew left braces. Jespers, Kubat, Van Antwerpen, and Vendramin [11] studied factorisations of skew left braces and proved a sort of analogue of Itô's celebrated theorem on metabelian groups.

This paper is a contribution to the structural study of skew left braces and tries to take the above studies further. Our first main goal is study the connection between left and right π-nilpotency and the structure of finite skew left braces, where π is a set of prime numbers. Our second main goal is to study factorisations of skew left braces and analyse their impact on the skew left brace structure. A Fitting-type ideal of left braces appears naturally as a consequence of our study.

Our approach heavily depends on a description of skew left braces in terms of triply factorised groups, following an idea of Sysak [19].

2 Preliminaries

In order to keep this paper self-contained, we provide in this section some basic results. A certain amount of what is here should be considered as folklore although probably some bits are new.

We will refer to a skew left brace $(B,+,\cdot)$ as simply B if the operations are understood. We will denote the identity element of $(B,+)$ by 0.

Let $\mathfrak{X}$ be a class of groups. If $(B,+)$ belongs to $\mathfrak{X}$, then B is called a skew left brace of $\mathfrak{X}$-type.

Rump's braces are exactly the skew left braces of abelian type.

Let $(B,\cdot)$ be a group. Then $(B,\cdot,\cdot)$ is a skew left brace. These braces are called *trivial* skew left braces.

Lemma 2.1 ([9])**.** *Let B be a skew left brace. Then:*

1. *The identity elements of $(B,+)$ and $(B,\cdot)$ coincide.*
2. *If a, $c \in B$, then $-ac = -a + a(-c) - a$.*
3. *The multiplicative group $(B,\cdot)$ of B induces an action $\lambda\colon (B,\cdot) \longrightarrow \operatorname{Aut}(B,+)$ given by $\lambda(a) = \lambda_a\colon (B,+) \longrightarrow (B,+)$, with $\lambda_a(b) = -a+ab$ for every a, $b \in B$.*
4. *For every a, $b \in B$, it holds that $a + b = a\lambda_a^{-1}(b)$.*

Recall that if a group $(G,\cdot)$ acts on a group $(A,+)$ via $\lambda\colon (G,\cdot) \longrightarrow \operatorname{Aut}(A,+)$, a *derivation* or 1-*cocycle* $\delta\colon (G,\cdot) \longrightarrow (A,+)$ with respect to λ is a map satisfying that $\delta(bc) = \delta(b) + \lambda_b(\delta(c))$ for every b, $c \in G$.

We see now that skew left braces are closely related with bijective derivations with respect to an action from a multiplicative group on an additive group (see [9, Proposition 1.11]).

Lemma 2.2. *If B is a skew left brace, then the identity map $\mathrm{id}_B\colon (B,\cdot) \longrightarrow (B,+)$ is a bijective derivation with respect to the action λ.*

Lemma 2.3. *Suppose that we have an action λ of a group $(B,\cdot)$ on a group $(A,+)$, and a bijective derivation $\delta\colon (B,\cdot) \longrightarrow (A,+)$ with respect to λ. Then we can define an addition on B via $b + c = \delta^{-1}(\delta(b) + \delta(c))$ and $(B,+,\cdot)$ becomes a skew left brace.*

We conclude that a skew left brace gives an action and a bijective derivation between $(B,\cdot)$ and $(B,+)$, and that given an action and a bijective derivation between a group $(B,\cdot)$ and a group $(A,+)$, then B can receive a skew left brace structure.

The following result about derivations is useful in some calculations.

Lemma 2.4. *Let $(C, \cdot)$ and $(K, +)$ be two groups. Suppose that $\delta\colon C \longrightarrow K$ is a derivation with respect to an action λ of C on K. Then $\delta(1) = 0$ and $\delta(c^{-1}) = -\lambda_{c^{-1}}(\delta(c))$ for every $c \in C$.*

Proof. Let $c \in C$, then $\delta(1) = \delta(1) + \lambda_1(\delta(1)) = \delta(1) + \delta(1)$, which implies that $\delta(1) = 0$, and so $0 = \delta(1) = \delta(c^{-1}c) = \delta(c^{-1}) + \lambda_{c^{-1}}(\delta(c))$, which implies that $\delta(c^{-1}) = -\lambda_{c^{-1}}(\delta(c))$. □

The following lemma is an interesting property about a derivation $\delta\colon C \longrightarrow K$ that allows us to obtain subgroups of C from C-invariant subgroups of K.

Lemma 2.5. *Let $(C, \cdot)$ and $(K, +)$ be two groups. Suppose that $\delta\colon C \longrightarrow K$ is a derivation associated to an action λ of C on K and that L is a C-invariant subgroup of K (for instance, this happens when L is a characteristic subgroup of K). Then $\delta^{-1}(L) \leqslant C$.*

Proof. Let e_1, $e_2 \in \delta^{-1}(L)$, then $\delta(e_1e_2) = \delta(e_1) + \lambda_{e_1}(\delta(e_2)) \in L$ because $\delta(e_1)$, $\delta(e_2) \in L$ and $\lambda_{e_1}(\delta(e_2)) \in L$ because L is C-invariant. Therefore $e_1e_2 \in \delta^{-1}(L)$. Since $\delta(e_1^{-1}) = -\lambda_{e_1^{-1}}(\delta(e_1))$, $\delta(e_1) \in L$ and L is C-invariant, it follows that $\delta(e_1^{-1}) \in L$ and so $e_1^{-1} \in \delta^{-1}(L)$. We conclude that $\delta^{-1}(L)$ is a subgroup of C. □

As a consequence of the above lemma, it follows that if B is a finite skew left brace of nilpotent type and π is a set of primes, then every Hall π-subgroup of $(B, +)$ is also a Hall π-subgroup of $(B, \cdot)$. In particular, $(B, \cdot)$ is soluble. This result was proved by Smoktunowicz and Vendramin (see [17, Corollary 2.23]) and also by Byott in the context of Hopf-Galois extensions (see [4, Theorem 1]).

The following basic results about commutators will be used several times in this paper without further reference.

Lemma 2.6 ([10, Kapitel III, Hilfssatz 1.10])**.** *Let A, B, and C be subgroups of a group G. If $B \leq \mathrm{N}_G(A) \cap \mathrm{N}_G(C)$, then $[AB, C] = [A, C][B, C]$.*

Lemma 2.7 ([10, Kapitel III, Hilfssatz 1.6])**.** *If A and B are two subgroups of G, then $[A, B] \trianglelefteq \langle A, B\rangle$.*

3 Triply factorised groups

There is an interesting connection between skew left braces and trifactorised groups as Sysak [19] shows. In fact, such groups can be used to answer some questions about skew left braces. In this section, we present some results on trifactorised groups that will be quite useful to proved our main results. Here as a *triply factorised grooup* or a *trifactorised group* we will understand a group G with three subgroups A, B, and K such that K is normal in G and $G = AB = AK = BK$. Some results of this section can also be found in [17, Section 3.2].

The next two lemmas were proved in [19] for skew braces of abelian type. They still hold in the general case and we include their proofs here for the sake of completeness.

Assume that a group $(C, \cdot)$ acts on a group $(K, +)$ via λ. Assume further that $\delta\colon C \longrightarrow K$ is a bijective derivation associated to λ.

We can consider the corresponding semidirect product $G = [K]C$. As in Doerk and Hawkes [6], we follow the compact notation $G = [K]C$ for the semidirect product, which is represented in other texts as $G = K \rtimes C$. The operation of this group will be denoted as a product. As usual, we identify K with the normal subgroup $\{(k, 1) \mid k \in K\} \leq G$ and C with the subgroup $\{(0, c) \mid c \in C\}$. Note that $(k_1, c_1)(k_2, c_2) = (k_1+\lambda_{c_1}(k_2), c_1c_2)$ for k_1, $k_2 \in K$, c_1, $c_2 \in C$. If needed, the passage from the multiplicative group C to the additive group K and vice versa will be done by means of the derivation $\delta\colon C \longrightarrow K$.

