GROUPS WHOSE SUBGROUPS SATISFY THE WEAK SUBNORMALIZER CONDITION*

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Abstract

A subgroup X of a group G is said to satisfy the *weak subnormalizer condition* if $N_G(Y) \leqslant N_G(X)$ for each non-normal subgroup Y of G such that $X \leqslant Y \leqslant N_G(X)$. The behaviour of generalized soluble groups whose (cyclic) subgroups satisfy the weak subnormalizer condition is investigated.

Mathematics Subject Classification (2010): 20E15, 20F16
Keywords: weak subnormalizer condition; T-group; weakly radical group

1 Introduction

A group G is said to be a T-*group* (or to have the T-*property*) if normality in G is a transitive relation, i.e. if all subnormal subgroups of G are normal. The structure of soluble T-groups was described by W. Gaschütz [3] in the finite case and by D.J.S. Robinson [9] for arbitrary groups. In particular, it was proved that all soluble groups with the T-property are metabelian and locally supersoluble, and that a finitely generated soluble T-group is either finite or abelian. Obviously, any simple group has the T-property, so that the class of T-groups is not subgroup closed, and a group G is called a $\overline{T}$-*group* if all subgroups of G have the T-property. It is known that every finite soluble T-group is a $\overline{T}$-group and that all finite groups with the $\overline{T}$-property are soluble. Recall also that a subgroup X satisfies the *subnormalizer condition* if it is normal in the normalizer $N_G(Y)$ of every subgroup Y of G such that $X \leqslant Y \leqslant N_G(X)$ (see [4],[6],[8] and also the recent survey [2], where subgroups satisfying the subnormalizer condition were considered, although under different denominations). Of course, self-normalizing subgroups and Sylow subgroups of arbitrary groups satisfy the subnormalizer condition, and it turns out that a group G has the $\overline{T}$-property if and only if all subgroups of G satisfy the subnormalizer condition.

We shall say that a subgroup X of a group G satisfies the *weak subnormalizer condition* if $N_G(Y) \leqslant N_G(X)$ whenever Y is a non-normal subgroup of G and $X \leqslant Y \leqslant N_G(X)$. It is clear that if the subgroup X satisfies the weak subnormalizer condition, then its normalizer $N_G(X)$ is either self-normalizing or normal in G. In

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particular, if X is a non-subnormal subgroup of G which satisfies the weak subnormalizer condition, then $N_G(N_G(X)) = N_G(X)$.

The aim of this paper is to study the class $\mathcal{W}$ of all groups in which every subgroup satisfies the weak subnormalizer condition, and the class $\mathcal{W}_c$ consisting of all groups whose cyclic subgroups satisfy the weak subnormalizer condition. It is quite obvious that subgroups and homomorphic images of $\mathcal{W}$-groups likewise belong to $\mathcal{W}$, while the the class $\mathcal{W}_c$ is closed with respect to subgroups but not with respect to homomorphic images. In fact, although the infinite dihedral group obviously has the $\mathcal{W}_c$-property, for each positive integer $n \geqslant 4$ the dihedral group of order 2^n does not belong to $\mathcal{W}_c$.

In order to avoid *Tarski groups* (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) and other similar pathologies that belong to the class $\mathcal{W}$, our main results will be proved within a suitable universe of generalized soluble groups; we shall say that a group G is *weakly radical* if it has an ascending normal series whose factors are either locally soluble or locally finite. The class of weakly radical groups is quite large, and contains in particular all locally soluble groups and locally finite groups. Notice also that any periodic weakly radical group is locally finite, because every group admitting an ascending series with locally finite factors is itself localy finite (see for instance [11] Part 1, p.35).

Most of our notation is standard and can be found in [11].

2 $\mathcal{W}_c$-groups

Recall that a subgroup X of a group G is said to be *ascendant* if there exists an ascending series from X to G. Clearly, all subnormal subgroups of an arbitrary group are ascendant, while subnormal subgroups and ascendant subgroups coincide in any finite group. It is also easy to show that all subgroups of hypercentral groups are ascendant.

