Large characteristically simple sections of finite groups

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Abstract

In this paper we prove that if G is a group for which there are k non-Frattini chief factors isomorphic to a characteristically simple group A, then G has a normal section C/R that is the direct product of k minimal normal subgroups of G/R isomorphic to A. This is a significant extension of the notion of crown for isomorphic chief factors.

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1 Introduction and statement of results

All groups considered in this paper will be finite.

Gaschütz [Gas62] introduced the notion of crown associated with a complemented chief factor of a soluble group G. Given a G-module A, he discovered an important section of G, called the A-*crown* of G, which is a completely reducible and homogeneous G-module and the length of its G-composition series is the number of complemented chief factors of G which are G-isomorphic to A in a given chief series of G. These crowns turn out

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to be complemented sections of G. Gaschütz applied his significant result to construct a characteristic conjugacy class of subgroups in every soluble group: the *prefrattini* subgroups. Later Hawkes [Haw73] used this notion to define a closure operation for Schunck classes of finite soluble groups.

Lafuente [Laf84] defined the crown associated with a non-Frattini chief factors of an arbitrary group G. His approach depends on a equivalence relation in the set of all chief factors of G, called G-*connection*, which is a natural extension of G-isomorphism. We say that two chief factors of G are G-*connected* (or G-*equivalent*) when they are G-isomorphic or there exists a normal subgroup N of G such that G/N is a primitive group of type 3 whose minimal normal subgroups are G-isomorphic to the given chief factors. It is clear that if two chief factors of G that are G-connected and non-G-isomorphic, then they are non-abelian and there is a primitive epimorphic image of G connecting them. Lafuente discovered the existence of some sections associated with the non-abelian chief factors with similar properties to Gaschütz's crowns (see [Laf84, Laf85a, Laf85b]), and he also used this notion to defined in [Laf84] a new closure operation of Schunck classes of arbitrary groups which allows us to discover new relations between Schunck classes and saturated formations. Later, Förster [För88] used the crowns to give an alternative appoach of the generalised Jordan-Hölder theorem, and Ezquerro and the first author [BBE91] used them to introduce the prefrattini subgroups in every group.

Crowns are also important in probabilistic group theory. Hall [Hal36] gave a formula for the probability $P_G(t)$ that t elements taken at random from a group G with a uniform probability distribution generate G and Gaschütz [Gas59] developed a formula for the conditional probability $P_{G,N}(t)$ that a t-tuple generates G modulo a normal subgroup N, given that the corresponding elements of the quotient group G/N generate G/N. The concept of crown has become crucial in the work of Detomi and Lucchini [DL03] to obtain factorisations of $P_G(t)$. Given a monolithic primitive group L with a unique minimal normal subgroup A, for each positive integer k we can consider the direct product L^k of k copies of L and its subgroup

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \pmod{A}\},$$

called in [DVL98] the *kth crown-based power* of L. The crown-based powers play a key role in [DVL98] to understand the groups that need more generators than their proper quotients.

There is a close relation between crowns and crown-based powers. In order to state it, we need to recall the definition of the primitive group associated with a chief factor.

Definition 1.1 (see [BBE06, Definition 1.2.9]). Given a chief factor H/K of a group G, the *primitive group* $[H/K] * G$ *associated with* H/K *in* G is the semidirect product $[H/K]\big(G/\mathrm{C}_G(H/K)\big)$ if H/K is abelian and the quotient group $G/\mathrm{C}_G(H/K)$ if H/K is non-abelian.

Theorem 1.2 ([DL03, Proposition 9]). *Let H/K be a non-Frattini chief factor of a finite group G and let C/R be its crown. Then G/R is isomorphic to a crown-based power L_k, where $L = [H/K] * G$ and k is the number of chief factors of G that are G-related to H/K in a given chief series of G.*

The following result shows the relation between crowns and crown-based powers.