Let us consider $D = \{(\delta(c), c) \mid c \in C\}$. We will use the symbol 0 to denote the neutral element of $(K, +)$ and the symbol 1 to denote the neutral element of $(C, \cdot)$.

Lemma 3.1. *The set D is a subgroup of G such that $G = KD = DC$ and $K \cap D = D \cap C = \{(0, 1)\}$.*

Proof. Let c_1, $c_2 \in C$, then $(\delta(c_1), c_1)(\delta(c_2), c_2) = (\delta(c_1) + \lambda_{c_1}(\delta(c_2)), c_1c_2) = (\delta(c_1c_2), c_1c_2) \in D$. Let $c \in C$. Then $(\delta(c), c)^{-1} = (-\lambda_{c^{-1}}(\delta(c)), c^{-1}) = (\delta(c^{-1}), c^{-1}) \in D$ by Lemma 2.4. It follows that D is a subgroup of G.

Given $(k_1, c_2) \in G$, with $k_1 \in K$ and $c_2 \in C$, there exists $c_1 \in C$ such that $k_1 = \delta(c_1)$. Hence we have that $(k_1, c_2) = (k_1 - \delta(c_2) + \delta(c_2), c_2) = (k_1 - \delta(c_2), 1)(\delta(c_2), c_2) \in KD$ and $(k_1, c_2) = (\delta(c_1), c_1)(0, c_1^{-1}c_2) \in DC$. It follows that $G = KD = DC$. Finally, if $(\delta(c), c) = (0, c) \in D \cap C$, then $\delta(c) = 0$ and $c = 1$, and if $(\delta(c), c) = (k, 1) \in K \cap D$, then $c = 1$ and $k = \delta(c) = 0$ by Lemma 2.4. Consequently, $D \cap C = K \cap C = \{(0, 1)\}$. $\square$

The group G is an example of a triply factorised group or trifactorised group. Since $C \cong C/(K \cap C) \cong KC/K = G/K = KD/K \cong D/(D \cap K) \cong D$, we have that C and D are isomorphic groups. The map $\alpha\colon C \longrightarrow D$ given by $\alpha(c) = (\delta(c), c)$ for $c \in C$ is a group isomorphism.

Trifactorised groups of this form are also a source of bijective derivations.

We assume in the sequel that $(C, \cdot)$ acts on a group $(K, +)$ via λ, and $G = [K]C$ is the corresponding semidirect product.

Lemma 3.2. *Suppose that $G = [K]C = KD = DC$ is a trifactorised group such that $K \cap D = D \cap C = \{(0,1)\}$. Then there exists a bijective derivation $\delta\colon C \longrightarrow K$ associated with the action of C on K such that $D = \{(\delta(c), c) \mid c \in C\}$.*

Proof. Let $c \in C$. Since $G = KD$, there exists $k_1 \in K$ and $(k_2, c_2) \in D$ such that $(0, c) = (k_1, 1)(k_2, c_2) = (k_1 + k_2, c_2)$. It follows that $c_2 = c$ and so there exists an element of D whose second component is c. Suppose that (k_1, c), $(k_2, c) \in D$, with k_1, $k_2 \in K$, $c \in C$. Then $(k_1, c)(k_2, c)^{-1} = (k_1 - k_2, 1) \in K \cap D = \{(0,1)\}$, which implies that $k_1 - k_2 = 0$, that is, $k_1 = k_2$. Consequently there is a unique element $k \in K$ such that $(k, c) \in D$. This defines a map $\delta\colon C \longrightarrow K$ by letting $\delta(c)$ be the unique element $k \in K$ such that $(k, c) \in D$.

We must prove that δ is bijective. Suppose that $\delta(c_1) = \delta(c_2)$ with c_1, $c_2 \in C$. Then $(\delta(c_1), c_1)^{-1}(\delta(c_2), c_2) = (0, c_1^{-1}c_2) \in D \cap C = \{(0,1)\}$, which implies that $c_1^{-1}c_2 = 1$ and so $c_1 = c_2$. It follows that δ is injective. Now let $k_1 \in K$. Since $G = DC$, there exist $(\delta(c_3), c_3) \in D$ and $c_4 \in C$ such that $(k_1, 1) = (\delta(c_3), c_3)(0, c_4)$. In particular, $k_1 = \delta(c_3)$. This implies that δ is surjective and so δ is bijective and $D = \{(\delta(c), c) \mid c \in C\}$.

Let us prove now that δ is a derivation. Let c_1, $c_2 \in C$. Then

$$(\delta(c_1), c_1)(\delta(c_2), c_2) = (\delta(c_1) + \lambda_{c_1}(\delta(c_2)), c_1c_2) \in D.$$

This implies that $\delta(c_1c_2) = \delta(c_1) + \lambda_{c_1}(\delta(c_2))$ and so δ is a derivation with respect to the action λ. ◻

On some occasions, it is interesting to obtain the image of a subset of C or the preimage of a subset of K under the derivation δ. We compute them in the semidirect product G.

Lemma 3.3. *Let $G = [K]C = KD = DC$ with $K \cap D = D \cap C = \{(0,1)\}$.*

1. *If $L \subseteq K$, then $\delta^{-1}(L) = (-L)D \cap C$.*
2. *If $E \subseteq C$, then $\delta(E) = DE^{-1} \cap K$.*

Proof. 1. Let $c \in \delta^{-1}(L)$. Then $(0, c) = (-\delta(c), 1)(\delta(c), c) \in (-L)D \cap C$. Conversely, suppose that for some $l \in L$, and $c \in C$, $(-l, 1)(\delta(c), c) \in (-L)D \cap C$. Then $(-l + \delta(c), c) \in C$, which implies that $-l + \delta(c) = 0$, that is, $l = \delta(c)$, in particular, $c \in \delta^{-1}(L)$. It follows that $\delta^{-1}(L) = (-L)D \cap C$.

2. Suppose that $k \in \delta(E)$. Then there exists $c \in E$ such that $\delta(c) = k$. Hence $(k, 1) = (k, c)(0, c^{-1}) \in DE^{-1} \cap K$. Conversely, suppose that for some $e \in E$ and $c \in C$, $(k, 1) = (\delta(c), c)(0, e^{-1}) \in K$. Then $ce^{-1} = 1$ and so $c = e$, which implies that $k = \delta(e) \in \delta(E)$. We conclude that $\delta(E) = DE^{-1} \cap K$. □

In the following, we will avoid the usage of ordered pairs to refer to elements of the semidirect product $G = [K]C$. We will use the same sign for the operations of K, C, and G, so that, for k, $l \in K$ and $c \in C$, $(k+l, 1)$ will be written as kl, $(-k, 1)$ will be written as k^{-1}, and $(\lambda_c(k), 1)$ will correspond to the conjugation $ckc^{-1} = k^{c^{-1}}$ in the semidirect product G (as usual, u^g denotes $g^{-1}ug$). The neutral element of G will be denoted by 1 and, as it is usual in group theory, we will also denote with the symbol 1 the trivial subgroup.

The following results turn out to be crucial to study skew left braces by means of trifactorised groups.

