



# Maximal subgroups of small index of finite almost simple groups

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## Abstract

We prove in this paper that every almost simple group  $R$  with socle isomorphic to a simple group  $S$  possesses a conjugacy class of core-free maximal subgroups whose index coincides with the smallest index  $l(S)$  of a maximal subgroup of  $S$  or a conjugacy class of core-free maximal subgroups with a fixed index  $v_S \leq l(S)^2$ , depending only on  $S$ . We also prove that the number of subgroups of the outer automorphism group of  $S$  is bounded by  $\log^3 l(S)$  and  $l(S)^2 < |S|$ .

**Keywords** Finite group · Maximal subgroup · Simple group · Almost simple group

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## 1 Introduction

All groups considered in this paper will be finite.

Given a group  $G$ , one can ask how many elements one should choose uniformly and at random to generate  $G$  with a certain given probability. The fact that an ordered  $r$ -tuple  $(g_1, \dots, g_r)$  generates  $G$  is equivalent to the fact that  $\{g_1, \dots, g_r\}$  is not contained in any maximal subgroup  $M$  of  $G$ . The probability that  $\{g_1, \dots, g_r\}$  is contained in a maximal subgroup  $M$  of  $G$  is  $1/|G : M|^r$ . Consequently, it is of relevant interest to find good bounds for the number  $m_n(G)$  of maximal subgroups of a group  $G$  of a given index  $n$ .

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Note that if  $M$  is a maximal subgroup of  $G$ , then  $G/M_G$ , where  $M_G$  denotes the core of  $M$  in  $G$ , is a primitive group. Consequently, the proof of many results in this field relies on the subgroup structure of such groups.

According to the theorem of Baer [1] (see also [2, Theorem 1.1.7]), there are three types of primitive groups, according to whether they have a unique abelian minimal normal subgroup (type 1), a unique non-abelian minimal normal subgroup (type 2), or two non-abelian minimal normal subgroups (type 3). The theorem of O’Nan and Scott (see [2, Theorem 1.1.52]) describes the different possibilities for a primitive pair  $(G, U)$  composed of a primitive group  $G$  of type 2 and a core-free maximal subgroup  $U$  of  $G$ . In all cases, the corresponding primitive pair is related to a primitive pair corresponding to an almost simple group with socle  $S$ , where the minimal normal subgroup of  $G$  is a direct product of copies of  $S$ . This makes crucial the study of the indices of core-free maximal subgroups of almost simple groups in the study of core-free maximal subgroups of primitive groups of type 2.

**Notation 1** We denote by  $l(X)$  the least degree of a faithful transitive permutation representation of a group  $X$ , that is, the smallest index of a core-free subgroup of  $G$ .

The aim of this paper is to prove that every almost simple group  $R$  with socle isomorphic to a simple group  $S$  possesses a conjugacy class of core-free maximal subgroups whose index coincides with the smallest index  $l(S)$  of a maximal subgroup of  $S$  or a conjugacy class of core-free maximal subgroups with a fixed index  $v_S \leq l(S)^2$ , depending only on  $S$ . We also prove that the number of subgroups of the outer automorphism group of  $S$  is bounded by  $\log^3 l(S)$  and that  $l(S)^2 < |S|$ .

These results will be applied in [3] to obtain lower bounds for the number of elements needed to generate a group with a certain probability and to obtain good lower bounds for the number of maximal subgroups of a given index of a group. They are also useful to estimate the number of possible socles of primitive groups of type 2 with a core-free maximal subgroup of a given index.

Our first main result includes relevant information over the smallest index  $l(S)$  of a maximal subgroup of a non-abelian simple group  $S$  and shows the existence of relevant subgroups of small index in an almost simple group with socle  $S$ . Moreover, we see that the order of the outer automorphism group of  $S$  is bounded by  $3 \log |S|$ . Here we reserve the symbol  $\log$  to denote the logarithm to the base 2. This last bound clarifies and improves the one used in the proof of [4, Lemma 2.3],  $|\text{Out } S| \leq O(\log^2 n)$ , and, as we will show in Remark 6, this bound is best possible. The bound  $|\text{Out } S| \leq 3 \log l(S)$  also appears in [5, Lemma 7.7], we present here a proof for completeness.

We say that a maximal subgroup of a simple group  $S$  is *ordinary* if its conjugacy class in  $S$  coincides with its conjugacy class in  $\text{Aut}(S)$ . Most simple groups  $S$  possess a conjugacy class of ordinary maximal subgroups of the smallest possible index  $l(S)$  and, by Lemma 2.1 below, every almost simple group with socle  $S$  possesses a maximal subgroup of index  $l(S)$ . However, some simple groups do not have ordinary maximal subgroups of the smallest possible index. These groups constitute the classes  $\mathfrak{X}$  and  $\mathfrak{Y}$  that we define below.

**Notation 2** Let  $\mathfrak{X}$  be the class of simple groups composed of the following groups:

1. the linear groups  $\text{PSL}_3(q)$ , where  $q = p^f > 3$  is a power of a prime  $p$  with  $f$  odd;
2. the linear groups  $\text{PSL}_n(q)$ , with  $q$  a prime power and  $n = 5$  or  $n \geq 7$ ;
3. the symplectic groups  $\text{PSp}_4(2^f)$ ,  $f \geq 2$ .

**Notation 3** Let  $\mathfrak{Y}$  be the class of simple groups composed of the following groups:

1. the Mathieu group  $M_{12}$ ;
2. the O’Nan group  $O’N$ ;
3. the Tits group  ${}^2F_4(2)'$ ;
4. the linear groups  $PSL_2(7) \cong PSL_3(2)$ ,  $PSL_2(9) \cong Alt(6)$ ,  $PSL_2(11)$ ,  $PSL_3(3)$ ;
5. the linear groups  $PSL_3(q_0^2)$ , with  $q_0$  a prime power;
6. the linear groups  $PSL_4(q)$ , with  $q$  a prime power,  $q > 2$ ;
7. the linear groups  $PSL_6(q)$ , with  $q$  a prime power;
8. the unitary group  $PSU_3(5)$ ;
9. the orthogonal groups  $O_8^+(q)$ , with  $q$  a prime power;
10. the orthogonal groups  $O_n^+(3)$ , with  $n \geq 10$ ;
11. the exceptional groups of Lie type  $G_2(3^f)$ , with  $f \geq 1$ ;
12. the exceptional groups of Lie type  $F_4(2^f)$ , with  $f \geq 1$ ;
13. the exceptional groups of Lie type  $E_6(q)$ , with  $q$  a prime power.

In the simple groups  $S$  of the class  $\mathfrak{Y}$ , there are no ordinary maximal subgroups of index  $l(S)$ , but we can find a number  $v_S \leq l(S)^2$  that depends only on  $S$  such that  $S$  has a conjugacy class of ordinary maximal subgroups of index  $v_S$ . Again by Lemma 2.1, every almost simple group with socle  $S$  possesses a conjugacy class of maximal subgroup of this index  $v_S$ . In other words,  $v_S$  appears as a common index of a core-free maximal subgroup for all almost simple groups with socle  $S$ . The class  $\mathfrak{X}$  is composed of the rest of the simple groups, that is, all simple groups that do not have a conjugacy class of ordinary maximal subgroups with index bounded by  $l(S)^2$ . However, we will prove that in the groups of the class  $\mathfrak{X}$ , we can find a number  $v_S \leq l(S)^2$  such that every almost simple group  $R$  with socle  $S$  possesses a maximal subgroup of order  $l(S)$  or a maximal subgroup of index  $v_S$ . We present this in detail in Theorem A.

**Theorem A** *Let  $S$  be a simple group.*

1. *If  $S$  does not belong to  $\mathfrak{X} \cup \mathfrak{Y}$ , then  $S$  has a conjugacy class of ordinary maximal subgroups. In particular, if  $R$  is an almost simple group with  $S \leq R \leq Aut(S)$ , then  $R$  has a conjugacy class of core-free maximal subgroups of index  $l(S)$ .*
2. *If  $S$  belongs to  $\mathfrak{Y}$ , then  $S$  has at least two conjugacy classes of maximal subgroups of the smallest index  $l(S)$  and there exists a number  $v_S \leq l(S)^2$ , depending only on  $S$ , such that if  $R$  is an almost simple group with  $S \leq R \leq Aut(S)$ , then  $R$  has a conjugacy class of core-free maximal subgroups with index  $v_S$ .*
3. *If  $S$  belongs to  $\mathfrak{X}$ , then  $S$  has at least two conjugacy classes of maximal subgroups of the smallest index  $l(S)$  and there exists a number  $v_S \leq l(S)^2$ , depending only on  $S$ , such that if  $R$  is an almost simple group with  $S \leq R \leq Aut(S)$ , then  $R$  has at least two conjugacy classes of core-free maximal subgroups with index  $l(S)$  or one conjugacy class of core-free maximal subgroups with index  $v_S$ .*
4. *In all cases,  $l(S)^2 < |S|$  and  $|Out S| \leq 3 \log l(S)$ .*
5. *If, in addition,*
  - (a)  $S \not\cong Alt(6)$ ;
  - (b)  $S$  is not of the form  $PSL_n(q)$  with  $q = p^f$  and
    - (i)  $n \geq 3$ ,  $p \in \{2, 3, 5, 7\}$ , and  $\gcd(n, q - 1) > 1$ , or
    - (ii)  $n = 2$  and  $q = 3^f$ ;
  - (c)  $S$  is not of the form  $PSU_n(q)$  with  $q = p^f$  and
    - (i)  $n = 3$  and  $p = 3$ , or
    - (ii)  $n = 3$  and  $q = 5$ , or
    - (iii)  $n \geq 4$ ,  $p = 2$ ,  $f > 1$  and  $\gcd(n, q + 1) > 1$ , or

- (iv)  $p = 3, n = 5,$  and
- (d)  $S \cong O_8^+(q)$  with  $q = p^f$  and  $p \in \{3, 5, 7, 11, 13\},$

then  $|\text{Out } S| \leq \log l(S).$

**Remark 1** According to [6], the automorphism group of the O’Nan simple group  $S \cong O^*N$  has all core-free maximal subgroups of index greater than its order, so Theorem A (4) cannot be extended to the core-free maximal subgroups of almost simple groups.