Theorem 1.3 ([DL03, Proposition 9]). *Let H/K be a non-Frattini chief factor of a finite group G and let C/R be its crown. Then G/R is isomorphic to a crown-based power L_k, where $L = [H/K] * G$ and k is the number of chief factors of G that are G-related to H/K in a given chief series of G.*

The main aim of this paper is to obtain an extension of the notion of crown for isomorphic chief factors, not necessarily related by G-connectedness. Our result will establish a relation between the number of non-Frattini chief factors isomorphic to a characteristically simple group A in a given chief series and the A-rank $\mathrm{r}_A(G)$, defined as the largest number k such that G has a normal section that is the direct product of k non-Frattini chief factors of G that are isomorphic to A.

Theorem A. *Let A be a non-Frattini chief factor of a group G and suppose that in a given chief series of G there are k non-Frattini chief factors isomorphic to A. Then there exist two normal subgroups C and R of G such that $R \leq C$ and C/R is isomorphic to a direct product of k minimal normal subgroups of G/R isomorphic to A.*

The proof of this result depends on the following property of monolithic primitive groups.

Theorem B. *If G is a primitive group with a unique minimal normal subgroup B, then G/B has no chief factors isomorphic to B.*

We thank one of the anonymous referees for informing us of a result (see Theorem 3.3 below) that has allowed us to present Theorem B in its broadest generality.

We bring the paper to a close with a consequence of Theorem A.

Corollary 1.4. *If a group G has k non-Frattini chief factors in a given chief series isomorphic to a characteristically simple group A, then $\mathrm{r}_A(G) = k$.*

2 Preliminaries

In this short section, we recall the definition of the precrowns and the crown associated with a non-Frattini chief factor and their main properties.

Definition 2.1. Let H/K be a supplemented chief factor of a group G. Assume that M is a maximal subgroup of G supplementing H/K in G such that G/M_G is a monolithic primitive group. We say that the chief factor $\mathrm{Soc}(G/M_G) = HM_G/M_G$ is the *precrown* of G associated with M and H/K, or simply a *precrown* of G associated with H/K.

Note that if C/R is the precrown of G associated with the maximal subgroup M and the supplemented chief factor H/K of G, then G/R is the primitive quotient group G/M_G of G associated with M.

Definition 2.2. Let H/K be a non-Frattini chief factor of a group G. Let $\mathcal{E}$ denote the set of all cores M_G of all maximal subgroups M of G such that the quotient G/M_G is a monolithic primitive group and M supplements chief factors G-connected to H/K, let

$$R = \bigcap\{N \mid N \in \mathcal{E}\},$$

and let $C^* = \mathrm{C}^*_G(H/K)$. We say that the factor C^*/R is the *crown* of G associated with H/K.

Here $\mathrm{C}^*_G(H/K) = H\mathrm{C}_G(H/K)$ denotes the *inneriser* of a chief factor H/K of G. The crown associated with a supplemented chief factor H/K of G possesses the following properties.

Theorem 2.3 (see [BBE06, Theorem 1.3.2])**.** *Let C^*/R be the crown of G associated to the supplemented chief factor H/K. Then $C^*/R = \mathrm{Soc}(G/R)$. Furthermore,*

1. *every minimal normal subgroup of G/R is a supplemented chief factor of G which is G-connected to H/K, and*
2. *no supplemented chief factor of G over C^* or below R is G-connected to H/K.*

Unless otherwise stated, we will follow the notation of the books [DH92] and [BBE06]. Detailed information about primitive groups and chief factors, crowns, and precrowns of a group can be found in [BBE06, Chapter 1].

3 Proofs of the theorems

We begin by proving Theorem B. In order to show that in a primitive group of type 2 the only chief factor isomorphic to the socle is the socle itself, we use the following elementary result.