Lemma 3.4. *Let $G = [K]C = KD = DC$ with $D \leq G$, $K \cap D = D \cap C = \{1\}$. Let k, $l \in K$ and c, $e \in C$. Then*

$$[kc, le] = [k, e]^c[k, l]^{ec}[c, l]^{c^{-1}ec}[c, e].$$

Proof. This can be easily checked by direct computation or by using twice [10, Kapitel III, Hilfssatz 1.2]. □

This formula gives interesting results when $k = \delta(c)$ and $l = \delta(e)$. We compute the commutators $[g, h] = g^{-1}h^{-1}gh$, for g, $h \in G$, in the semidirect product $G = [K]C$.

Lemma 3.5. *Let $G = [K]C = KD = DC$ with $D \leq G$, $K \cap D = D \cap C = \{1\}$, and let $\delta\colon C \longrightarrow K$ be the corresponding derivation. Let c, $e \in C$, $k = \delta(c)$, $l = \delta(e)$. Then*

$$\delta([c, e]) = [k, e]^c[k, l]^{ec}[c, l]^{c^{-1}ec}.$$

Proof. Note that kc, $le \in D$. By Lemma 3.4, we have that

$$[kc, le] = [k, e]^c[k, l]^{ec}[c, l]^{c^{-1}ec}[c, e] \in D,$$

since it is the commutator of two elements of D. But K is a normal subgroup of $G = [K]C$, this implies that $[k, e] \in [K, C] \subseteq K$, $[k, l] \in [K, K] \subseteq K$ and $[c, l] \in [C, K] = [K, C] \subseteq K$. Consequently, $[k, e]^c[k, l]^{ec}[c, l]^{c^{-1}ec} \in K$. As $[c, e] \in [C, C] \subseteq C$, the conclusion follows. □

Theorem 3.6. *Let $G = [K]C = KD = DC$ with $D \leq G$, $K \cap D = D \cap C = \{1\}$, and let $\delta\colon C \longrightarrow K$ be the corresponding derivation. Suppose that H is a subgroup of K such that H is normalised by C. Let c, $e \in C$, $k = \delta(c)$, $l = \delta(e)$. Suppose that three of the elements $[k, e]$, $[k, l]$, $[c, l]$ and $\delta([c, e])$ belong to H. Then so does the other one.*

Proof. Since H is normalised by C, we have that $[k, e] \in H$ if and only if $[k, e]^c \in H$, $[k, l] \in H$ if and only if $[k, l]^{ec} \in H$, and $[c, l] \in H$ if and only if $[c, l]^{c^{-1}ec} \in H$. The result then is an immediate consequence of Lemma 3.5. $\square$

Lemma 3.7. *Let $G = [K]C = KD = DC$ with $D \leq G$, $K \cap D = D \cap C = \{1\}$, and let $\delta\colon C \longrightarrow K$ be the corresponding derivation. Suppose that $E \leq C$ and that $L = \delta(E)$ is a normal subgroup of G. Then the following statements are equivalent:*

1. *E is a normal subgroup of C.*
2. *$[E, C] \subseteq E$.*
3. *$[K, E] \subseteq L$.*

Proof. The equivalence between $E \trianglelefteq C$ and $[E, C] \subseteq E$ is straightforward.

Let $c \in C$, $e \in E$, $k = \delta(c) \in K$, $l = \delta(e) \in L$. Since L is normal in G, we have that $[k, l] \in L$ and $[c, l] \in L$. By Theorem 3.6, we have that $[k, e] \in L$ if and only if $\delta([c, e]) \in L$, and this is equivalent to stating that $[c, e] \in E$. Therefore $[K, E] \subseteq L$ if and only if $[C, E] \subseteq E$. $\square$

Definition 3.8. Let $(B, +, \cdot)$ be a skew left brace. The 5-tuple (G, K, C, D, δ) is said to be a *trifactorised group* associated with B if $K = (B, +)$, $C = (B, \cdot)$, $G = [K]C = KD = DC$, $K \cap D = D \cap C = \{1\}$, $\delta\colon C \longrightarrow K$ is a bijective derivation associated with the action λ such that $D = \{\delta(c)c \mid c \in C\}$.

4 (Left) Ideals of skew left braces

As we noted in the introduction, skew left braces can be considered as a generalisation of radical rings (see, for instance Rump [16]). This motivated to apply ring theoretical methods to the study of skew left braces and inspired the definitions of left ideal and ideal in [9], and the definition of strong left ideal in [11] in a skew left brace $(B, +, \cdot)$. In this section, we will recall their definitions and we will interpret them in terms of its trifactorised group (G, K, C, D, δ). Each such ideal can be expressed as a subset of each of the groups K and C. We note that Sysak (see, for instance, [18]) characterised modules over radical rings in terms of trifactorised groups.

We first recall the definition of the $*$ operation, which plays an analogous role to the product in associative rings.

Let B be a skew left brace. Denote the operation $a * b = -a + ab - b = \lambda_a(b) - b$ for any a, $b \in B$. If a is regarded as an element of the multiplicative group C and b is regarded as an element of the additive group K, then this can be represented in the semidirect product $G = [K]C$ as $aba^{-1}b^{-1} = [a^{-1}, b^{-1}] \in [C, K] \subseteq K$ as C normalises K.

Given two subsets X and Y of B, we define $X * Y$ as the subgroup of K generated by $\{x * y \mid x \in X, y \in Y\}$. If we identify X as a subgroup E of C and Y as a subgroup H of K, this subgroup can be identified as $\langle\{[e^{-1}, h^{-1}] \mid e \in E, h \in H\}\rangle = [E, H] \leq K$.

Definition 4.1. A subgroup I of K is said to be a *left ideal* if $\lambda_a(I) \subseteq I$ for all $a \in B$, or equivalently, if $B * I$ is a subgroup of I. Moreover, I is called a *strong left ideal* if I is a normal subgroup of K.

Assume that I is a left ideal of B and suppose that I corresponds to the subgroup L of K in the semidirect product. Since L is C-invariant, by Lemma 2.5 we have that $E = \delta^{-1}(L)$ is a subgroup of C. Then we can consider the semidirect product $[L]E$, that admits a triple factorisation through the subgroup $[L]E \cap D$ and C normalises L or, equivalently, that $[L, C] \subseteq L$. If, moreover, I is a strong ideal of B, then L is a normal subgroup of K and so is in G, that is, $[L, G] \subseteq L$.

As it is shown in [11], strong left ideals have a considerable impact on the structure of the solution associated with the skew left brace.

Definition 4.2. An *ideal* of B is a left ideal I of B such that $aI = Ia$ and $a + I = I + a$ for all $a \in B$.

Ideals of skew left braces are true analogues of normal subgroups in groups and ideals in rings. In fact, if I is an ideal of B, we can construct the quotient skew left brace B/I as it was shown in [9].

Now we characterise ideals of a brace in terms of its trifactorised group (G, K, C, D, δ). Let I be an ideal of B and assume that I corresponds to the subgroup L of K and to the subgroup E of C, with $\delta(E) = L$. Since I is a strong left ideal, we have that L is normal in G. By Lemma 3.7, as E is normal in C, we have that $[K, E] \subseteq L$. Hence $[LE, G] = [L, G][E, G] = [L, G][E, KC] = [L, G][E, K][E, C] \subseteq LLE = LE$, which implies that LE is a normal subgroup of G.

We give now an extension of [13, Lemma 6] to skew left braces (see also [1, Proposition 5.2]).

Lemma 4.3. *Let B be a skew left brace. Suppose that S is a left ideal of B and I is an ideal of B. Then $I * S$ is a left ideal of B. Moreover, $I * B$ is an ideal of B.*

Proof. Suppose that the left ideal S corresponds to $E \leq C$ and $L = \delta(E) \leq K$, and that the ideal I corresponds to $F \leq C$ and $H = \delta(F) \leq K$. Then $I * S$ corresponds to $[L, F] \leq K$. It remains to show that $[L, F]$ is normalised by C, but this is true because L and F are normalised by C.