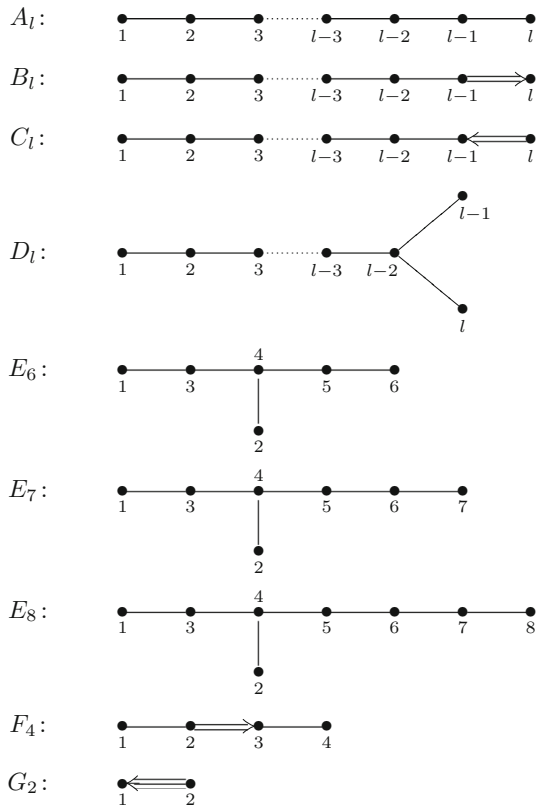
**Theorem B** *The number of subgroups of the outer automorphism group of a non-abelian simple group  $S$  is bounded by  $\log^3 l(S).$*

Unless otherwise stated, we will follow the notation of the books [7] and [2]. Detailed information about primitive groups and chief factors of a group can be found in [2, Chapter 1].

## 2 Proofs

Our results will depend heavily on the classification of simple groups. For the simple groups of Lie type, we number the nodes of the corresponding Dynkin diagrams as in Fig. 1 and denote accordingly the associated parabolic subgroups. The values of  $l(S)$  for the simple groups of Lie type have been computed in the series of papers of Mazurov [8], Vasil’ev and Mazurov [9], and Vasilyev [10–12].

**Fig. 1** Dynkin diagrams for the simple groups of Lie type



**Lemma 2.1** *Suppose that  $S$  is a non-abelian simple group and let  $A = \text{Aut}(S)$ . We identify  $S$  with the subgroup of  $A$  composed by all inner automorphisms induced by  $S$ . If  $M$  is a maximal subgroup of  $S$  such that the conjugacy class of  $M$  in  $S$  is invariant under the action of  $A$ , then  $W = N_A(M)$  is a maximal subgroup of  $A$ ,  $W \cap S = M$ , and  $|S : M| = |A : W|$ .*

**Proof** Note that the length of the conjugacy class of  $M$  in  $S$  is equal to the length of the conjugacy class of  $M$  in  $A$ . Since this length coincides with the index of the corresponding normalisers, we have that  $|S : N_S(M)| = |A : N_A(M)|$ . In particular, if  $S \leq T \leq A$ , then the conjugacy class of  $M$  in  $T$  coincides with the conjugacy class of  $M$  in  $S$ . Since the length of the conjugacy class coincides with the index of the normaliser and  $M = N_S(M)$ , we have that

$$|S : M| = |T : N_T(M)| = |A : N_A(M)| \tag{1}$$

for every  $T$  with  $S \leq T \leq A$ . Let  $W = N_A(M)$ . Then  $W \cap S = N_S(M) = M$  since  $M$  is a maximal subgroup and  $S$  is a non-abelian simple group. We prove now that  $W$  is a maximal subgroup of  $A$ . Suppose that  $W \leq V \leq A$ . By taking intersections with  $S$ , we obtain that  $M \leq V \cap S \leq S$ . Since  $M$  is maximal in  $S$ ,  $M = V \cap S$  or  $V \cap S = S$ . In the first case, since  $M = V \cap S$  is a normal subgroup of  $V$ , we obtain that  $V \leq N_A(M) = W$  and so  $V = W$ . In the second case,  $S \leq V$ . As  $N_A(M) = N_V(M)$ , by (1), it turns out that  $V = A$ .  $\square$

By Lemma 2.1, if  $X$  is an almost simple group with  $\text{Soc}(X) \cong S$  and  $M$  is an ordinary maximal subgroup of  $S$ , then  $N_X(M)$  is a maximal subgroup of  $X$  of index  $|S : M|$ . We will use this fact without mentioning it explicitly.

**Proof of Theorem A** We will analyse the different possibilities for  $S$  in the classification of finite simple groups. We note that the condition  $l(S)^2 < |S|$  is equivalent to affirming that there is a maximal subgroup of  $S$  with index less than its order. We warn the reader that the information about the maximal subgroups comes from several sources and, in order to make it easier to check the results, we have preferred to adhere to the notation of the corresponding source, even if in some cases there appear some inconsistencies in the notation.

*Sporadic simple groups*

Suppose first that  $S$  is a sporadic simple group. It is clear that if the outer automorphism group of  $S$  is trivial, then  $S \notin \mathfrak{X} \cup \mathfrak{Y}$  and the result is trivially valid. In the other cases, the outer automorphism group has order 2. In the sporadic simple groups  $M_{22}$ ,  $J_2$ ,  $\text{Suz}$ ,  $\text{HS}$ ,  $\text{McL}$ ,  $\text{He}$ ,  $\text{Fi}_{22}$ ,  $\text{HN}$ , and  $J_3$ , according to the Atlas [6], the largest maximal subgroups are ordinary and so  $l^*(A) = l(S)$ . The maximal subgroups of the group  $\text{Fi}'_{24}$  and its automorphism group  $\text{Fi}_{24}$  have been studied in [13]. The smallest index maximal subgroup  $\text{Fi}_{23}$  is ordinary. In the Mathieu group  $M_{12}$ , there are two classes of the smallest index maximal subgroup, with index 12 and there is a class of ordinary maximal subgroups of index  $144 = 12^2$  (see [6]). In the O’Nan group  $S \cong \text{O}'N$ , according again to [6], there are two conjugacy classes of maximal subgroups of type  $L_3(7) : 2$ , of the smallest index 122 760, fused under the outer automorphism, giving a conjugacy class of novel maximal subgroups of type  $7^{1+2}_+ : (3 \times D_{16})$  and index  $55\,978\,560 \leq l(S)^2$ .

For all the sporadic groups  $S$ , we also see that the inequality  $l(S)^2 < |S|$  holds for all groups whose maximal subgroups have been described in [6]. The conclusion for the Baby Monster group  $B$  follows, by [14], for the smallest index maximal subgroup  $2.^2E_6(2) : 2$ . The conclusion for the Monster group  $M$  is also true, by [15, 16], with the maximal subgroup  $2.B$ . Finally, it is clear that  $|\text{Out } S| \leq 2 \leq \log l(S)$  for all sporadic simple groups  $S$ .

According to [6], the Tits group  ${}^2F_4(2)'$  has an outer automorphism group of order 2 and two conjugacy classes of subgroups of type  $\text{PSL}_3(3) : 2$  of the smallest possible index, 1 600,

fused under the graph automorphism, and an ordinary maximal subgroup of type  $2.[2^8] : 5 : 4$  of index  $1\,755 \leq 1\,600^2$ . Furthermore,  $l(S)^2 < |S|$  and  $|\text{Out } S| \leq 2 \leq \log l(S)$ .

*Alternating groups*

Suppose now that  $S \cong \text{Alt}(n)$  with  $n \geq 7$  or  $n = 5$ . Then  $\text{Alt}(n - 1)$  is an ordinary maximal subgroup of  $S$  of the smallest possible order,  $n$ . If  $S \cong \text{Alt}(6) \cong \text{PSL}_2(9)$ , we see in [6] that the smallest index of a maximal subgroup of  $S$  is 6, while  $S$  possesses an ordinary maximal subgroup of type  $3^2 : 4$  and index  $10 < 6^2$ . Clearly,  $l(\text{Alt}(n))^2 < |\text{Alt}(n)|$  for all  $n \geq 5$  and  $|\text{Out Alt}(n)| = 2 \leq \log l(\text{Alt}(n))$  if  $n \neq 6$ , and  $|\text{Out Alt}(6)| = 4 \leq 3 \log l(\text{Alt}(6))$ .

*Linear groups*

We start with the linear groups on dimension 2. Suppose first that  $S \cong \text{PSL}_2(q)$  with  $q \in \{5, 7, 8, 9, 11\}$ . We have that  $S = \text{PSL}_2(5) \cong \text{Alt}(5)$  and  $S = \text{PSL}_2(9) \cong \text{Alt}(6)$  have been studied before. Moreover, according to [6],  $S = \text{PSL}_2(7)$  has two conjugacy classes of maximal subgroups of index  $l(S) = 7$  and a class of ordinary maximal subgroups of index 8,  $S = \text{PSL}_2(8)$  has an ordinary maximal subgroup of index  $l(S) = 9$ , and  $S \cong \text{PSL}_2(11)$  has two conjugacy classes of maximal subgroups of index  $l(S) = 11$  and another conjugacy class of ordinary maximal subgroups of index  $12 < l(S)^2$ . We can see in [6] the existence of maximal subgroups in  $S$  and in all almost simple groups with the prescribed indices and that  $|\text{Out PSL}(2, q)| \leq \log l(S)$  when  $q \in \{5, 7, 8, 11\}$  and that  $|\text{PSL}(2, 9)| \leq 3 \log l(S)$ . Since the parabolic subgroups have index  $q + 1$  and order  $q(q - 1)/d$  with  $d = \text{gcd}(q - 1, 2)$ , we see that  $l(S) \leq q + 1 \leq q(q - 1)/d$  and so  $l(S)^2 < |S|$ .