Lemma 2.1 *Let G be a group, and let X be an ascendant subgroup of G which satisfies the weak subnormalizer condition. Then X is subnormal in G with defect at most 2.*

Proof — Assume for a contradiction that the statement is false, and let

$$X = X_0 \lhd X_1 \lhd \ldots \lhd X_\alpha \lhd X_{\alpha+1} \lhd \ldots \lhd \ldots X_\tau = G$$

be an ascending series from X to G of shortest length τ. Let $\mu \leqslant \tau$ be the first ordinal such that X is not normal in X_μ. Then μ cannot be a limit ordinal, and X is a normal subgroup of $X_{\mu-1}$. On the other hand, X is not normal in the normalizer $N_G(X_{\mu-1})$, and hence $X_{\mu-1}$ is normal in G because X satisfies the weak subnormalizer condition. Therefore X is subnormal in G with defect at most 2, and this contradiction proves the statement. □

Lemma 2.2 *Let* G *be a group, and let* X *be a periodic locally nilpotent subgroup of* G *whose primary components satisfy the weak subnormalizer condition. Then also* X *satisfies the weak subnormalizer condition.*

Proof — Let Y be a non-normal subgroup of G such that $X \leqslant Y \leqslant N_G(X)$. Each Sylow subgroup of X is characteristic in X, so that it is normal in Y, and hence even normal in $N_G(Y)$. It follows that also X itself is a normal subgroup of $N_G(Y)$, whence X satisfies the weak subnormalizer condition. □

Corollary 2.3 *Let* G *be a periodic group in which every cyclic subgroup of prime-power order satisfies the weak subnormalizer condition. Then* G *is a* $\mathcal{W}_c$*-group.*

Lemma 2.4 *Let* G *be a group, and let* $\langle x\rangle$ *be a cyclic subgroup of* G *satisfying the weak subnormalizer condition. Then either the centralizer* $C_G(x)$ *is a normal subgroup of* G *or* $N_G(\langle x\rangle) = N_G\big(C_G(x)\big)$.

Proof — Suppose that the centralizer $C_G(x)$ is not normal in G, so that $\langle x\rangle$ is a normal subgroup of $N_G\big(C_G(x)\big)$ by the weak subnormalizer condition, and hence $N_G\big(C_G(x)\big) \leqslant N_G(\langle x\rangle)$. On the other hand, the subgroup $C_G(x) = C_G(\langle x\rangle)$ is obviously normal in $N_G(\langle x\rangle)$, and so $N_G(\langle x\rangle) = N_G\big(C_G(x)\big)$. □

Our next resut shows that groups with a *very small* commutator subgroup have the $\mathcal{W}$-property.

Lemma 2.5 *Let* G *be a group whose commutator subgroup* G' *has prime order. Then* G *is a* $\mathcal{W}$*-group.*

Proof — Let X and Y be subgroups of G such that $X \leqslant Y \leqslant N_G(X)$ and Y is not normal in G. Then Y cannot contain G', and so

$$[Y, N_G(Y)] \leqslant Y \cap G' = \{1\}.$$

It follows that $N_G(Y) = C_G(Y)$, so that in particular X is normal in $N_G(Y)$, and hence G belongs to the class $\mathcal{W}$. □

If G is any group, the subgroup $H(G)$ generated by all locally nilpotent normal subgroups of G is called the *Hirsch–Plotkin radical* of G. It is well known that $H(G)$ is likewise locally nilpotent, and contains all locally nilpotent ascendant subgroups of G.

Lemma 2.6 *Let* G *be a* $\mathcal{W}_c$*-group, and let* $H = H(G)$ *be the Hirsch–Plotkin radical of* G*. Then* H *is nilpotent of class at most* 3 *and all cyclic subgroups of* H *are subnormal in* G *of defect at most* 2*.*

Proof — Let E be any finitely generated subgroup of H. As E is nilpotent, it follows from Lemma 2.1 that all cyclic subgroups of E have defect at most 2. Then E has nilpotency class at most 3 by a result of H. Heineken [5] and S.K. Mahdavianary [7], and hence H itself is nilpotent and has class at most 3. In particular, all cyclic subgroups of H are subnormal in G, and so a further application of Lemma 2.1 yields that they have defect at most 2 in G. □

Corollary 2.7 *Let G be a locally nilpotent $\mathcal{W}_c$-group. Then G is nilpotent of class at most 3.*

The consideration of the quaternion group Q_{16} of order 16 shows that a finite 2-group with the $\mathcal{W}_c$-property may have nilpotency class 3. On the other hand, our next result proves that the situation is much better in the case of torsion-free groups.