Now consider $I * B$, which corresponds to the subgroup $[K, F]$ of K. Note that $[K, F]$ is normalised by K and, since both K and F are normalised by C, we obtain that $[K, F] \trianglelefteq G$, that is, $I * B$ is a strong left ideal of B. Since $F \trianglelefteq C$, by Lemma 3.7 it follows that $[K, F] \leq H$, that is, $\delta^{-1}([K, F]) \leq F$. Therefore $[K, \delta^{-1}([K, F])] \leq [K, F]$ and, again by Lemma 3.7, we obtain that $\delta^{-1}([K, F]) \trianglelefteq C$. Hence $I * B$ is an ideal of B. $\square$

We have seen above that the star operation can be regarded as an analogue of a commutator of a group. In a similar way as nilpotency in groups can be defined in terms of iterated commutators, we can define nilpotency in skew left braces and some generalisations of nilpotency in terms of iterated star operations. We will use the notation of [13]. Let X, Y be two subsets of a brace B. Then we define inductively

$$\begin{aligned} &\mathrm{L}_0(X, Y) = Y; \qquad &&\mathrm{L}_n(X, Y) = X * \mathrm{L}_{n-1}(X, Y), \qquad &&n \geq 1; \\ &\mathrm{R}_0(X, Y) = X; \qquad &&\mathrm{R}_n(X, Y) = \mathrm{R}_{n-1}(X, Y) * Y, \qquad &&n \geq 1. \end{aligned}$$

When $X = Y = B$, we obtain that $\mathrm{L}_n(B, B)$ coincides with what Rump denoted as B^{n+1} in [16] and $\mathrm{R}_n(B, B)$ coincides with what Rump called $B^{(n+1)}$. By Lemma 4.3, $\mathrm{L}_n(B, B)$ is a left ideal of B and $\mathrm{R}_n(B, B)$ is an ideal of B.

We analyse the series $\mathrm{L}_n(X, Y)$ and $\mathrm{R}_n(X, Y)$ in terms of commutators of the semidirect product $G = [K]C$.

Let us start with the $\mathrm{L}_n(X, Y)$. Note that $\mathrm{L}_1(X, Y) = X * Y$. Suppose that X corresponds to the subgroup E of C and that Y corresponds to the subgroup H of K. Then $X * Y$ corresponds to $\langle\{[e^{-1}, h^{-1}] \mid e \in E, h \in H\}\rangle = [E, H] = [H, E] \leq K$. Now $\mathrm{L}_2(X, Y) = X * (X * Y)$ corresponds to $\langle\{[e^{-1}, t^{-1}] \mid e \in E, t \in [H, E]\}\rangle = [E, [H, E]] = [[H, E], E]$ and we can show by induction that $\mathrm{L}_n(X, Y)$ corresponds to $[\cdots[[H, E], E], \dots, E]$, where E appears n times.

Let us study now the terms $\mathrm{R}_n(X, Y)$. For $n = 1$, we have that $\mathrm{R}_1(X, Y) = X * Y$ can be identified with the subgroup $[E, H]$ of K if, as before, X corresponds to the subgroup E of C and Y corresponds to the subgroup H of K. However, in order to compute $\mathrm{R}_2(X, Y) = (X * Y) * Y$, we need to interpret

$X * Y$ as a subgroup of C, namely, $\delta^{-1}([H, E])$. Hence it seems convenient to identify $\mathrm{R}_1(X, Y)$ with the subgroup $\delta^{-1}([E, H])$. Then $\mathrm{R}_2(X, Y)$ can be interpreted as $\delta^{-1}([\delta^{-1}([E, H]), H])$. We can see by induction that $\mathrm{R}_n(X, Y)$ can be interpreted as $\delta^{-1}([\cdots[\delta^{-1}([E, H]), H], \dots, H])$, with exactly n commutators (and, for coherence, $\mathrm{R}_0(X, Y) = X$ should be identified with $E \leq C$). For short, we write $\rho_0(E, H) = E$, $\rho_n(E, H) = \delta^{-1}([\rho_{n-1}(E, H), H])$ for $n \geq 1$.

Following [16], we call a skew left brace B *right nilpotent* of class m if $\mathrm{R}_m(B, B) = 0$ and $\mathrm{R}_{m-1}(B, B) \neq 0$, and *left nilpotent* of class m if $\mathrm{L}_m(B, B) = 0$ and $\mathrm{L}_{m-1}(B, B) \neq 0$. The trivial brace $B = \{0\}$ is said to be left nilpotent of class 0 and right nilpotent of class 0.

5 Left and right π-nilpotency of finite skew left braces

Given a prime p, we say that a finite group G is p-nilpotent if G has a normal Hall p'-subgroup. Let π be a set of primes. As in [15], we say that a finite group G is π*-nilpotent* if it is p-nilpotent for each prime $p \in \pi$. If G is a π-soluble group, this is equivalent to stating that G has a normal Hall π'-subgroup and that G has a nilpotent Hall π-subgroup. It is known that the class of all finite π-nilpotent groups is a saturated formation [6].

In this section, we will present analogues of π-nilpotency of groups in the scope of skew left braces and characterise these properties in terms of the multiplicative group of the skew left brace. Our results extend the results of [13] and [1, Theorems 5.8 and 6.4] to a set of primes π with a totally different approach.

We will suppose in the sequel B is a finite skew left brace of nilpotent type. Let B_π be the Hall π-subgroup of $K = (B, +)$.

Definition 5.1. We say that B is *left* π*-nilpotent* if for some n we have that $L_n(B, B_\pi) = 0$.

Notation 5.2. Suppose that $G = [K]C$ is the semidirect product of K and C and that $\delta\colon C \longrightarrow K$ is a bijective derivation with respect to the action of C on K. If K is finite and K_π, for a set of primes π, is a C-invariant Hall π-subgroup of K, we denote by $C_\pi = \delta^{-1}(K_\pi)$. In this case, G_π denotes $[K_\pi]C_\pi$. Furthermore, if $\pi = \{p\}$, where p is a prime, we denote $K_{\{p\}}$ by K_p, $C_{\{p\}}$ by C_p and $G_{\{p\}}$ by G_p.

The following result about trifactorised groups will be crucial in the proof of Theorem 5.4. It is a consequence of Theorem 6.5.4 and the remarks after its proof in [2].

Theorem 5.3. *Let $\mathfrak{F}$ be a saturated formation of finite groups, and let the group $G = AB = AK = BK$ be the product of three subgroups A, B, and K, where K is normal in G. If A and G are $\mathfrak{F}$-groups and K is nilpotent, then G is an $\mathfrak{F}$-group.*

Our first result in this section extends [13, Theorem 14] and [1, Theorem 6.4] to a set of primes π.

Theorem 5.4. *Suppose that $C = (B, \cdot)$ has a nilpotent Hall π-subgroup. Then B is left π-nilpotent if and only if C is π-nilpotent.*

Proof. Assume that $L_n(B, B_\pi) = 0$ for some n. Let (G, K, C, D, δ) be the trifactorised group corresponding to B. Then $[K_\pi, C, C, \dots, C] = 1$. In particular, $[K_\pi, C_{\pi'}, \dots, C_{\pi'}] = 1$. Since $[K_\pi, C_{\pi'}] = [K_\pi, C_{\pi'}, C_{\pi'}]$ by [6, Chapter A, Proposition 12.4], we have that $[K_\pi, C_{\pi'}] = 1$. Let $c \in C_{\pi'}$ and $e \in C$. Call $k = \delta(c) \in K_{\pi'}$, $l = \delta(e) \in K$. Since $K_{\pi'}$ is a normal subgroup of G, $[k, e]$, $[k, l] \in K_{\pi'}$. Write $l = l_{\pi'} l_\pi$, with $l_{\pi'} \in K_{\pi'}$, $l_\pi \in K_\pi$, then $[c, l] = [c, l_\pi][c, l_{\pi'}]^{l_\pi} = [c, l_{\pi'}]^{l_\pi} \in K_{\pi'}$. By Lemma 3.7, we have that $C_{\pi'}$ is a normal subgroup of C. This implies that C is π-nilpotent.