Suppose that  $S \cong \text{PSL}_2(q)$ , with  $q = p^f \geq 13$ . Then the parabolic (Borel) subgroups are the smallest index maximal subgroups and are ordinary by [8, Theorem 1]. Their index is  $l(S) = q + 1$  and their order is  $q(q - 1)/d$ . Then  $q^2 - 3q - 2 = q(q - 3) - 2 \geq 0$ , which implies that  $l(S) = q + 1 \leq q(q - 1)/2 \leq q(q - 1)/d$  and so  $l(S)^2 \leq |S|$ . Furthermore,  $|\text{Out } S| = \text{gcd}(2, q - 1) \cdot f \leq 2f \leq 2 \log p^f \leq 2 \log l(S)$ . If, in addition  $p \geq 5$ , then  $2f \leq \log p^f \leq \log l(S)$ , while if  $p = 2$ , then  $|\text{Out } S| = 1 \cdot f \cdot 1 \leq \log l(S)$ .

We consider now the linear groups on dimension greater than 2. The groups  $\text{PSL}_3(2) \cong \text{PSL}_2(7)$  and  $\text{PSL}_4(2) \cong \text{Alt}(8)$  have been considered before. Let  $R$  be an almost simple group with  $S = \text{Soc}(R) \cong \text{PSL}_n(q)$ , where  $n \geq 3$ , that is,  $S \leq R \leq A$  with  $A \cong \text{Aut}(S)$ , and suppose that  $(n, q) \notin \{(3, 2), (4, 2)\}$ . According to [8, Theorem 1],  $l(S) = (q^n - 1)/(q - 1)$  is the index of a parabolic subgroup. Then the outer automorphism group of  $S$  is isomorphic to  $[C_d][C_f]C_2 = \langle \delta, \phi, \gamma \rangle$ , with  $q = p^f$ ,  $p$  a prime, and  $d = \text{gcd}(n, q - 1)$  (see, for example, [6]). If  $R/S$  is contained in  $\langle \delta, \phi \rangle$ , the parabolic subgroups of type  $P_1$ , which are the stabilisers of a 1-dimensional subspace, induce maximal subgroups of  $R$  of index  $(q^n - 1)/(q - 1)$  since they are stabilised by  $\langle \delta, \phi \rangle$ . If  $R/S$  is not contained in  $\langle \delta, \phi \rangle$ , then the double parabolic subgroups  $P_{1, n-1}$ , stabilisers of pairs of subspaces  $(W, U)$  with  $W < U$  and  $1 = \dim W = n - \dim U$ , induce maximal subgroups of  $R$  of index  $v_S = (q^n - 1)(q^{n-1} - 1)/(q - 1)^2$  since they are stabilised by  $\langle \delta, \phi, \gamma \rangle$ . Then  $R$  has a maximal subgroup of index  $l(S) = (q^n - 1)/(q - 1)$  or of index  $v_S = (q^n - 1)(q^{n-1} - 1)/(q - 1)^2 \leq l(S)^2$ , according to whether or not  $R/S$  is contained in  $\langle \delta, \phi \rangle$ , respectively. In particular,  $l^*(R) \leq l(S)^2$ . It is clear that

$$l(S)^2 = \left( \frac{q^n - 1}{q - 1} \right)^2 < \frac{q^{n(n-1)/2} (q^n - 1)(q^{n-1} - 1) \cdots (q^2 - 1)}{\text{gcd}(n, q - 1)} = |S|.$$

If  $S \cong \text{PSL}_4(2)$ , then  $|\text{Out } S| = 2 < \log l(S)$ . Furthermore, for  $S = \text{PSL}_n(q)$  with  $n \geq 3$ ,  $(n, q) \neq (4, 2)$ , we have that  $l(S) = (q^n - 1)/(q - 1)$  and so

$$\begin{aligned}
 |\text{Out } S| &\leq n \cdot f \cdot 2 \leq (3/2)(n - 1) \cdot f \cdot 2 \\
 &= 3 \log((2^f)^{n-1}) \leq 3 \log q^{n-1} < 3 \log l(S).
 \end{aligned}$$

If, in addition,  $p \geq 11$ , then  $3 \log(2^f)^{n-1} \leq \log(p^f)^{n-1} \leq \log l(S)$ . Furthermore, if  $p \geq 2$  and  $\gcd(n, q - 1) = 1$ , then  $|\text{Out } S| = 2f \leq q^2 + 1 \leq l(S)$ .

Now we analyse the cases for which  $G \notin \mathfrak{X}$ . Note that, according to [6], the group  $S \cong \text{PSL}_3(3)$  has two conjugacy classes of maximal subgroups of index  $l(S) = 13$  and a conjugacy class of ordinary maximal subgroups of type  $13 : 3$  and index  $144 < l(S)^2 < |S|$  and  $S \cong \text{PSL}_3(4)$  has two conjugacy classes of maximal subgroups of index  $l(S) = 21$  and a conjugacy class of ordinary maximal subgroups of type  $3^2 : Q_8$  and index  $280 < l(S)^2 < |S|$ .

Suppose that  $S = \text{PSL}_3(q)$  with  $q = q_0^2$ , where  $q_0$  a prime power,  $q_0 > 2$ . We have that  $l(S) = (q^3 - 1)(q - 1) = (q_0^6 - 1)/(q_0^2 - 1)$ . Then  $\gcd(q_0 + 1, 3) = 1$  or  $\gcd(q_0 - 1, 3) = 1$ . By [17, Table 8.3], in the first case,  $\text{SL}_3(q)$  has an ordinary maximal subgroup isomorphic to  $\text{SL}_3(q_0)$  of index  $q_0^3(q_0^3 + 1)(q_0^2 + 1) = q_0^8 + q_0^6 + q_0^5 + q_0^3 < (q_0^4 + q_0^2 + q_0)^2 = l(S)^2$ , while, in the second case,  $\text{SL}_3(q)$  has an ordinary maximal subgroup isomorphic to  $\text{SU}_3(q_0)$ , of index  $q_0^3(q_0^3 - 1)(q_0^2 + 1) < ((q_0^6 - 1)/(q_0^2 - 1))^2 = l(S)^2$ .

For the group  $S \cong \text{PSL}_4(q)$ ,  $q > 2$ , according to [17, Table 8.8], we have that  $\text{SL}_4(q)$  has two classes of maximal subgroups of the smallest index  $l(S) = (q^4 - 1)/(q - 1)$  and a conjugacy class of ordinary maximal subgroups of type  $E_q^4 : \text{SL}_2(q) \times \text{SL}_2(q) : (q - 1)$  and index  $v_S = (q^2 + 1)(q^3 - 1)/(q - 1)$ . Therefore  $v_S/l(S)^2 = (q - 1)(q^3 - 1)/((q^2 - 1)(q^4 - 1)) < 1$  and so  $v_S < l(S)^2$ .

For the group  $S \cong \text{PSL}_6(q)$ , according to [17, Table 8.24], we have that  $\text{SL}_6(q)$  has two classes of maximal subgroups of the smallest index  $l(S) = (q^6 - 1)/(q - 1)$  and a conjugacy class of ordinary maximal subgroups of type  $E_q^9 : \text{SL}_3(q) \times \text{SL}_3(q) : (q - 1)$  and index  $v_S = (q^5 - 1)(q^4 - 1)(q^3 + 1)/((q - 1)^2(q + 1))$ . Therefore  $v_S/l(S)^2 = (q^5 - 1)(q^2 + 1)(q^2 - 1)/((q^6 - 1)(q + 1)^2(q^3 - 1)) < 1$  and so  $v_S < l(S)^2$ .

*Symplectic groups*

Now suppose that  $S \cong \text{PSp}_n(q)$  with  $n \geq 4$  even. Then

$$|S| = \frac{1}{d} q^{(n/2)^2} \left( \prod_{i=1}^{n/2} (q^{2i} - 1) \right),$$

where  $d = \gcd(2, q - 1)$ . The smallest index maximal subgroups of  $S$  are described in [8, Theorem 2]. If  $(n, q) = (4, 3)$  then  $S \cong \text{PSU}_4(2)$  and  $G$  has an ordinary maximal subgroup of type  $2^4 : \text{Alt}(5)$ , of index  $l(S) = 27$  and order 960. In this case,  $|\text{Out } S| = 2 < \log l(S)$ .

Suppose that  $n \geq 6$  and  $q = 2$ . Then  $S \cong \text{PSp}_n(2) \cong \text{O}_{n+1}(2)$  and the smallest index maximal subgroup of  $S$  is isomorphic to  $\text{O}_n^-(2)$  and has index  $2^{n/2-1}(2^{n/2} - 1)$ . By [17, Tables 8.28, 8.48, 8.64, 8.80] and [18, Table 3.5.C], this subgroup is ordinary. It is clear that this index is smaller than the order of this subgroup, namely  $2^{(n/2)(n/2-1)}(2^{n/2} + 1) \prod_{i=1}^{n/2-1} (q^{2i} - 1)$ , and that  $|\text{Out } S| = 1 < \log l(S)$ .

If  $n = 4$  and  $q = 2^f$ , then  $|S| = q^4(q^4 - 1)(q^2 - 1)$  and there are two conjugacy classes of parabolic maximal subgroups of type  $E_q^3 : \text{GL}_2(q)$  and index  $l(S) = (q^4 - 1)/(q - 1)$  fused under the graph automorphism, by [17, Table 8.14]. There is a novelty subgroup  $[q^4] : (C_{q-1})^2$ , maximal under subgroups not contained in the subgroup  $\langle \phi \rangle$  generated by the field automorphism, of index

$$\frac{(q^4 - 1)(q + 1)}{q - 1} < \left( \frac{q^4 - 1}{q - 1} \right)^2 = l(S)^2.$$

In this case,

$$|\text{Out } S| = 1 \cdot f \cdot 2 = 2f \leq \log q^2 \leq \log l(S).$$

For the rest of the values of  $(n, q)$ , the smallest index maximal subgroup is a parabolic subgroup, which can be taken to have the form  $[q^{n-1}] : ((q - 1). \text{PSp}_{n-2}(q))$ , with index  $l(S) = (q^n - 1)/(q - 1)$  and order

$$\frac{1}{d} q^{n-1} (q - 1) q^{((n-2)/2)^2} \prod_{i=1}^{(n-2)/2} (q^{2i} - 1) > q^n \geq \frac{q^n - 1}{q - 1} = l(S).$$

Here  $|\text{Out } S| = \gcd(2, q - 1) \cdot f \cdot 1 = 2f$  if  $p$  is odd and  $|\text{Out } S| = \gcd(2, q - 1) \cdot f \cdot 1 = f$  if  $p = 2$ . In any case,

$$|\text{Out } S| \leq 2f \leq \log q^{n-1} \leq \log l(S).$$

*Unitary groups*

Suppose now that  $S \cong \text{PSU}_n(q)$  with  $n \geq 3$  and  $q > 2$  if  $n = 3$ ; then  $|\text{Out } S| \leq n \cdot f \cdot 1$  with  $p^f = q^2$ . The smallest index of a maximal subgroup of  $S$  is given in [8, Theorem 3].