Theorem 2.8 *Let G be a torsion-free locally nilpotent $\mathcal{W}_c$-group. Then G is abelian.*

Proof — The group G is nilpotent by Corollary 2.7. If x is any element of G, the normalizer $N_G(\langle x\rangle)$ must be normal in G, since it is subnormal and $\langle x\rangle$ satisfies the weak subnormalizer condition. On the other hand, G does not have infinite dihedral sections, and hence the centralizer $C_G(x) = N_G(\langle x\rangle)$ is a normal subgroup of G. Therefore G is a 2-Engel group, and so it has nilpotency class at most 2 (see [11] Part 2, Theorem 7.15).

Assume for a contradiction that the statement is false, so that there are elements a and b of G such that $[a,b] \neq 1$. Clearly, we may suppose that $G = \langle a,b\rangle$ is a 2-generator group. Fix two coprime integers $m,n > 1$. As

$$[a^m, b^n] = [a,b]^{mn} \neq 1,$$

the element b^n cannot normalize $\langle a^m\rangle$. On the other hand,

$$\langle a^m\rangle \lhd \langle a^m, [a^m,b^n]\rangle \lhd \langle a^m, b^n\rangle,$$

and hence it follows from the $\mathcal{W}_c$-property that the subgroup $\langle a^m, [a^m,b^n]\rangle$ is normal in G. A similar argument shows that $\langle b^n, [a^m,b^n]\rangle$ is normal in G, so that also $\langle a^m, b^n\rangle$ is a normal subgroup of G. Since m and n are coprime, the factor group $G/\langle a^m,b^n\rangle$ is abelian, and so

$$[a,b] = a^{mr}b^{ns}[a,b]^{mnt}$$

for suitable integers r,s,t. Thus

$$\begin{aligned}[a,b]^{1-mnt} &= a^{-1}[a,b]^{1-mnt}a = a^{-1}a^{mr}b^{ns}a \\ &= a^{mr}b^{ns}a^{-1}[a^{-1},b^{ns}]a = [a,b]^{1-mnt}[a,b]^{-ns},\end{aligned}$$

so that $[a,b]^{-ns} = 1$ and hence $s = 0$. Similarly, we obtain that $r = 0$, whence

$$[a,b] = [a,b]^{mnt},$$

which is of course impossible. This contradiction proves the statement. □

Notice that the assumption that the group G is locally nilpotent in the above statement cannot be weakened. To prove this, consider the semidirect product $G = \langle y \rangle \ltimes \langle x \rangle$, where $\langle x \rangle$ and $\langle y \rangle$ are infinite cyclic groups and $x^y = x^{-1}$. Then G is a torsion-free non-abelian $\mathcal{W}_c$-group, which is also metabelian and supersoluble.

Moreover, it is known that there exist simple non-abelian groups whose proper non-trivial subgroups are infinite cyclic, so that torsion-free $\overline{T}$-groups need not be abelian. On the other hand, Theorem 2.8 can be improved at least when the group G belongs to the relevant subclass T_c of $\mathcal{W}_c$. Here a group G is called a T_c-*group* if every cyclic subnormal subgroup of G is normal, while G is said to be a $\overline{T}_c$-*group* if all its subgroups have the T_c-property. It is clear that every group with the $\overline{T}_c$-property belongs to the class $\mathcal{W}_c$. The class of T_c-groups was first studied by M. Xu [14] and T. Sakamoto [13], who proved that soluble T_c-groups are metabelian and locally supersoluble, while D.J.S. Robinson [12] provided later a full description of finite T_c-groups.