Suppose that C is π-nilpotent. Since $G = [K]C$ is a trifactorised group, with K normal nilpotent and C and D π-nilpotent, we can apply Theorem 5.3 to conclude that G is π-nilpotent. Let $c \in C_\pi$ and $e \in C_{\pi'}$, and let $k = \delta(c) \in K_\pi$ and $l = \delta(e) \in K_{\pi'}$. Then $[k, e] \in K_\pi$, $[k, l] = 1$, $[c, l] \in K_{\pi'}$, and, since C is π-nilpotent, $[c, e] \in C_{\pi'}$, and so $\delta([c, e]) \in K_{\pi'}$. By Theorem 3.6, we have that $[k, e] \in K_{\pi'}$, and so $[k, e] = 1$. We conclude that $[K_\pi, C_{\pi'}] = 1$. Now $[K_\pi, C] = [K_\pi, C_\pi C_{\pi'}] = [K_\pi, C_\pi][K_\pi, C_{\pi'}] = [K_\pi, C_\pi]$. Suppose that $[K_\pi, \overbrace{C, \dots, C}^{n-1}] = [K_\pi, \overbrace{C_\pi, \dots, C_\pi}^{n-1}]$. Then

$$\begin{aligned}
[K_\pi, \overbrace{C, \dots, C}^{(n)}] &= [K_\pi, \overbrace{C_\pi, \dots, C_\pi}^{(n-1)}, C_\pi C_{\pi'}] \\
&= [K_\pi, \overbrace{C_\pi, \dots, C_\pi}^{(n)}][K_\pi, \overbrace{C_\pi, \dots, C_\pi}^{(n-1)}, C_{\pi'}] \\
&= [K_\pi, \overbrace{C_\pi, \dots, C_\pi}^{(n)}].
\end{aligned}$$

Since G is π-nilpotent, G_π is nilpotent and there exists an n such that $[\overbrace{G_\pi, \dots, G_\pi}^{(n+1)}] = 1$. Consequently $\mathrm{L}_n(B, B_\pi)$ can be identified with the subgroup of K

$$[K_\pi, \overbrace{C, \dots, C}^{(n)}] = [K_\pi, \overbrace{C_\pi, \dots, C_\pi}^{(n)}] \subseteq [\overbrace{G_\pi, \dots, G_\pi}^{(n+1)}] = 0.$$

This completes the proof. $\square$

Definition 5.5. We say that B is *right π-nilpotent* if for some n we have that $\mathrm{R}_n(B_\pi, B) = 0$.

We say that B is *nilpotent with respect to* π if the Hall π-subgroup $G_\pi = K_\pi C_\pi$ of a trifactorised group (G, K, C, D, δ) associated with B is nilpotent.

It is clear that every brace of nilpotent type is nilpotent with respect to p for all primes p.

Our second theorem is this section extends [13, Theorem 18] and [1, Theorem 5.8] to a set of primes π.

Theorem 5.6. *Suppose that B is nilpotent with respect to π and its multiplicative group C has an abelian normal Hall π-subgroup. Then B is right π-nilpotent.*

Proof. Let (G, K, C, D, δ) be the trifactorised group associated with B such that $G_\pi = K_\pi C_\pi$ is nilpotent. Let $e \in C_{\pi'}$, $l = \delta(e) \in K_{\pi'}$, $c \in C_\pi$, $k = \delta(c) \in K_\pi$. Note that $[l, k] = 1$ because K is nilpotent, $[k, e] \in K_\pi$ and $[c, e] \in C_\pi$ because C_π is a normal subgroup of C. By Theorem 3.6, $[l, c] \in K_\pi$. But $[l, c] \in K_{\pi'}$ and so $[l, c] = 1$. It follows that $[K, C_\pi] = [K_{\pi'}K_\pi, C_\pi] = [K_\pi, C_\pi]$.

Now consider $c \in \delta^{-1}([C_\pi, K]) = \delta^{-1}([C_\pi, K_\pi]) \subseteq C_\pi$, $k = \delta(c) \in [C_\pi, K_\pi]$, $e \in C_\pi$, $l = \delta(e) \in K_\pi$. Since C_π is abelian, $[c, e] = 1$. We have that $[k, e] \in [C_\pi, [C_\pi, K_\pi]] \subseteq [K_\pi, G_\pi, G_\pi]$ and $[k, l] \in [K_\pi, [C_\pi, K_\pi]] \subseteq [K_\pi, G_\pi, G_\pi]$. By Theorem 3.6, we obtain that $\delta([c, l]) \in [K_\pi, G_\pi, G_\pi]$. In particular, $[\delta^{-1}([C_\pi, K_\pi]), K_\pi] \subseteq \delta^{-1}([K_\pi, G_\pi, G_\pi])$.

Suppose, by induction, that $E_{(n)} = \delta^{-1}([\delta^{-1}([\dots \delta^{-1}([C_\pi, K_\pi]), \dots, K_\pi])])$, with n δ signs, is contained in $\delta^{-1}([K_\pi, \overbrace{G_\pi, \dots, G_\pi}^{(n)}])$. Take $e \in E_{(n)} \subseteq C_\pi$, $l = \delta(e) \in \delta(E_{(n)}) \subseteq [K_\pi, \overbrace{G_\pi, \dots, G_\pi}^{(n)}]$, $c \in C_\pi$, $k = \delta(c) \in K_\pi$. Then $[c, e] = 1$ since $E_{(n)} \subseteq C_p$, $[l, c] \in [K_p, \overbrace{G_p, \dots, G_p}^{(n)}, C_p] \subseteq [K_p, \overbrace{G_p, \dots, G_\pi}^{(n+1)}]$, $[k, l] \in [K_\pi, \overbrace{G_\pi, \dots, G_\pi}^{(n)}, K_\pi] \subseteq [K_\pi, \overbrace{G_\pi, \dots, G_\pi}^{(n+1)}]$. By Theorem 3.6, $[k, e] \in [K_\pi, \overbrace{G_\pi, \dots, G_\pi}^{(n+1)}]$. Since $[K_{\pi'}, C_\pi] = 1$, we have that $\delta(E_{(n+1)}) = [K, E_{(n)}] = [K_\pi, E_{(n)}] \subseteq [K_\pi, \overbrace{G_\pi, \dots, G_\pi}^{(n+1)}]$.

Since G_π is nilpotent, there exists an m such that $[G_\pi, \overbrace{G_\pi, \dots, G_\pi}^{(m)}] =$

1. Hence $E_{(n)} \subseteq \delta^{-1}([K_\pi, \overbrace{G_\pi, \ldots, G_\pi}^{(m)}]) = 1$, and so $\mathrm{R}_{m+1}(B_\pi, B) = 0$, as desired. $\square$

6 Factorisations. A Fitting-like ideal

Motivated by factorisations of groups, Jespers, Kubat, Van Antwerpen and Vendramin [11] have introduced factorizations of skew left braces.

Definition 6.1. Let $(B, +, \cdot)$ be a skew left brace and let B_1 and B_2 be left ideals of B. We say that B admits a *factorisation* through B_1 and B_2 if $B = B_1 + B_2$.

In [11], the authors study factorisations of skew left braces as a sum of two trivial braces. Some results of this paper are simply structural properties of a particular semidirect product.