We consider first the case  $S \cong \text{PSU}_3(5)$ . We see in [6] that the automorphism group of  $S$  is isomorphic to  $S_3$  and that there are three conjugacy classes of maximal subgroups of the smallest possible index, of type  $\text{Alt}(7)$  and index  $l(S) = 50$  and order 2520. Moreover, there is an ordinary maximal subgroup of type  $5_+^{1+2} : 8$  and index  $126 \leq 50^2$ . In this case,  $|\text{Out } S| = 3 \cdot 2 \cdot 1 = 6 \leq 3 \log l(S)$ .

Suppose that  $n = 3, q \notin \{2, 5\}$ . Then  $S$  has a conjugacy class of parabolic ordinary maximal subgroups of type  $[q^3] : ((q^2 - 1)/d)$ , with  $d = \gcd(3, q + 1)$  and index  $l(S) = q^3 + 1$ , as we can see in [17, Table 8.5]. Clearly,  $l(S)^2 < |S|$  and  $|\text{Out } S| \leq \gcd(3, q - 1) \cdot 2f \leq \log 2^{6f}$ . If  $p \geq 5, \log 2^{6f} \leq \log q^3 \leq \log l(S)$ . If  $p = 3$ , then  $\log 2^{6f} \leq \log 3^{4f} = (4/3) \log 3^{3f} \leq (4/3) \log l(S)$ .

Now assume that  $n = 4$ . Then  $S$  has a conjugacy class of parabolic maximal subgroups of type  $[q^4].\text{SL}_2(q^2) : ((q - 1)/d)$ , with  $d = \gcd(q + 1, 4)$ , whose index is  $l(S) = (q^3 + 1)(q + 1)$  and whose order is  $q^6(q^4 - 1)(q - 1) > l(S)$ . These subgroups are ordinary, as shown in [17, Table 8.10]. Moreover,  $|\text{Out } S| = \gcd(4, q + 1) \cdot 2f \cdot 1 = 8f \leq 2 \log 2^{4f} \leq 2 \log q^4 \leq 2 \log l(S)$ . If, in addition,  $p \notin \{2, 3\}$ , then  $|\text{Out } S| = 8f \leq \log q^4 \leq \log l(S)$ .

Assume now that  $n > 4$ , and that  $q > 2$  if  $n$  is even. In this case, the smallest index maximal subgroups are the parabolic subgroups of type  $[q^{2n-3}] : \text{SU}_{n-2}(q) : ((q^2 - 1)/d)$ , with  $d = \gcd(n, q + 1)$ . These subgroups have index  $l(S) = (q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})/(q^2 - 1)$ . By [17, Tables 8.20, 8.26, 8.37, 8.46, 8.56, 8.62, 8.72, and 8.78] and [18, Table 3.5.B], we see that these maximal subgroups are ordinary. Since they have order

$$\frac{1}{d} q^{n(n-1)/2} (q^2 - 1) \prod_{i=1}^{n-3} (q^{i+1} - (-1)^{i+1}),$$

$(q^n + 1) < q^2(q^{n-2} + 1)$  and  $q^{n-1} - 1 < q^{n-1}$  if  $n$  is odd, and  $(q^{n-1} + 1) < q^2(q^{n-3} + 1)$  and  $q^n - 1 < q^n$  if  $n$  is even, we see that the index of these subgroups is smaller than their order. Observe that



$$\begin{aligned}
 l(S) &= \begin{cases} (q^{n-1} - q^{n-2} + \dots - q + 1)(q^{n-2} + q^{n-3} + \dots + q + 1) & (n \text{ odd}) \\ (q^{n-1} + q^{n-2} + \dots + q + 1)(q^{n-2} - q^{n-3} + \dots - q + 1) & (\text{even}) \end{cases} \\
 &= \begin{cases} ((q - 1)(q^{n-2} + q^{n-4} + \dots + q) + 1)(q^{n-2} + q^{n-3} + \dots + 1) & (n \text{ odd}) \\ (q^{n-1} + q^{n-2} + \dots + 1)((q - 1)(q^{n-3} + q^{n-5} + \dots + q) + 1) & (n \text{ even}) \end{cases} \\
 &\geq \begin{cases} q^{n-2}q^{n-2} & (n \text{ odd}) \\ q^{n-1}q^{n-3} & (n \text{ even}) \end{cases} \\
 &= q^{2n-4}.
 \end{aligned}$$

It follows that  $\log l(S) \geq (2n - 4) \log q = (2n - 4)f \log p$ . Suppose that  $p \geq 5$  or that  $p = 3$  and  $n \geq 6$ . Then  $n \leq (n - 2) \log p$  and so

$$|\text{Out } S| \leq n \cdot 2f \cdot 1 \leq (2n - 4)f \log p \leq \log l(S).$$

Suppose now that  $p = 2$  and  $n \geq 5$ , or that  $p = 3$  and  $n = 5$ , and  $\gcd(n, q + 1) = 1$ . In this case,  $|\text{Out } S| = 2f$  and so

$$|\text{Out } S| \leq 2f \cdot (2n - 4)f \log 2 \leq \log l(S).$$

Suppose now that  $p = 2$  and  $n \geq 5$  or that  $p = 3$  and  $n = 5$ , and that  $\gcd(n, q + 1) > 1$ . In this case,  $n \leq 3(n - 2) \log p$  and so

$$|\text{Out } S| \leq 2nf \leq 3(2n - 4)f \log p \leq 3 \log l(S).$$

Finally, assume that  $n \geq 6$ ,  $n$  is even, and  $q = 2$ . Then the smallest index maximal subgroups of  $S$  have type  $\text{SU}_{n-1}(2) : (3/d)$ , with  $d = \gcd(3, m)$ , and  $\log l(S) = 2^{n-1}(2^n - 1)/3$ , and, since  $2^{n-1} \leq 2^{(n-1)(n-2)/2}$  and  $2^n - 1 < 2^n + 2 = 2(2^{n-1} + 1)$ , we obtain that  $l(S)^2 < |S|$ . By [17, Tables 8.26, 8.46, 8.62, and 8.78] and [18, Table 3.5.B], we conclude that these maximal subgroups are ordinary. In this case,  $|\text{Out } S| = 2 \leq \log l(S)$  if  $n$  is not divisible by 6, while  $|\text{Out } S| = 3 \cdot 2 = 6 \leq \log l(S)$  if  $n$  is divisible by 6.

*Orthogonal groups*

Suppose now that  $S \cong \text{O}_n^\epsilon(q)$  is an orthogonal group with  $n \geq 7$ ,  $n$  even if  $q = 2^f$ . The smallest index maximal subgroups of  $S$  have been described in [9, Theorem].

Assume first that  $n = 8$ ,  $\epsilon = +$  and  $q > 3$ . Then we have that  $l(S) = (q^4 - 1)(q^3 + 1)/(q - 1)$  and we can take a maximal subgroup  $H$  of type  $q^6 \cdot (\Omega_6^+(q) \times (q - 1)/d) \cdot e$ , where  $d = \gcd(q^4 - 1, 4)$ ,  $e = \gcd(q^4 - 1, 2)$ . Hence  $|H| = q^{12}(q^3 - 1)(q^4 - 1)(q^2 - 1)(q - 1)/e$ . Since  $(q^3 - 1)(q - 1) > 2$ , we conclude that  $q(q^3 - 1) = q^4 - q > q^3 + 1$ , and so  $l(S) < |H|$ . By [17, Table 8.50],  $S$  possesses an ordinary maximal subgroup in the Aschbacher class  $\mathcal{C}_1$  of type  $E_q^{1+8}(\frac{1}{e}\text{GL}_2(q) \times \Omega_4^+(q)) \cdot e$  with index  $v = (q + 1)(q^2 - q + 1)(q^2 + 1)^2(q^2 + q + 1)$ . Therefore

$$\frac{v}{l(S)^2} = \frac{q^3 - 1}{(q + 1)^2(q^3 + 1)(q - 1)} < 1.$$

It follows that  $v < l(S)^2$ . In this case, we have three conjugacy classes of maximal subgroups of index  $l(S)$ , fused under the triality outer automorphism. Moreover,  $|\text{Out } S| = \gcd(2, q - 1)^2 \cdot f \cdot 3!$ . If  $p \neq 2$ , then  $2^8 < 3^6$  and so  $|\text{Out } S| \leq 24f \leq 3 \log 2^{8f} \leq 3 \log p^{6f} \leq 3 \log l(S)$ . If  $p \geq 17$ , then  $2^4 \leq p$  and so  $|\text{Out } S| \leq 24f \leq \log 2^{24f} \leq \log p^{6f} \leq \log l(S)$ . If  $p = 2$ , then  $|\text{Out } S| = 6f \leq \log q^6 \leq \log l(S)$ .

Suppose now that  $S \cong O_n^+(2)$  with  $n = 2t$  even. Then there is at least one conjugacy class of maximal subgroups of smallest index of type  $H \cong \Omega_{n-1}(2)$  and index  $l(S) = 2^{t-1}(2^t - 1)$ , and order

$$|H| = 2^{(n/2-1)^2} (2^{n/2} - 1)(2^{n/2-1} - 1) \dots (2^2 - 1);$$

clearly  $l(S) < |H|$ . In this case,  $|\text{Out } S| \leq 6 \leq \log l(S)$ .