Theorem 2.9 *Let G be a torsion-free weakly radical group with the $\overline{T}_c$-property. Then G is abelian.*

Proof — Assume for a contradiction that the statement is false, and suppose first that G contains a soluble non-abelian subgroup X. If Y is the Fitting subgroup of X, we have that all subgroups of Y are normal in X, so that $Y = C_X(Y)$ is abelian and X/Y is isomorphic to a group of power automorphisms of Y. In particular, X/Y has order 2 and $y^x = y^{-1}$ for all elements y of Y and $x \in X \setminus Y$, so that $x^4 = 1$. This contradiction shows that all soluble subgroups, and so even all locally soluble subgroups of G are abelian.

Let A be a maximal abelian normal subgroup of G. As G is weakly radical and A is a proper subgroup of G, there exists a non-trivial normal subgroup K/A of G/A which is either locally soluble or locally finite. Clearly, K cannot be locally soluble, and hence K/A must be locally finite. If g is any element of K, the subgroup $\langle g, A \rangle$ is soluble, and so abelian. Thus A is contained in $Z(K)$, so that $K/Z(K)$ is locally finite and hence also the commutator subgroup K' is locally finite (see [11] Part 1, p.102). It follows that $K' = \{1\}$, and K is abelian. This contradiction completes the proof of the statement. □

In relation to Theorem 2.9, we mention that it is an open question whether torsion-free locally graded $\overline{T}$-groups must be abelian. Here a group G is called *locally graded* if each finitely generated non-trivial subgrup of G contains a proper subgroup of finite index.

Following Robinson [10] (see also [1], Chapter 2), we shall say that a group G satisfies the property $\mathcal{C}_p$, where p is any prime number, if it contains a Sylow p-subgroup P such that all subgroups of P are normal in $N_G(P)$. It is clear that if G is any finite $\mathcal{C}_p$-group and N is a normal subgroup of G whose order is prime to p, then also the factor group G/N has the $\mathcal{C}_p$-property.

For our purposes, we need the following lemma on finite $\overline{T}$-groups, that was proved in [10].

Lemma 2.10 *A finite group G has the $\mathcal{C}_p$-property for al prime numbers p if and only if it is a soluble T-group.*

Lemma 2.11 *Let G be a finite group, and p be a prime number such that all cyclic p-subgroups of G satisfy the weak subnormalizer condition. If F is the Fitting subgroup of G, then the factor group G/F satisfies the $\mathcal{C}_p$-property.*

Proof — Let P be a Sylow p-subgroup of G. If P is normal in G, then it is contained in F and so G/F obviously satisfies the $\mathcal{C}_p$-property because its order is prime to p. Suppose now that P is not normal in G, or equivalently that P is not contained in F. Let x be any element of $P \setminus F$, and let

$$\langle x \rangle = X_0 < X_1 < \ldots < X_k = P$$

be a series from $\langle x \rangle$ to P. As $\langle x \rangle$ satisfies the weak subnormalizer condition and each X_i is not normal in G, we have that the subgroup $\langle x \rangle$ is normal in the normalizer $N_G(P)$. It follows that $\langle x \rangle O_p(F)$ is normal in $N_G(P)$, and hence all cyclic subgroups of $P/O_p(F)$ are normal in $N_G(P)/O_p(F)$. Thus $G/O_p(F)$ is a $\mathcal{C}_p$-group. As

$$F = O_p(F) \times O_{p'}(F),$$

the group G/F has the $\mathcal{C}_p$-property. □

Corollary 2.12 *Let G be a finite $\mathcal{W}_c$-group, and let F be the Fitting subgroup of G. Then the factor group G/F has the $\overline{T}$-property.*

Proof — It follows from Lemma 2.11 that G/F satisfies the $\mathcal{C}_p$-property for each prime number p, and hence G/F is a $\overline{T}$-group by Lemma 2.10. □

We point out here that it was remarked by J.S. Rose that a finite group G has the $\mathcal{C}_p$-property if and only if all its p-subgroups are pronormal (see [10]), so that in particular subgroups and homomorphic images of a finite group with the $\mathcal{C}_p$-property likewise are $\mathcal{C}_p$-groups.