Assume that we have a factorisation of the form $B = B_1 + B_2$, where B_1 and B_2 are left ideals. In terms of the trifactorised group (G, K, C, D, δ), this means that $K = K_1K_2$, $C = C_1C_2$, $\delta(C_1) = K_1$, $\delta(C_2) = K_2$.

Moreover, B_i is a trivial skew left subbrace of B if and only if $[K_i, C_i] = 1$, $i = 1, 2$.

Our first result provides some structural information about the factorised group associated with a factorised skew left brace.

Lemma 6.2. *Assume that in the trifactorised group (G, K, C, D, δ) we have that $K = K_1K_2$, $C = C_1C_2$, $\delta(C_1) = K_1$, $\delta(C_2) = K_2$, $[K_1, C_1] = [K_2, C_2] = 1$, $[K_1, C] \subseteq K_1$, $[K_2, C] \subseteq K_2$. Then:*

1. $[K_2, C_1] \trianglelefteq G$, $[K_1, C_2] \trianglelefteq G$.
2. $[[K_2, C_1], \delta^{-1}([K_2, C_1])] = 1$, $[[K_1, C_2], \delta^{-1}([K_1, C_2])] = 1$.
3. $[K, C] = [K_1, C_2][K_2, C_1] = [K_2, C_1][K_1, C_2]$.
4. $[C_1, C_2] \leq \mathrm{C}_C(K)$.

Proof. Let $k_1 \in K_1$, $k_2 \in K_2$, $c_1 \in C_1$, $c_2 \in C_2$.

1. We have that $[K_1, C_2]$ is normalised by C_2 and since $[K_1, C_2] \subseteq K_1$, $[[K_1, C_2], C_1] = 1$. Then C normalises $[K_1, C_2]$.

   Note that K_1 normalises $[K_1, C_2]$. Furthermore, as $K = K_1K_2$, there exist $\hat{k}_1 \in K_1$, $\hat{k}_2 \in K_2$ such that $k_2^{-1}k_1k_2 = \hat{k}_1\hat{k}_2$. Hence $[k_1, c_2]^{k_2} = [k_2^{-1}k_1k_2, c_2] = [\hat{k}_1\hat{k}_2, c_2] = [\hat{k}_1, c_2] \in [K_1, C_2]$ and so $[K_1, C_2]$ is also normalised by K_2. We conclude that $[K_1, C_2] \trianglelefteq G$.

   The proof for $[K_2, C_1]$ is similar.

2. Since $[K_1, C_2] \subseteq K_1$, we have that $\delta^{-1}([K_1, C_2]) \subseteq C_1$ and so we have that $[[K_1, C_2], \delta^{-1}([K_1, C_2])] = 1$. The proof for $[K_2, C_1]$ is analogous.

3. We have:

$$\begin{aligned} [k_1k_2, c_1c_2] &= [k_1k_2, c_2][k_1k_2, c_1]^{c_2} = [k_1, c_2]^{k_1}[k_2, c_2][k_1, c_1]^{k_1c_2}[k_2, c_1]^{c_2} \\ &= [k_1, c_2]^{k_1}[k_2, c_1]^{c_2} \in [K_1, C_2][K_2, C_1] \end{aligned}$$

since $[K_1, C_2]$, $[K_2, C_1] \trianglelefteq G$. It follows that $[K, C] \subseteq [K_1, C_2][K_2, C_1]$. Since $[K_1, C_2][K_2, C_1] \subseteq [K, C]$, the equality holds. The other equality follows in the same way since $K = K_2K_1$ and $C = C_2C_1$.

4. Let $k = k_1k_2 \in K$, with $k_1 \in K_1$, $k_2 \in K_2$, and let $c_1 \in C_1$, $c_2 \in C_2$. Then $(k_1k_2)^{c_1c_2} = k_1^{c_1c_2}k_2^{c_1c_2} = k_1^{c_2}k_2^{c_1}$ and $(k_1k_2)^{c_2c_1} = k_1^{c_2c_1}k_2^{c_1c_2} = k_1^{c_2}k_2^{c_1}$. It follows that $c_1c_2c_1^{-1}c_2^{-1} \in \mathrm{C}_C(K)$. □

Theorem 6.3. *Assume that in the trifactorised group (G, K, C, D, δ) we have that $K = K_1K_2$, $C = C_1C_2$, $\delta(C_1) = K_1$, $\delta(C_2) = K_2$, $[K_1, C_1] = [K_2, C_2] = 1$, $[K_2, C] \subseteq K_2$, $K_1 \trianglelefteq G$. Then $\delta^{-1}([K_2, C_1]) \trianglelefteq C$ and $[\delta^{-1}([K_2, C_1]), K] = 1$.*

Proof. By Lemma 6.2, $[K_2, C_1] \trianglelefteq G$.

Let $c_1 \in C_1$, $c_2 \in C_2$. Then

$$\begin{aligned} \delta(c_1c_2c_1^{-1}c_2^{-1}) &= \delta(c_1)\delta(c_2)^{c_1^{-1}}\left(\delta(c_1)^{c_1c_2^{-1}c_1^{-1}}\right)^{-1}\left((\delta(c_2)^{c_2c_1c_1^{-1}c_2^{-1}}\right)^{-1} \\ &= \delta(c_1)\delta(c_2)^{c_1^{-1}}\left(\delta(c_1)^{c_2^{-1}}\right)^{-1}\delta(c_2)^{-1}, \end{aligned}$$

As $K_1 \trianglelefteq G$, we have that there exists $\hat{k}_1 \in K_1$ such that $\delta(c_1c_2c_1^{-1}c_2^{-1}) = [c_1^{-1}, \delta(c_2)^{-1}]\hat{k}_1$, that is, $\delta(c_1c_2c_1^{-1}c_2^{-1})\hat{k}_1^{-1} = [c_1^{-1}, \delta(c_2)^{-1}]$. Let $\hat{c}_1 \in C_2$ such that $\hat{k}_1 = \delta(\hat{c}_1)$. By Lemma 6.2, $[C_1, C_2] \subseteq \mathrm{C}_C(K)$. Hence

$$\delta^{-1}([c_1^{-1}, \delta(c_2)^{-1}]) = (c_1c_2c_1^{-1}c_2^{-1})\hat{c}_1^{-1}.$$

Let $k_1 \in K_1$. Then $[\delta^{-1}([c_1^{-1}, \delta(c_2)^{-1}]), k_1] = [(c_1c_2c_2^{-1}c_2^{-1})\hat{c}_1^{-1}, k_1] = 1$. On the other hand, from $[c_1^{-1}, \delta(c_2)^{-1}] \in K_2$, we get that $\delta^{-1}([c_1^{-1}, \delta(c_2)^{-1}]) \in C_2$ and so $[\delta^{-1}([c_1^{-1}, \delta(c_2)^{-1}]), k_2] = 1$ for each $k_2 \in K_2$. Consequently, $[\delta^{-1}([K_2, C_1]), K] = 1 \subseteq [K_2, C_1]$, which is a normal subgroup of G by Lemma 6.2. We can apply Lemma 3.7 to conclude that $\delta^{-1}([K_2, C_1]) \trianglelefteq C$ and $[\delta^{-1}([K_2, C_1]), K] = 1$. □

Corollary 6.4 ([11, Theorem 3.9])**.** *Let B be a skew left brace. If $B = B_1 + B_2$ is a factorisation through left ideals B_1 and B_2 that are trivial as skew left braces and B_1 is a strong left ideal of B, then $B_1 * B_2$ is an ideal of B that acts trivially on B.*