By [6], if  $n = 8$ , then there are three conjugacy classes of maximal subgroups of smallest index fused under the triality automorphism; moreover,  $l(S) = 120$  and  $S$  has an ordinary maximal subgroup of type  $2_+^{1+8} : (S_3 \times S_3 \times S_3)$  and index  $1575 < 120^2 = l(S)^2$ . Assume that  $n \geq 10$ . By [17, Tables 8.66 and 8.82] and [18, Table 3.5.E], this subgroup is ordinary. Furthermore,  $|\text{Out } S| = 6 < \log l(S)$ .

Assume now that  $S \cong O_n(3)$ , where  $n = 2t + 1$  is odd. Then there exists a conjugacy class of maximal subgroups isomorphic to  $H \cong \Omega_{n-1}^-(3).2$ , with index  $l(S) = 3^t(3^t - 1)/2$  and

$$|H| = 3^{t(t-1)}(3^t + 1)(3^{2t-2} - 1)(3^{2t-4} - 1) \dots (3^4 - 1)(3^2 - 1),$$

so that  $l(S) < |H|$ . Furthermore, by [17, Tables 8.39, 8.58, and 8.74] and [18, Table 3.5.D], these maximal subgroups are ordinary. In this case,  $|\text{Out } S| = 2 \cdot 1 \cdot 2 = 4 < \log l(S)$ .

Assume that  $S \cong O_n^+(3)$ , where  $n = 2t$  is even. The maximal subgroups of the smallest index have type  $H \cong \Omega_{n-1}(3).h$ , where  $h = \gcd(t - 1, 2)$ . Their index is  $l(S) = 3^{t-1}(3^t - 1)/2$ , clearly smaller than their order  $|H| = (1/2)3^{t(t-1)^2} \prod_{i=1}^{t-1} (3^{2i} - 1)$ . In this case,  $|\text{Out } S| = 2^2 \cdot 1 \cdot 2 = 8 < \log l(S)$  if  $n \geq 10$ .

In the case that  $S \cong O_8^+(3)$ ,  $l(S) = 1080$ , there are six conjugacy classes of maximal subgroups isomorphic to  $H$ , the outer automorphism group of  $S$  is isomorphic to  $S_4$  and there is an ordinary subgroup of type  $3_+^{1+8} : 2(\text{Alt}(4) \times \text{Alt}(4) \times \text{Alt}(4)).2$ , according to [6], of index  $36400 < l(S)^2$ . Moreover,  $|\text{Out } S| = 24 < 3 \log 1080 = 3 \log l(S)$ .

Assume that  $n = 2t \geq 10$ . By [17, Tables 8.66 and 8.82] and [18, Table 3.5.E], there are two conjugacy classes of subgroups of the smallest index  $l(S) = 3^t(3^t - 1)/2$ . On the other hand, there is an ordinary maximal parabolic subgroup of type  $3^{n-2}.(\Omega_{n-2}^+(3).2)$ , of index  $(3^t - 1)(3^{t-1} + 1)/2 < l(S)^2$ . In this case,  $|\text{Out } S| = \gcd(2, 3 - 1)^2 \cdot 1 \cdot 2 = 8$  and, since  $l(S) \geq 3^5(3^5 - 1)/2 > 2^8$ , we conclude that the inequality  $|\text{Out } S| < \log l(S)$  also holds in this case.

Suppose that  $S \cong O_n(q)$  where  $n = 2t + 1$  is odd,  $q = p^f \neq 3$ , and  $p$  is an odd prime. Then the smallest index of a maximal subgroup of  $S$  corresponds to  $H \cong [q^{n-2}].((\Omega_{n-2}(q) \times (q - 1)/2).2)$ , of index  $(q^{n-1} - 1)/(q - 1)$ . Since  $|H| = (1/2)q^{n-2}q^{(t-1)^2} \prod_{i=1}^{t-1} (q^{2i} - 1)$ , it is clear that  $|H| > l(S)$ . By [17, Tables 8.39, 8.58, and 8.74] and [18, Table 3.5.D], these maximal subgroups are ordinary. Moreover,  $|\text{Out } S| = 2^2 \cdot f \cdot 1 \leq 2 \log 2^f \leq 2 \log q < \log l(S)$ .

Suppose now that  $n = 2t$ ,  $q = 2^f$ ,  $f \geq 2$ , and that  $(n, \varepsilon) \neq (8, +)$ . In this case, the smallest index of a maximal subgroup corresponds to subgroups of type  $H = [q^{n-2}].(\Omega_{n-2}^\varepsilon(q) \times (q - 1))$ , of index  $l(S) = (q^t - \varepsilon)(q^{t+1} + \varepsilon)/(q - 1)$  and order  $|H| = q^{n-2}q^{(t-1)(t-2)}(q^{t-1} - \varepsilon) \prod_{i=1}^{t-1} (q^{2i} - 1)$ . We can see that  $|H| > l(S)$ . By [17, Tables 8.52, 8.66, 8.68, 8.82, and 8.84] and [18, Tables 3.5.E and 3.5.F], we see that these subgroups are ordinary. Furthermore,  $|\text{Out } S| = f \cdot 2$  if  $\varepsilon = +$  and  $|\text{Out } S| = 2f \cdot 1$  if  $\varepsilon = -$ . In both cases,  $|\text{Out } S| \leq \log l(S)$ .

Finally, suppose that  $n = 2t$ ,  $q = p^f$ ,  $p$  is an odd prime, the pair  $(m, \varepsilon)$  is different from  $(8, +)$ , and  $(q, \varepsilon)$  is different from  $(3, +)$ . Then the smallest index of a maximal subgroup corresponds to the subgroups of type  $H \cong p^{f(m-2)}.(\Omega_{m-2}^\varepsilon(q) \times (q - 1)/h).j$ , where  $(h, j) = (2, 2)$  if  $\gcd(q^t - \varepsilon, 4) = 2$ ,  $(h, j) = (2, 1)$  if  $\gcd(q^t - \varepsilon, 4) = 4$  and

$\varepsilon(q^{t-1} - \varepsilon, 4) = 2$ , and  $(h, j) = (4, 2)$  if  $\gcd(q^t - \varepsilon, 4) = 4$  and  $\gcd(q^{t-1} - \varepsilon, 4) = 1$ . This index is  $l(S) = (q^t - \varepsilon)(q^{t-1} + \varepsilon)/(q - 1)$ . As in the previous cases,  $l(S) < |H|$ . This subgroup is ordinary by [17, Tables 8.52, 8.66, 8.68, 8.82, and 8.84] and [18, Tables 3.5.E and 3.5.F]. Furthermore,  $|\text{Out } S| \leq 2^2 \cdot f \cdot 2 \leq \log q^6 \leq \log l(S)$ .

*Groups of type  $G_2(q)$*

The maximal subgroups of smallest index of the simple groups  $G_2(q)$ ,  $q > 2$ , have been studied in [10, Theorem 1].

Assume that  $S \cong G_2(3)$ . Then  $P \cong \text{PSU}_3(3) : 2$  is a maximal subgroup of the smallest possible index  $l(S) = 351$  and order  $12\,096 > l(S)$ . By [6], there are two conjugacy classes of maximal subgroups of this index and there is a conjugacy class of ordinary maximal subgroups of type  $\text{PSL}_2(8) : 3$  and index  $2\,808 < l(S)^2$ . In this case,  $|\text{Out } S| = 2 < \log l(S)$ .

Assume that  $S \cong G_2(4)$ . Then  $P \cong J_2$  is a maximal subgroup of the smallest possible index  $l(S) = 416$  and order  $604\,800 > l(S)$ . According to [6], this subgroup is ordinary. Moreover,  $|\text{Out } S| = 2 < \log l(S)$ .

Suppose now that  $S \cong G_2(q)$  with  $q \geq 5$ . Then  $l(S) = (q^6 - 1)/(q - 1)$ .

Assume that  $S \cong G_2(q)$  with  $q = 2^f$ ,  $f \geq 3$ . Then  $P_1 \cong (2^f \cdot 2^{4f}) : (\text{PSL}_2(q) \times (q - 1))$  and  $P_2 \cong (2^{2f} \cdot 2^{3f}) : (\text{PSL}_2(q) \times (q - 1))$  are maximal subgroups of  $S$  of the smallest possible index. We see in [17, Table 8.30] that these subgroups are ordinary. Clearly,  $l(S) < |P|$  and  $|\text{Out } S| = f \leq \log l(S)$ .

Assume that  $S \cong G_2(3^f)$  with  $f \geq 2$ . Then the smallest index maximal subgroups of  $S$  are of type  $P \cong (3^f \cdot 3^{2f} \times 3^{2f}) : (2 \cdot (\text{PSL}_2(q) \times (q - 1)/2) \cdot 2)$ . Note that  $|P| > l(S)$ . By [17, Table 8.42], there are two conjugacy classes of subgroups of this type. There is a conjugacy class of ordinary subgroups of type  $(\text{SL}_2(q) \circ \text{SL}_2(q)) \cdot 2$  and index  $q^4(q^4 + q^2 + 1) = q^4(q^6 - 1)/(q^2 - 1) < l(S)^2$ . Moreover,  $|\text{Out } T| = 2f \leq \log l(S)$ .

Now assume that  $q = p^s$ , with  $p$  a prime,  $p > 3$ . Then there are two conjugacy classes of maximal subgroups of the smallest index, namely  $P_1 \cong (p^{2f} \cdot (p^f \cdot p^{2f})) : (2 \cdot (\text{PSL}_2(q) \times (q - 1)/2) \cdot 2)$  and  $P_2 \cong (p^f \cdot p^{4f}) : (2 \cdot (\text{PSL}_2(q) \times (q - 1)/2) \cdot 2)$ . Again,  $|P_1| > l(S)$ ,  $|\text{Out } S| = f \leq \log l(S)$  and, by [17, Table 8.41], these subgroups are ordinary.

*Groups of type  $F_4(q)$*

The smallest index maximal subgroups of  $F_4(q)$  have been studied in [10, Theorem 2]. This index is

$$l(S) = \frac{(q^{12} - 1)(q^4 + 1)}{q - 1}$$

and is attained by a parabolic subgroup. Since the order of this subgroup is  $|P| = q^{24}(q^4 - 1)(q^6 - 1)(q^2 - 1)(q - 1)$ , we have that  $|P| > l(S)$ . Moreover,  $|\text{Out } S|$  is  $f$  if  $p \neq 2$  and  $2f$  if  $p = 2$ . In both cases,  $|\text{Out } S| \leq \log q^4 \leq \log l(S)$ .