Lemma 3.1. *Let n be a natural number. Then 2^n does not divide $n!$.*

Proof. Suppose that the result is false. Note that 2^1 does not divide $1!$. Suppose that n is the smallest natural number such that 2^n divides $n!$. We have that $n > 1$ and that n must be even. Hence 2^n divides the product

$$2 \cdot 4 \cdot 6 \cdot 8 \cdots n = 2^{n/2}(n/2)!.$$

Therefore $2^{n/2}$ divides $(n/2)!$. This contradicts the minimality of n. $\square$

Theorem 3.2. *Let G be a monolithic primitive group in which $B = \mathrm{Soc}(G)$ is non-abelian. Then G/B has no chief factors isomorphic to B.*

Proof. Let $B = S_1 \times \cdots \times S_n$ be the decomposition of B as a product of isomorphic non-abelian simple groups $S_i \cong S$, $1 \leq i \leq n$. Let $Y = \bigcap_{i=1}^{n} \mathrm{N}_G(S_i)$. By [BBE06, Remarks 1.1.40 (13)], G is isomorphic to a subgroup of $X \wr P_n$, which is in turn isomorphic to a subgroup of $\mathrm{Aut}(S) \wr \mathrm{Sym}(n) \cong \mathrm{Aut}(S^n)$, where P_n is a transitive subgroup of the symmetric group $\mathrm{Sym}(n)$ and $X = \mathrm{N}_G(S_1)/\mathrm{C}_G(S_1)$ is isomorphic to a subgroup or $\mathrm{Aut}(S)$ containing the inner automorphism group. Hence we can assume that G is in fact a subgroup of $W = \mathrm{Aut}(S) \wr \mathrm{Sym}(n)$. Let M be the intersection of G with the base group of W, then $M/B = (G \cap W^\natural)/B$ is isomorphic to a subgroup of $\mathrm{Aut}(S)^\natural/B \cong \big(\mathrm{Out}\, S\big)^n$, which is a soluble group by the Schreier conjecture, whose validity has been checked with the classification of finite simple groups (see [KS04, page 151]).

Assume now that there exists a chief factor F of G/B such that $F \cong B$. Since M/B is soluble, there exist normal subgroups N, K of G such that $M \leq N \leq K$ and $K/N \cong B$. In particular,

$$G/M = G/\Big(G \cap \big(\mathrm{Aut}(S)\big)^\natural\Big) \cong G\big(\mathrm{Aut}(S)\big)^\natural/\big(\mathrm{Aut}(S)\big)^\natural,$$

which is a subgroup of $P_n \leq \mathrm{Sym}(n)$. It follows that the order of G/M divides $n!$ and, in particular, $|K/N|$ divides $n!$. Since $K/N \cong S_1 \times \cdots \times S_n$ and all non-abelian simple groups have order divisible by 2, 2^n divides $|K/N|$. Consequently, 2^n divides $n!$. This contradicts Lemma 3.1. $\square$

Now let us consider non-Frattini abelian chief factors. Recall that a Fermat prime is a prime of the form $2^{2^n} + 1$ for some $n \geq 0$. We apply the following consequence of a result of Giudici, Glasby, Li, and Verret [GGLV17, Theorem 1]. We thank one of the referees for drawing our attention to this beautiful result. Without it, we would not have been able to prove Theorem B in its current form and the proofs would have been longer.

Theorem 3.3. *If G is a primitive group with a unique minimal normal subgroup of order $q = p^d$, where p is a prime, then the number of composition factors of G of order p is at most $d + \frac{\varepsilon_p d - 1}{p-1}$, where*

$$\varepsilon_p = \begin{cases} \frac{p}{p-1} & \text{if } p \text{ is a Fermat prime,} \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 3.4. *Let G be a prime and let G be a primitive group in which $B = \mathrm{Soc}(G)$ is an elementary abelian p-group of order p^d. Then G/B has no chief factors isomorphic to B.*

Proof. By Theorem 3.3, the number of composition factors of order p of G is bounded by

$$d + \frac{\varepsilon_p d - 1}{p-1} \leq \begin{cases} d + \frac{d-1}{2-1} = 2d - 1 < 2d, & \text{if } p = 2, \\ d + \frac{(3/2)d - 1}{p-1} < 2d, & \text{if } p \geq 3. \end{cases}$$

Hence the number of composition factors of order p of G/B is less than d. If G/B had a chief factor of order p^d, then by refining the corresponding chief series to a composition series we would obtain that the number of composition factors of order p of G/B is at least d. This contradiction shows that G/B does not have chief factors isomorphic to B. □

We are now in a position to prove Theorem A.