Theorem 6.5. *Assume that in the trifactorised group (G, K, C, D, δ) we have that $K = K_1K_2$, $C = C_1C_2$, $\delta(C_1) = K_1$, $\delta(C_2) = K_2$, $[K_1, C_1] = [K_2, C_2] = 1$, $K_1 \trianglelefteq G$, $K_2 \trianglelefteq G$. Then $[\delta^{-1}([K, C]), K] = 1$. In particular, G has a normal trifactorised subgroup $[L]E$, with $L \trianglelefteq K$ and $E \trianglelefteq C$, such that $[L]E$ and $[K/L](C/E)$ satisfy $[L, E] = 1$ and $[K/L, C/E] = 1$.*

Proof. Since $K_1 \trianglelefteq G$ and $K_2 \trianglelefteq G$, we apply Theorem 6.3 to conclude that $[\delta^{-1}([K_2, C_1]), K] = 1$ and $[\delta^{-1}([K_1, C_2]), K] = 1$. Moreover, $\delta^{-1}([K_2, C_1]) \trianglelefteq C$ and $\delta^{-1}([K_1, C_2]) \trianglelefteq C$. Since $[K, C] = [K_1, C_2][K_2, C_1]$, $[K_1, C_2] \trianglelefteq G$, and $[K_2, C_1] \trianglelefteq G$ by Lemma 6.2, we have that

$$[\delta^{-1}([K, C]), K] = [\delta^{-1}([K_1, C_2])\delta^{-1}([K_2, C_1]), K] = 1.$$

Let $E = \delta^{-1}([K_1, C_2])\delta^{-1}([K_2, C_1])$. Then E is a normal subgroup of C by Theorem 6.3, and if $L = \delta(E) = [K_1, C_2][K_2, C_1]$, we have that $[L, E] = 1$. Since $[K, C] = L$, it follows that C/E acts trivially on K/L. □

Corollary 6.6 ([11, Theorem 3.5])**.** *Let B be a skew left brace. If $B = B_1 + B_2$ is a factorisation through strong left ideals B_1 and B_2 that are trivial as skew left braces, then B is right nilpotent of class at most three. In particular, B is meta-trivial, that is, it possesses an ideal I such that I and B/I are trivial.*

Theorem 6.7. *Assume that in the trifactorised group (G, K, C, D, δ) we have that $K = K_1K_2 \neq 1$, $C = C_1C_2$, $\delta(C_1) = K_1$, $\delta(C_2) = K_2$, $[K_1, C_1] = [K_2, C_2] = 1$, $K_1 \trianglelefteq G$, $[K_2, C] \subseteq K_2$. Then K_1 or K_2 contains a non-trivial normal subgroup L of G such that $\delta^{-1}(L) \trianglelefteq C$ and $[\delta^{-1}(L), K] = 1$.*

Proof. If $[K_2, C_1] \neq 1$, then by Theorem 6.3, $L = [K_2, C_1]$ satisfies that $\delta^{-1}([L) \trianglelefteq C$ and $[\delta^{-1}(L), K] = 1$ and so it satisfies the conditions. Suppose that $[K_2, C_1] = 1$. Then $[K, C] = [K_1, C_2]$ by Lemma 6.2. Since obviously $[\delta^{-1}([K, C]), K] \subseteq [C, K] = [K, C]$, by Lemma 3.7 it follows that $\delta^{-1}([K, C]) \trianglelefteq C$. If $[K, C] = [K_1, C_2] \neq 1$, then it is the desired subgroup. Otherwise $[K, C] = 1$ and then δ is an isomorphism, $K \cong C$ and C_1 and C_2 are normal subgroups of C. Then $[K_1]C_1$ and $[K_2]C_2$ are normal subgroups of G. □

Corollary 6.8 ([11, Theorem 3.9])**.** *Let B be a non-zero skew left brace that has a factorisation $B = B_1 + B_2$ through left ideals B_1 and B_2, where both are trivial as skew left braces. If B_1 is a strong left ideal of B, then B_1 or B_2 contains a non-zero ideal I of B that acts trivially on B.*

Theorem 6.9. *Assume that in the trifactorised group (G, K, C, D, δ) we have that $K = K_1K_2$, $C = C_1C_2$, $\delta(C_1) = K_1$, $\delta(C_2) = K_2$, $[K_1, C_1] = [K_2, C_2] = 1$, $K_1 \trianglelefteq G$, $[K_2, C] \subseteq K_2$. Then*

$$[\delta^{-1}([\delta^{-1}([K, C]), K]), K] = 1.$$

Proof. By Lemma 6.2 and Theorem 6.3, $[[K_2, C_1]](\delta^{-1}([K_2, C_1])$ is normal in G. Suppose first that $[K_2, C_1] = 1$. Then $[K, C_1] = [K_1K_2, C_1] = [K_1, C_1][K_2, C_1] = 1$. Moreover $[K, C] = [K_1, C_2] \leq K_1$ by Lemma 6.2. Consequently

$$[\delta^{-1}([K_1, C_2]), K] \leq [C_1, K] = 1.$$

Suppose now that $[K_2, C_1] \neq 1$. Then we can consider $\bar{G} = G/N$, where $N = [K_2, C_1](\delta^{-1}([K_2, C_1]) \trianglelefteq G$. We use the bar notation to denote quotients by N. Then $\bar{G}$ admits a triple factorisation through $\bar{K}$, $\bar{C}$ and $\bar{D}$, $\bar{K}_1 \trianglelefteq \bar{G}$, and $[\bar{K}_2, \bar{C}_1] = 1$. By the previous case, $[\bar{\delta}^{-1}([\bar{K}, \bar{C}]), \bar{K}] = [\bar{\delta}^{-1}([\bar{K}_1, \bar{C}_2]), \bar{K}] = 1$, that is, $[\delta^{-1}([K, C]), K] \leq [K_2, C_1]$. By Theorem 6.3, $[\delta^{-1}([K_2, C_1]), K] = 1$. Therefore $[\delta^{-1}([\delta^{-1}([K, C]), K]), K] = 1$. □

Corollary 6.10 ([11, Corollary 3.11])**.** *Let $B = B_1 + B_2$ be a skew left brace with a factorsation through its left ideals B_1 and B_2, which are trivial as skew left braces. If B_1 is a strong left ideal of B, then B is right nilpotent of class at most four.*

A well-known theorem of Fitting states that the product of two nilpotent normal subgroups of a finite group is nilpotent. It follows that every finite group has a largest normal nilpotent subgroup, known as its *Fitting subgroup.*

One natural question is then whether a brace that can be factorised as a sum of two right nilpotent (respectively, left nilpotent) left ideals or strong left ideals is right nilpotent (respectively, left nilpotent). However, it is pointed out in [11, Example 3.15] that the answer is negative for left ideals or strong left ideals.

Example 6.11. The brace `SmallBrace(72, 475)` of the YangBaxter library [20] of GAP [8] is a product of the strong left ideals corresponding to the Sylow 2-subgroup and the Sylow 3-subgroup of its additive group. Both are class 2 right nilpotent braces. Furthermore, both left ideals are left nilpotent. However, this brace is simple and so it is not right nilpotent nor left nilpotent.

Since ideals of skew braces correspond to normal subgroups every trifactorised group associated with B, a natural candidate for a Fitting-like theorem could be obtained by considering ideals. For left nilpotency and left braces we have the following positive theorem.

Theorem 6.12. *Suppose that a skew left brace of abelian type B can be descomposed as the sum of two ideals that are left nilpotent as left braces. Then B is left nilpotent.*

The proof of Theorem 6.12 depends on the following result, that will be used without further reference.