Assume first that  $q = 2^f$ . Then there are two conjugacy classes of parabolic maximal subgroup isomorphic to  $P \cong (2^f \cdot 2^{8f} \times 2^{6f}) : (\text{PSP}_6(q) \times (q - 1))$ . By [19, Table 5.1],  $S$  has an ordinary maximal subgroup of type  $H \cong e \cdot (L_3^\varepsilon(q) \times L_3^\varepsilon(q)) \cdot e \cdot 2$ , where  $\varepsilon = \pm 1$ ,  $e = \gcd(3, q - \varepsilon)$ ,  $L_3^{+1}(q) = \text{PSL}_3(q)$  and  $L_3^{-1}(q) = \text{PSU}_3(q)$ . Then  $v = |S : H| = q^{18}(q + 1)^2(q^2 - q + 1)^2(q^2 + 1)^2(q^4 - q^2 + 1)(q^4 + 1)$  and we can check that  $v \leq l(S)^2$ .

Assume that  $q = p^f$  with  $p$  a prime different from 2. Then  $P_1 = (p^f \cdot p^{14f}) : (2 \cdot (\text{PSp}_4(q) \times (q - 1)/2) \cdot 2)$  or  $P_4 = (p^{7f} \cdot p^{8f}) : (2 \cdot (\text{O}_7(q) \times (q - 1)/2) \cdot 2)$  are parabolic maximal subgroups of  $S$  of the smallest index. The conjugacy classes of both subgroups are fixed under the outer automorphism group of  $S$  since both are parabolic.

*Groups of type  $E_6(q)$*

The maximal subgroups of smallest index of  $S \cong E_6(q)$  have been studied in [11, Theorem 1]. The smallest index of a maximal subgroup of  $S$  is  $l(S) = (q^9 - 1)(q^8 + q^4 + 1)/(q - 1) = (q^9 - 1)(q^{12} - 1)/((q^4 - 1)(q - 1))$ , corresponding to two conjugacy classes of parabolic subgroups  $P_1 = p^{16f} : (e.O_{10}^+(q) \times (q - 1)/e'.e)$ , where  $e = \gcd(q - 1, 4)$ ,  $e' = e \cdot \gcd(q - 1, 3)$ , interchanged by the graph automorphism. Clearly,  $l(S) < |P_1|$ . Moreover,  $|\text{Out } S| = \gcd(3, q - 1) \cdot f \cdot 2 \leq 6f \leq \log q^8 \leq \log l(S)$ . The parabolic subgroup  $P_2$  is ordinary and is of type  $[q^{21}] : H$  where  $H$  has a section isomorphic to  $\text{PSL}_6(q)$  (see also [20, Table 7.3]) and so its index divides  $v = (q^{12} - 1)(q^4 + 1)(q^9 - 1)/(q^3 - 1)$ . Then

$$\frac{v}{l(S)^2} = \frac{(q^8 - 1)(q^4 - 1)(q - 1)^2}{(q^{12} - 1)(q^9 - 1)(q^3 - 1)} < 1,$$

therefore  $v < l(S)^2$ . We conclude that  $|S : P_2| \leq v \leq l(S)^2$ .

*Groups of type  $E_7(q)$*

The maximal subgroups of smallest index of  $S \cong E_7(q)$  for  $q = p^f$  have been studied in [11, Theorem 2]. They are the parabolic subgroups  $P \cong p^{27f} : (d'.(E_6(q) \times (q - 1)/c).d')$ , with  $d' = \gcd(q - 1, 3)$ ,  $c = \gcd(q - 1, 2) \cdot d'$ , of index  $l(S) = (q^{14} - 1)(q^9 + 1)(q^5 + 1)/(q - 1)$ . Clearly  $P_1$  is ordinary,  $l(S) < |P_1|$ , and  $|\text{Out } S| = \gcd(2, q - 1) \cdot f \cdot 1 \leq 2f \leq \log q^5 \leq \log l(S)$ .

*Groups of type  $E_8(q)$*

The maximal subgroups of smallest index of  $S \cong E_8(q)$  for  $q = p^f$  have been studied in [11, Theorem 3]. They are the parabolic subgroups  $P \cong (p^f.p^{56f}) : (d.(E_7(q) \times (q - 1)/d).d)$ , with  $d = \gcd(q - 1, 2)$ , of index  $l(S) = (q^{20} - 1)(q^{12} + 1)(q^{10} + 1)(q^6 + 1)/(q - 1)$ . Clearly  $P$  is ordinary and  $l(S) < |P_1|$ , and  $|\text{Out } T| = f \leq \log q^6 \leq \log l(S)$ .

*Twisted groups*

In [12, Theorem 1], it is shown that if  $S \cong {}^2B_2(q)$ , with  $q = 2^f$ ,  $f$  an odd integer greater than 1, the smallest index of a maximal subgroup of  $S$  corresponds to the parabolic subgroup  $P \cong (2^f.2^f) : (q - 1)$ , with index  $l(S) = q^2 + 1$ . By [17, Table 8.16], these subgroups are ordinary and, clearly,  $l(S) < |P|$  and  $|\text{Out } S| = f = \log q \leq \log l(S)$ .

In [12, Theorem 2], it is shown that if  $S \cong {}^2G_2(q)$ , with  $q = 3^f$  and  $f$  an odd integer greater than 1, there is a class of smallest index maximal subgroups isomorphic to  $P \cong (3^f.3^f.3^f) : (q - 1)$  and index  $q^3 + 1$ . By [17, Table 8.43], these subgroups are ordinary and, clearly,  $l(S) < |P|$  and  $|\text{Out } S| = f \leq \log q \leq \log l(S)$ .

In [12, Theorem 3], it is shown that if  $S \cong {}^3D_4(q)$ , with  $q = p^f$ , the smallest index maximal subgroups of  $S$  are isomorphic to  $P \cong (p^f.p^{8f}) : (d.(\text{PSL}_2(q^3) \times (q - 1)/d).d)$ , where  $d = \gcd(2, q - 1)$ , with index  $l(S) = (q^8 + q^4 + 1)(q + 1)$ . By [17, Table 8.51], these subgroups are ordinary. Moreover,  $|P| = dq^{12}(q^6 - 1)(q - 1) = dq^{12}(q - 1)^2(q + 1)(q^4 + q^2 + 1) > l(S)$  and  $|\text{Out } S| = f \leq \log q \leq \log l(S)$ .

In [12, Theorem 4], it is shown that for  $S \cong {}^2E_6(q)$ , with  $q = p^f$ , the smallest index maximal subgroups of  $S$  are isomorphic to  $P \cong (p^f.p^{20f}) : (d_+. \text{PSU}_6(q) \times (q - 1)/d'_+).d'_+$ , where  $d_+ = \gcd(2, q + 1)$ ,  $d'_+ = \gcd(3, q + 1)$ . Their index is  $l(S) = (q^{12} - 1)(q^6 - q^3 + 1)(q^4 + 1)/(q - 1)$ . These subgroups are clearly ordinary because they are parabolic. Clearly,  $|P| > l(S)$  and  $|\text{Out } S| = \gcd(3, q + 1) \cdot f \cdot 1 \leq 3f \leq \log q^{11} \leq \log l(S)$ .

By [12, Theorem 5], if  $S \cong {}^2F_4(q)$ , with  $q = 2^f$ ,  $f > 1$  odd, the smallest index maximal subgroups of  $S$  are isomorphic to  $P \cong (2^f.2^{4f}.2^{5f}) : ({}^2B_2(q) \times (q - 1))$ , with index  $l(S) = (q^6 + 1)(q^3 + 1)(q + 1)$  and order  $|P| = q^{12}(q^2 + 1)(q - 1)^2$ . It is clear that

$|P| < l(S)$  and  $|\text{Out } S| = f \leq \log q \leq \log l(S)$ . Moreover, the subgroup  $P$  is ordinary because it is a parabolic subgroup.  $\square$

**Remark 2** We thank one of the anonymous referees for drawing our attention to the interesting paper [21] of Alavi and Burness. These authors have obtained in their Theorems 2–5 for each simple group  $G$  and in their Theorem 7 for each almost simple group  $G$  the list of all maximal subgroups  $H$  of  $G$  with  $|H|^3 \geq |G|$ . They call them *large*. In fact, all maximal subgroups appearing in the proof of Theorem A are large in this sense and so all of them are mentioned in [21].

**Remark 3** Note that the smallest index of a smallest core-free maximal subgroup of an almost simple group with socle  $\text{PSL}_n(q)$  with  $n \geq 3$  can be different from the indices of the parabolic and the double parabolic subgroups. According to [6], if  $S = \text{PSL}_3(4)$ , then the extension  $S.2_1$  contains a maximal subgroup of type  $M_{10}$  and least index 56, different from the indices of the parabolic subgroups of type  $P_1$ , of index 21, that do not exist in this extension, and the double parabolic subgroups of type  $P_{1,2}$ , of index 105, that also appear as a maximal subgroup of  $S.2_1$ .

**Remark 4** Let  $S = \text{PSL}_n(2) \cong \text{GL}_n(2)$ , where  $n$  is a prime,  $n \geq 5$ . There is a unique class of ordinary maximal subgroups of  $S$  of geometric type by Tables 8.18, 8.19, 8.36, 8.37, 8.70, 8.71 of [17] and [18, Table 3.5.A], namely the subgroup  $M = \text{GL}_1(2^n) : n$  in the Aschbacher class  $\mathcal{C}_3$ . Note that  $|S| = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1})$ , while  $|M| = (2^n - 1)n$ . Consequently,  $|S : M| = (2^n - 2) \cdots (2^n - 2^{n-1})/n$ . The smallest index of a core-free maximal subgroup of  $S$  is smaller or equal than the index of the parabolic subgroup  $P_1$ , corresponding to the stabiliser of a vector subspace of dimension 1. Since  $|P_1| = 2^{n-1}(2^{n-1} - 1)(2^{n-1} - 2) \cdots (2^{n-1} - 2^{n-2})$ , we have that  $|S : P_1| = 2^n - 1$ . Now the largest power of 2 dividing  $|S : M|$  is  $2 \cdot 2^2 \cdots 2^{n-1} = 2^{n(n-1)/2}$ . Therefore  $|S : M| = (2^n)^{(n-1)/2} > (2^n - 1)^{(n-1)/2} \geq l(S)^{(n-1)/2}$ . In particular, there cannot exist a constant  $c$  such that if  $l_1(S)$  is the common index of a maximal subgroup of geometric type in all almost simple groups associated with the non-abelian simple group  $S$ ,  $l_1(S) \leq l(S)^c$  for all non-abelian simple groups  $S$ .