Proof of Theorem A. We will construct a sequence

$$(C_0, R_0), (C_1, R_1), (C_2, R_2), \dots, (C_l, R_l)$$

of pairs of subgroups of G satisfying the following conditions:

1. C_i and R_i are normal subgroups of G with $R_i \leq C_i$, $1 \leq i \leq l$, and $C_0 = R_0 = G$.

2. $C_i/R_i = (N_{i,1}/R_i) \times \cdots \times (N_{i,i}/R_i)$, where $N_{i,j}/R_i$ is a non-Frattini minimal normal subgroup of G/R_i isomorphic to A.

3. $C_i = \bigcap_{j=1}^{i} \mathrm{C}^*_G(N_{i,j}/R_i)$.

This will be done by induction on i. For $i = 0$, we construct $C_0 = R_0 = G$. Assume that for some $i \geq 0$ we have constructed (C_i, R_i) satisfying the previous conditions. Consider a chief series of G passing through R_i and suppose that in this series there exists a non-Frattini chief factor H/K of G isomorphic to A such that $H \leq R_i$. Let C/R be a precrown associated to H/K. Let $R_{i+1} = R_i \cap R$. Note that $C = \mathrm{C}^*_G(H/K) = \mathrm{C}^*_G(C/R)$. Let $C_{i+1} = C_i \cap C$. The unique minimal normal subgroup of G/R is C/R. Since $HR = C$, H is not contained in R and so R_i is not contained in R. Since G/R is monolithic, we have that

$$R < C \leq R_i R = R_i C.$$

Therefore

$$\begin{aligned}(C \cap R_i)/(R \cap R_i) &= (C \cap R_i)/(C \cap R_i \cap R) \\ &\cong_G R(C \cap R_i)/R = (C \cap RR_i)/R = C/R.\end{aligned}$$

Consequently $(C \cap R_i)/R_{i+1}$ is a normal subgroup of G/R_{i+1} that is G-isomorphic to C/R. Consider now $1 \leq j \leq i$, then

$$N_{i,j}R/RR_i = N_{i,j}RR_i/RR_i \cong_G N_{i,j}/(N_{i,j} \cap RR_i).$$

By the minimality of $N_{i,j}/R_i$, we have that either $N_{i,j} = N_{i,j} \cap RR_i$ or $N_{i,j} \cap RR_i = R_i$. Assume that the second case holds for a given j with $1 \leq j \leq i$. Hence $(N_{i,j}/R_i) \cap (RR_i/R_i) = 1$ and so $R < C \leq RR_i \leq \mathrm{C}_G(N_{i,j}/R_i) \leq \mathrm{C}^*_G(N_{i,j}/R_i)$. But now in a chief series of $G/\mathrm{C}^*_G(N_{i,j}/R_i)$ there are no chief factors isomorphic to A. Moreover, $C/R \cong A$ and $RR_i < RN_{i,j} \leq \mathrm{C}^*_G(N_{i,j}/R_i)$ with $RN_{i,j}/RR_i \cong A$. It follows that there are at least two chief factors isomorpic to A in a chief series of the primitive group associated with C/R, which is isomorphic to the primitive group associated with H/K. This contradicts Theorem B. Consequently, for all j with $1 \leq j \leq i$, the condition $N_{i,j} = N_{i,j} \cap RR_i$ holds, that is, $N_{i,j} \leq RR_i$. Therefore, $C_i \leq RR_i$. Now

$$\begin{aligned}(C_i \cap R)/(R_i \cap R) &= (C_i \cap R)/(R_1 \cap C_i \cap R) \\ &\cong_G R_1(C_i \cap R)/R_1 = (C_i \cap RR_i)/R_i = C_i/R_i.\end{aligned}$$