Lemma 6.13. *Suppose that K_1 and K_2 are subgroups of an abelian subgroup K of a group G and that E is a subgroup of G such that E normalises K_1 and K_2. Then $[K_1K_2, E] = [K_1, E][K_2, E]$.*

Proof. Let $k_1 \in K_1$, $k_2 \in K_2$, $e \in E$. Then $[k_1k_2, e] = [k_1, e]^{k_2}[k_2, e] = [k_1, e][k_2, e]$ because $[k_1, e] \in K_1$ and $k_2 \in K_2$ centralises K_1. $\square$

Proof of Theorem 6.12. Let (G, K, C, D, δ) be the trifactorised group associated with B, let C_1 and C_2 be the subgroups of C corresponding to both ideals and let $K_1 = \delta(C_1)$ and $K_2 = \delta(C_2)$. Then $K_1 \trianglelefteq G$, $K_2 \trianglelefteq G$, $C_1 \trianglelefteq C$ and $C_2 \trianglelefteq C$, $K_1C_1 \trianglelefteq G$ and $K_2C_2 \trianglelefteq G$. Note that $K = K_1K_2$. Furthermore, as C_2 normalises C_1 and K_1K_2, we have

$$\begin{aligned}[K, C] &= [K_1K_2, C_1C_2] = [K_1K_2, C_1][K_1K_2, C_2]\\ &= [K_1, C_1][K_2, C_1][K_1, C_2][K_2, C_2].\end{aligned}$$

Since each subgroup of each commutator is normalised by C_1 and by C_2, all commutators are normalised by C. By induction, we can prove that

$$[K, \overbrace{C, \dots, C}^{(r-1)}] = \prod_{\substack{i_j \in \{1,2\}\\ 1 \le j \le r}} [K_{i_1}, C_{i_2}, \dots, C_{i_r}] \tag{1}$$

Now we prove that

$$[K_{i_1}, C_{i_2}, \dots, C_{i_r}] \le \overbrace{[K_1, C_1, \dots, C_1]}^{(v_1)} \cap \overbrace{[K_2, C_2, \dots, C_2]}^{(v_2)}, \tag{2}$$

where

$$v_u = |\{j \in \{1, \dots, r\} : i_j = u\}|, \quad u \in \{1, 2\},$$

and a commutator with zero terms is understood to be equal to G. We argue by induction on r. For one subgroup in the commutator, the result is obvious. Assume that the result is true for a certain value of r, that is, that Equation (2) holds. Then

$$[K_{i_1}, C_{i_2}, \dots, C_{i_r}, C_{i_{r+1}}] \le [\overbrace{K_1, C_1, \dots, C_1}^{(v_1)}, C_{i_{r+1}}] \cap [\overbrace{K_2, C_2, \dots, C_2}^{(v_2)}, C_{i_{r+1}}].$$

Call $j = i_{r+1}$. Now C_j normalises $[\overbrace{K_{3-j}, C_{3-j}, \dots, C_{3-j}}^{(v_j)}]$ and so

$$[\overbrace{K_{3-j}, C_{3-j}, \dots, C_{3-j}}^{(v_{3-j})}, C_j] \leq [\overbrace{K_{3-j}, C_{3-j}, \dots, C_{3-j}}^{(v_{3-j})}],$$

and, clearly,

$$[\overbrace{K_j, C_j, \dots, C_j}^{(v_j)}, C_j] = [\overbrace{K_j, C_j, \dots, C_j}^{(v_j+1)}].$$

It follows that Equation (2) holds for all r.

Now assume that $[\overbrace{K_1, C_1, \dots, C_1}^{(w_1)}] = 1$ and $[\overbrace{K_2, C_2, \dots, C_2}^{(w_2)}] = 1$. Take $r = w_1 + w_2 - 1$. In every decomposition $r = v_1 + v_2$, we have that $v_1 \geq w_1$ or $v_2 \geq w_2$. By Equation (2), all commutators of the form $[K_{i_1}, C_{i_2}, \dots, C_{i_r}]$ are trivial. By Equation (1), $[K, \overbrace{C, \dots, C}^{(r-1)}] = 1$. We conclude that B is left nilpotent. $\square$

Theorem 6.12 allows us to define a Fitting-like ideal for every finite left brace.

Definition 6.14. Given a finite left brace B, the *left-Fitting ideal* $\mathrm{lF}(B)$ of B is the largest ideal that, as a left brace, is left nilpotent. It coincides with the ideal generated by all ideals of B that, as left braces, are left nilpotent.

We have not been able to prove or disprove that a brace generated by two ideals that are right nilpotent as left braces is right nilpotent. However, we have a positive answer when one of the ideals is trivial as a left brace. This is a consequence of the following slightly more general result.

Theorem 6.15. *Let B be a skew left brace of abelian type that can be factorised as the product of an ideal I_1 that is trivial as a left brace and a strong left ideal I_2 that is right nilpotent as a left brace. Then B is right nilpotent.*

Proof. Let (G, K, C, D, δ) be the trifactorised group associated with B, let C_1 be the subgroup of C corresponding to I_1, C_2 the subgroup of C corresponding to I_2, $K_1 = \delta(C_1)$ and $K_2 = \delta(C_2)$. Then $K_1C_1 \trianglelefteq G$ and $K_2 \trianglelefteq G$. Moreover, $K_1 \trianglelefteq G$, $C_1 \trianglelefteq C$, $K = K_1K_2$, and $C = C_1C_2$. There exists an r such that $\rho_r(C_2, K_2) = 1$. Therefore

$$\begin{aligned}\rho_r(C, K)/C_1 = \rho_r(C_2C_1, K_2K_1)/C_1 &= \rho_r(C_2C_1/C_1, K_2K_1/K_1) \\ &= \rho_r(C_2, K_2)C_1/C_1 = 1,\end{aligned}$$

which implies that $\rho_r(C, K) \le C_1$. Now

$$\begin{aligned}\rho_{r+1}(C, K) &= \delta^{-1}([\rho_r(C, K), K]) \\ &\le \delta^{-1}([C_1, K]) = \delta^{-1}([C_1, K_1][C_1, K_2]) = \delta^{-1}([C_1, K_2]).\end{aligned}$$

Since $[C_1, K_2] \le [K_1C_1, K_2] \le K_1C_1 \cap K_2 = K_1 \cap K_2$, we conclude that $\rho_{r+1}(C, K) \le C_1 \cap C_2$.

Let $t \ge 1$. We prove by induction on t that $\rho_{r+t}(C, K) \le C_1 \cap \rho_t(C_2, K_2)$. For $t = 1$, the result is clear, since

$$\rho_{r+1}(C, K) \le C_1 \cap C_2 \le C_1 \cap \rho_1(C_2, K_2).$$

Assume that $\rho_{r+t}(C, K) \le C_1 \cap \rho_t(C_2, K_2)$. Then

$$\begin{aligned}\rho_{r+t+1}(C, K) &= \delta^{-1}([\rho_{r+t}(C, K), K]) \le \delta^{-1}([C_1 \cap \rho_t(C_2, K_2), K_1K_2]) \\ &= \delta^{-1}([C_1 \cap \rho_t(C_2, K_2), K_1][C_1 \cap \rho_t(C_2, K_2), K_2]) \\ &= \delta^{-1}([C_1 \cap \rho_t(C_2, K_2), K_2]) \\ &\le \delta^{-1}([C_1, K_2] \cap [\rho_t(C_2, K_2), K_2]) \\ &\le \delta^{-1}(K_1 \cap [\rho_t(C_2, K_2), K_2]) = \delta^{-1}(K_1) \cap \delta^{-1}([\rho_t(C_2, K_2), K_2]) \\ &= C_1 \cap \rho_{t+1}(C_2, K_2).\end{aligned}$$

We conclude that $\rho_{2r}(C, K) \le C_1 \cap \rho_r(C_2, K_2) = 1$. Therefore, the brace is right nilpotent. □

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