**Remark 5** The groups  $\text{PSL}_n(q)$  for  $n \geq 3$ ,  $q = q_0^2$ ,  $q_0$  a prime power, contain always a maximal subgroup of the form  $\text{PSL}_n(q_0)$  or  $\text{PSU}_n(q_0)$ , but their indices in  $\text{PSL}_n(q)$  are polynomials on  $q_0$  of degree larger than the degree of  $l(\text{PSL}_n(q))^2 = (q_0^{2n} - 1)^2 / (q_0^2 - 1)^2$  when  $n \geq 4$ . Hence this construction cannot be extended further. The groups  $\text{PSL}_{2m}(q)$  for  $m \geq 2$ ,  $q$  a prime power,  $(2m, q) \neq (4, 2)$ , contain a parabolic subgroup  $P_m$  that is ordinary. However, for  $m \geq 4$ , this subgroup has index in  $\text{PSL}_{2m}(q)$  that is a polynomial on  $q$  of degree larger than the one of  $l(\text{PSL}_{2m}(q))^2 = (q^{2m} - 1)^2 / (q - 1)^2$ . This justifies that these constructions cannot be extended further and so  $\text{PSL}_3(q_0^2)$ ,  $\text{PSL}_4(q)$ ,  $\text{PSL}_6(q)$  belong to the class  $\mathfrak{Q}$ , but not the linear groups in larger dimensions.

**Remark 6** Let  $S = \text{PSL}_m(2^f)$  with  $m \geq 3$  and  $m \mid 2^f - 1$  (for example,  $m - 1 \mid f$ ). Then  $|\text{Out } S| = (2^f - 1) \cdot f \cdot 2$  and  $l(S) = (2^{mf} - 1) / (2^f - 1)$ . Since

$$\lim_f \frac{\log((2^{mf} - 1) / (2^f - 1))}{m \cdot f \cdot 2} = \frac{m - 1}{2m},$$

it follows that the bound  $|\text{Out } S| \leq 3 \log l(S)$  cannot be improved.

**Proof of Theorem B** The result is clear for sporadic and alternating groups, since then the outer automorphism group is trivial, isomorphic to  $C_2$ , or isomorphic to  $C_2 \times C_2$ . It only

remains to consider the case when  $S$  is a simple group of Lie type. According to [6, Table 5], the outer automorphism group is isomorphic to an extension of a metacyclic group by a cyclic group, with the possible exception of  $O_{2m}^+(q)$  with  $m \geq 4$ ,  $m$  even. By [22, page 181],

$$\text{Out}(O_8^+(q)) \cong \begin{cases} \text{Sym}(3) \times C_f & \text{if } q \text{ is even,} \\ \text{Sym}(4) \times C_f & \text{if } q \text{ is odd.} \end{cases}$$

If  $m \geq 6$  is even, the same arguments show that

$$\text{Out}(O_{2m}^+(q)) \cong \begin{cases} D_8 \times C_f & \text{if } q \text{ is even,} \\ C_2 \times C_f & \text{if } q \text{ is odd.} \end{cases}$$

By considering the normal subgroup isomorphic to  $C_f$ , we see that all subgroups of  $\text{Out}(O_{2m}^+(q))$  for  $m \geq 4$ ,  $m$  even, can also be generated by at most 3 generators. In all other cases, all subgroups of  $\text{Out } S$  are extensions of a metacyclic group by a cyclic group and so they are also 3-generated.

If  $|\text{Out } S| \leq \log l(S)$ , then given a subgroup of  $G$ , we have at most  $\log^3 l(S)$  possibilities for a generating set. It follows that the number of subgroups of  $\text{Out } S$  is at most  $\log^3 l(S)$ . Therefore we must study the cases in which the inequality  $|\text{Out } S| \leq \log l(S)$  can fail, that is, the cases mentioned in Theorem A (5).

We begin with the linear groups  $S = \text{PSL}_m(q)$ . Suppose first that  $m = 2$  and that  $q = 3^f$ . Then

$$|\text{Out } S| = 2 \cdot f \cdot 1 = \frac{2}{\log 3} \log 3^f \leq \frac{2}{\log 3} \log l(S).$$

Note that  $\text{Out } S$  and all its subgroups are 2-generated. One of the generators can be taken in the subgroup of order 2, while the other one can be chosen in  $2f$  different ways. This gives for the number of subgroups an upper bound of

$$4f \leq 2 \cdot \frac{2}{\log 3} \log l(S) = \frac{4}{\log 3} \log l(S).$$

Since  $4/\log 3 \leq \log^2 5 \leq \log^2 l(S)$ , we conclude that the number of subgroups of  $\text{Out } S$  is bounded by  $\log^3 l(S)$ .

Let  $S = \text{PSL}_m(q)$  with  $m \geq 3$ . Then

$$\begin{aligned} |\text{Out } S| &= \gcd(m, q - 1) \cdot f \cdot 2 \leq 2mf \\ &= \frac{2m}{(m - 1) \log p} \log p^{(m-1)f} \leq \frac{2m}{(m - 1) \log p} \log l(S). \end{aligned}$$

Note that

$$\frac{2m}{(m - 1) \log p} \leq 3.$$

Suppose that  $f = 1$ . Since  $l(\text{PSL}_3(2)) = 7$  and  $|\text{Out } \text{PSL}_3(2)| = 2$ , and  $l(\text{PSL}_4(2)) = 8$  and  $|\text{Out } \text{PSL}_4(2)| = 2$ , we can assume that  $(m, q) \notin \{(3, 2), (4, 2)\}$ . Consequently,  $l(S) = (q^m - 1)/(q - 1) = q^{m-1} + q^{m-2} + \dots + q + 1$  and so  $\log l(S) > (m - 1) \log q$ . We have that  $\text{Out } S$  is 2-generated and has order  $\gcd(m, q - 1) \cdot 2$ . Moreover, all subgroups of  $\text{Out } S$  are 2-generated, the first generator can be taken in the normal cyclic subgroup of order  $\gcd(m, q - 1)$  and the second one in  $\text{Out } S$ . This gives at most  $\gcd(m, q - 1)^2 \cdot 2$

possibilities for a subgroup of  $\text{Out } S$ . Since  $(m - 1)^3 \log^3 q - 2m^2 > 0$  for  $m \geq 3$  if  $q \geq 3$  and  $(m - 1)^3 \log^3 2 - 2m^2 = (m - 1)^3 - 2m^2 > 0$  for  $m \geq 5$ , we have that

$$\gcd(m, q - 1)^2 \cdot 2 \leq 2m^2 \leq (m - 1)^3 \log^3 q \leq \log^3 l(S)$$

for  $m \geq 3, (m, q) \notin \{(3, 2), (4, 2)\}$ .

Hence we can assume that  $f \geq 2$ . Every subgroup of  $\text{Out } S$  is 3-generated, and the generators can be taken one in the group of diagonal automorphisms of order  $\gcd(m, q - 1)$ , another one in the group generated by the diagonal and field automorphisms of order  $\gcd(m, q - 1)f$ , and the third one in  $\text{Out } S$ . It follows that the number of possible choices is at most  $2f^2 \gcd(m, q - 1)^3$ . Suppose first that  $\gcd(m, q - 1) < m$ , then  $\gcd(m, q - 1) \leq m/2$  and so the number of possible subgroups is bounded by

$$2f^2 \frac{m^3}{8} \leq \frac{1}{8} \frac{(2fm)^3}{8} \leq \frac{1}{8} \frac{m^3}{(m - 1)^3 \log^3 p} \log^3 l(S) < \log^3 l(S).$$

Therefore we can assume that  $\gcd(m, q - 1) = m$ , that is,  $m \mid q - 1$ . The number of possible subgroups of  $\text{Out } S$  is bounded by

$$2f^2 m^3 \leq \frac{(2fm)^3}{8} \leq \frac{m^3}{(m - 1)^3 \log^3 p} \log^3 l(S).$$

If  $p \geq 3$ , then  $m^3 < (m - 1)^3 \log^3 p$  and so the number of possible subgroups of  $\text{Out } S$  is again bounded by  $\log^3 l(S)$ . Therefore we can suppose that  $p = 2$ . In particular,  $m$  must be odd. Assume that  $m$  is a prime. Then the number of choices of the element of the group of diagonal automorphisms can be reduced from  $m$  to 2, namely the trivial element and a generator. This gives that the number of possible subgroups of  $\text{Out } S$  is bounded by

$$2 \cdot mf \cdot 2mf = (2mf)^2 \leq \frac{4m^2}{(m - 1)^2} \log^2 l(S).$$

If  $m = 5$ , then  $l(S) \geq (2^4)^4 = 2^{16}$ , and so  $\log l(S) \geq 16$ . It follows that  $4m^2/(m - 1)^2 < \log l(S)$ . If  $m = 7$ , then  $l(S) \geq (2^3)^6 = 2^{18}$  and so  $\log l(S) \geq 18$ . Consequently,  $4m^2/(m - 1)^2 < \log l(S)$ . Hence for  $m \in \{5, 7\}$ , the number of subgroups of  $\text{Out } S$  is bounded by  $\log^3 l(S)$ . Assume now that  $m \geq 9$  is odd. The number of choices of the element of the group of diagonal automorphisms can be reduced to the number of subgroups of this cyclic group, which coincides with the number of divisors of  $m$ . Since  $m$  is odd, this number is not greater than  $2m/3$ . The number of possible choices for the generators of a subgroup of  $\text{Out } S$  is bounded by

$$(2m/3) \cdot mf \cdot 2mf = \frac{1}{6f} (2mf)^3 \leq \frac{1}{12} \frac{8m^3}{(m - 1)^3} \log^3 l(S),$$

and  $8m^3/(12(m - 1)^3) \leq 243/256$ , so that this number is bounded by  $\log^3 l(S)$ . It only remains the case  $m = 3$ . In this case, the group of outer automorphisms of  $S$  has the presentation

$$\text{Out } S = \langle x, y, z \mid x^3 = y^f = z^2 = 1, x^y = x^{-1}, x^z = x^{-1}, y^z = y^{-1} \rangle.$$