Moreover, $(C_i \cap R) \cap (C \cap R_i) = R \cap R_i = R_{i+1}$. It follows that G/R_{i+1} has a normal subgroup $N_{i+1,i+1}/R_{i+1} = (C \cap R_i)/R_{i+1}$ G-isomorphic to C/R, which is in turn G-isomorphic to H/K, and another normal subgroup $(C_i \cap R)/R_{i+1}$ which is G-isomorphic to C_i/R_i. Hence $(C_i \cap R)/R_{i+1}$ is isomorphic to a direct product of minimal normal subgroups $N_{i+1,j}/R_{i+1}$ of G/R_{i+1} G-isomorphic, respectively, to $N_{i,j}/R_i$, $1 \leq j \leq i$. By construction,

$$C_{i+1} = C_i \cap C = \bigcap_{j=1}^{i} \mathrm{C}^*_G(N_{i,j}/R_i) \cap \mathrm{C}^*_G(H/K) = \bigcap_{j=1}^{i+1} \mathrm{C}^*_G(N_{i+1,j}/R_{i+1}).$$

This construction can be done until we reach an $i = l$ such that there are no non-Frattini chief factors isomorphic to A below R_l. Note that, in this case, $l = k$, because all primitive groups associated to non-Frattini chief factors isomorphic to A have no chief factors isomorphic to A and, hence, $G/\mathrm{C}^*_G(H/K)$ has no chief factor isomorhic to A, consequently, there are no chief factor of G isomorphic to A that could appear above C_i. □

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4. The normal section C/R obtained in Theorem A is the product of k minimal normal subgroups of G/R isomorphic to A, hence $k \leq \mathrm{r}_A(G)$. Since obviously $\mathrm{r}_A(G) \leq k$, we obtain the result. □

Example 3.5. Let S be a non-abelian simple group, then the group $H = \mathrm{Inn}(S) \cong S$ of inner automorphisms of S acts on S and we can consider the corresponding semidirect product $G = [S]H$. As a consequence of Theorem A we can obtain the well-known fact that G is isomorphic to the direct product $S \times S$ of two copies of S.

Example 3.6. Consider the group

$$G = \langle (1, 2, 3), (1, 4, 5), (1, 2), (6, 7, 8), (6, 9, 10), (6, 7) \rangle$$

isomorphic to the direct product of two copies of the symmetric group of degree 5 acting naturally on the sets $\{1, 2, 3, 4, 5\}$ and $\{6, 7, 8, 9, 10\}$. Then G has two crowns corresponding to chief factors isomorphic to the alternating group $\mathrm{Alt}(5)$ of degree 5, namely C_1/R_1 with

$$C_1 = \langle (1, 2, 3), (1, 4, 5), (6, 7, 8), (6, 9, 10), (6, 7) \rangle$$

and

$$R_1 = \langle (6, 7, 8), (6, 9, 10), (6, 7) \rangle$$

and C_2/R_2 with

$$C_2 = \langle (1, 2, 3), (1, 4, 5), (1, 2), (6, 7, 8), (6, 9, 10) \rangle$$

and

$$R_2 = \langle (1, 2, 3), (1, 4, 5), (1, 2) \rangle.$$

We note that the chief factors C_1/R_1 and C_2/R_2 correspond to different crowns because their innerisers, C_1 and C_2, respectively, are different. In this case, Theorem A gives the normal section C/R with $C = 1$ and $R = \langle(1,2,3),(1,4,5),(6,7,8),(6,9,10)\rangle$ isomorphic to the direct product of two copies of the alternating group of degree 5.

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