Note that  $\langle y^2 \rangle$  centralises  $\langle x \rangle$ . Now

$$\text{Out } S / \langle y^2 \rangle \cong \langle a, b \mid a^3 = b^2 = 1, a^b = a^{-1} \rangle \times \langle c \mid c^2 = 1 \rangle \cong \text{Sym}(3) \times C_2,$$

where  $a = \bar{x}, b = \bar{z}, c = \bar{y}\bar{z}$ . Note that every subgroup of  $\text{Sym}(3) \times C_2$  is 2-generated. A pair of generators can be obtained by taking an element of  $\langle a, b \rangle$  and an element of the set

$\{1, c, ac, bc, abc, a^2bc\}$ . The preimages of these sets under the natural epimorphism from  $\text{Out } S$  onto  $(\text{Out } S)/\langle y^2 \rangle$  have  $6(f/2) = 3f$  elements each. Hence every element of  $\text{Out } S$  can be obtained by considering an element of  $\langle y^2 \rangle$ , for which we have  $f/2$  choices, and the  $3f$  choices for each element of the preimages. This gives a bound for the number of subgroups of  $(f/2)(3f)^2 = 9f^3/2 = 9(2f)^3/16 < (9/16) \log^3 l(S) < \log^3 l(S)$ . This completes the proof for the linear case.

If  $S \cong \text{PSU}_3(5)$ , then  $\text{Out } S \cong \text{Sym}(3)$  has 6 subgroups, clearly  $6 \leq \log^3 50 = \log^3 l(S)$  by [6]. Suppose that  $S \cong \text{PSU}_3(q)$  with  $q \notin \{2, 5\}$ . Then  $l(S) = q^3 + 1$  by [8, Theorem 3] and  $\text{Out } S$  is a metacyclic group of order  $3 \cdot 2f = 6f$  and all its subgroups are also 2-generated. Then the number of choices for a couple of generators for a subgroup of  $G$  is bounded by  $3f \cdot 6f \leq (2/3) \log l(S) \cdot (4/3) \log l(S) < \log^2 l(S) < \log^3 l(S)$ . Assume now that  $S \cong \text{PSU}_m(q)$  with  $q > 2$  if  $m$  is even and  $q^2 = p^f$ , with  $p$  a prime. By [8, Theorem 3],  $l(S) = (q^m - (-1)^m)(q^{m-1} - (-1)^{m-1})/(q^2 - 1)$ . Moreover,  $|\text{Out } S| = m \cdot (2f) \cdot 1$  is 2-generated. Since  $2mf \leq 3 \log l(S)^2$ , we have that the number of choices for the pair of generators of a subgroup of  $\text{Out } S$  is bounded by  $mf \cdot 2mf \leq (9/2) \log^2 l(S) \leq \log^3 l(S)$ , because  $9/2 \leq \log 28 = \log^3 1(\text{PSU}_3(3)) < \log l(S)$  by [6].

Finally, suppose that  $G \cong \text{O}_8^+(q)$ , where  $q = p^f$  and  $p \in \{3, 5, 7, 11, 13\}$ . Now  $l(S) = (q^3 + q^2 + q + 1)(q^3 + 1)$  by [9, Theorem] and  $\text{Out } S \cong \text{Sym}(4) \times C_f$ . All subgroups of  $\text{Out } S$  are 3-generated, and one generator can be taken in  $C_f$ , another one in  $\text{Alt}(4) \times C_f$  and the other one in the whole group. This gives at most  $f \cdot (12f) \cdot (24f) = 288f^3$  possibilities for a subgroup of  $\text{Out } S$ . Since  $24f \leq 3 \log l(S)$ , we have that  $8f \leq \log l(S)$ . Consequently,  $288f^3 \leq 512f^3 \leq \log^3 l(S)$ . This completes the proof of the theorem.  $\square$

**Remark 7** We note that in the Janko sporadic group  $S \cong J_3$  of order 50 232 960 and with  $l(S) = 6 156$ , if  $\xi = \log 50 232 960 / \log 6 156 \approx 2.0323$ , we have that  $|S| = l(S)^\xi$ . Therefore the exponent 2 in  $l(S)^2 \leq |S|$  cannot be increased too much.

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## Declarations

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## References

- Baer, R.: Classes of finite groups and their properties. *Illinois J. Math.* **1**, 115–187 (1957). <https://doi.org/10.1215/IJM/1255379396>
- Ballester-Bolínches, A., Ezquerro, L.M.: *Classes of Finite Groups. Mathematics and Its Applications*, vol. 584. Springer, Dordrecht (2006). <https://doi.org/10.1007/1-4020-4719-3>
- Ballester-Bolínches, A., Esteban-Romero, R., Jiménez-Seral, P.: Bounds on the number of maximal subgroups of finite groups: applications. *Mathematics* **10**, 1153–25 (2022). <https://doi.org/10.3390/math10071153>
- Borovik, A.V., Pyber, L., Shalev, A.: Maximal subgroups in finite and profinite groups. *Trans. Amer. Math. Soc.* **348**(9), 3745–3761 (1996). <https://doi.org/10.1090/S0002-9947-96-01665-0>
- Guralnick, R.M., Maróti, A., Pyber, L.: Normalizers of primitive permutation groups. *Adv. Math.* **310**, 1017–1063 (2017). <https://doi.org/10.1016/j.aim.2017.02.012>
- Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: *Atlas of Finite Groups*. Oxford Univ. Press, London (1985)
- Doerk, K., Hawkes, T.: Finite Soluble Groups. *De Gruyter Expositions in Mathematics*, vol. 4, p. 891. Walter de Gruyter & Co., Berlin (1992). <https://doi.org/10.1515/9783110870138>
- Mazurov, V.D.: Minimal permutation representations of finite simple classical groups. *Special linear, symplectic, and unitary groups. Algebra Logic* **32**(3), 142–153 (1993). <https://doi.org/10.1007/BF02261693>
- Vasil'ev, V.A., Mazurov, V.D.: Minimal permutation representations of finite simple orthogonal groups. *Algebra Logic* **33**(6), 1994 (1994). <https://doi.org/10.1007/BF00756348>
- Vasilyev, A.V.: Minimal permutation representations of finite simple exceptional groups of types  $G_4$  and  $F_4$ . *Algebra Logic* **35**(6), 371–383 (1996). <https://doi.org/10.1007/BF02366397>
- Vasilyev, A.V.: Minimal permutation representations of finite simple exceptional groups of types  $E_6$ ,  $E_7$ , and  $E_8$ . *Algebra Logic* **36**(5), 302–310 (1997). <https://doi.org/10.1007/bf02671607>
- Vasilyev, A.V.: Minimal permutation representations of finite simple exceptional twisted groups. *Algebra Logic* **37**(1), 9–20 (1998). <https://doi.org/10.1007/BF02684081>
- Linton, S.A., Wilson, R.A.: The maximal subgroups of the Fischer groups  $Fi_{24}$  and  $Fi'_{24}$ . *Proc. Lond. Math. Soc.* **3**(63), 113–164 (1991). <https://doi.org/10.1112/plms/s3-63.1.113>
- Wilson, R.A.: The maximal subgroups of the Baby Monster. *I. J. Algebra* **211**, 1–14 (1999). <https://doi.org/10.1006/jabr.1998.7601>
- Wilson, R.A.: New computations in the Monster. In: Lepowsky, J., McKay, J., Tuite, M.P. (eds.) *Moonshine: the First Quarter Century and Beyond*. *London Math. Soc. Lecture Note Ser.*, vol. 372, pp. 393–403. Cambridge Univ. Press, Cambridge (2010)
- Norton, S.P., Wilson, R.A.: A correction to the 41-structure of the Monster, a construction of a new maximal subgroup  $L_2(41)$  and a new Moonshine phenomenon. *J. Lond. Math. Soc.* **2**(87), 943–962 (2013). <https://doi.org/10.1112/jlms/jds078>
- Bray, J., Holt, D., Roney-Dougal, C.: *The Maximal Subgroups of the Low-dimensional Finite Classical Groups*. *London Math. Soc. Lect. Note Ser.*, vol. 407. Cambridge Univ. Press, Cambridge, UK (2013). <https://doi.org/10.1017/CBO9781139192576>
- Kleidman, P.B., Liebeck, M.W.: *The Subgroup Structure of the Finite Classical Groups*. *London Math. Soc. Lecture Notes Series*, vol. 129. Cambridge Univ. Press, Cambridge, UK (1990). <https://doi.org/10.1017/CBO9780511629235>
- Liebeck, M.W., Saxl, J., Seitz, G.M.: Subgroups of maximal rank in finite exceptional groups of Lie type. *Proc. London Math. Soc.* **3**(65), 297–325 (1992). <https://doi.org/10.1112/plms/s3-65.2.297>
- Craven, D.A.: The maximal subgroups of the exceptional groups  $F_4(q)$ ,  $E_6(q)$  and  ${}^2E_6(q)$  and related almost simple groups. 10.48550
- Alavi, S.H., Burness, T.C.: Large subgroups of simple groups. *J. Algebra* **421**, 187–233 (2015). <https://doi.org/10.1016/j.jalgebra.2014.08.026>

22. Kleidman, P.B.: The maximal subgroups of the finite 8-dimensional orthogonal groups  $P\Omega^+(q)$  and of their automorphism groups. *J. Algebra* **110**(1), 173–242 (1987). [https://doi.org/10.1016/0021-8693\(87\)90042-1](https://doi.org/10.1016/0021-8693(87)90042-1)

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