Clearly, Corollary 2.7 shows in particular that locally nilpotent $\mathcal{W}_c$-groups are metabelian. Our next result proves that also locally finite groups with the $\mathcal{W}_c$-property are soluble and have bounded derived length.

Theorem 2.13 *Let* G *be a locally finite* $\mathcal{W}_c$*-group. Then* G *is soluble and has derived length at most* 4.

Proof — Let E be any finite subgroup of G. It follows from Lemma 2.6 that the Fitting subgroup F of E has nilpotency class at most 3, so that it has derived length at most 2. Moreover, E/F is a $\overline{T}$-group by Corollary 2.12, and hence it is metabelian. Therefore E is soluble and has derived length at most 4, and the same conclusion obviously holds for G. □

Notice that the special linear group $SL(2,3)$ satisfies the $\mathcal{W}_c$-property and has derived length 3. On the other hand, w leave here as an open question whether there exist finite $\mathcal{W}_c$-groups of derived length 4.

3 $\mathcal{W}$-groups

The main purpose of this section is to prove that weakly radical $\mathcal{W}$-groups have a periodic commutator subgroup and are soluble of bounded derived length. Of course, in any group with a periodic commutator subgroup all torsion-free abelian normal subgroups lie in the centre, and in fact next lemma shows that this property holds for certain torsion-free abelian normal subgroups of $\mathcal{W}$-groups.

Lemma 3.1 *Let* G *be a* $\mathcal{W}$*-group, and let* A *be a torsion-free abelian normal subgroup of* G *such that* G/A *is periodic. Then* A *is contained in the centre of* G.

Proof — It is clearly enough to prove that the subgroup $\langle a, g\rangle$ is abelian for all elements a of A and g of G. Thus it can be assumed, without loss of generality, that $G = \langle a, g\rangle$, so that $A = \langle a\rangle^G$ is free abelian of finite rank and $G/A = \langle gA\rangle$ is a finite cyclic group. Moreover, if T is the largest periodic normal subgroup of G, we may replace G by the factor group G/T, and hence suppose that G has no periodic non-trivial normal subgroups.

Assume now that the statement is false, and choose a counterexample G such that the factor group G/A has smallest possible order, n say. Let p be a prime divisor of n. Then $\langle g^p, A\rangle$ is a proper normal subgroup of G, so that it is torsion-free abelian and hence $g^p \in A$. Therefore G/A has order p. For each positive integer k, we have that G/A^{p^k} is a finite p-group with the $\mathcal{W}$-property and hence it has nilpotency class at most 3 by Corollary 2.7. As

$$\bigcap_{k>0} A^{p^k} = \{1\},$$

it follows that G is a torsion-free nilpotent group, and hence it is abelian by Theorem 2.8. This contradiction proves the statement. □

Lemma 3.2 *Let G be a locally (soluble-by-finite) $\mathcal{W}$-group. Then the elements of finite order of G form a subgroup.*

Proof — Assume for a contradiction that G contains elements x and y of finite order such that the product xy has infinite order, and put $E = \langle x, y\rangle$. Since E is soluble-by-finite, it contains a largest soluble normal subgroup S, and the index $|E : S|$ is finite. Clearly, the counterexample G can be chosen in such a way that the derived length m of S is smallest possible. Write $A = S^{(m-1)}$. The minimal assumption on m applied to the $\mathcal{W}$-group E/A yields that E/A is periodic, and so even finite. Let a be any element of infinite order of A. Then a has only finitely many conjugates in G, so that the normal closure $\langle a\rangle^G$ is a finitely generated abelian normal subgroup, and in particular its largest periodic subgroup T is finite. Put $X = \langle x\rangle\langle a\rangle^G$ and $Y = \langle y\rangle\langle a\rangle^G$. It follows from Lemma 3.1 that X/T and Y/T are abelian, so that X and Y have finite commutator subgroups and hence they are even finite over their centres. Thus there exists a positive integer k such that a^k belongs to $Z(X) \cap Z(Y) \leqslant Z(E)$, so that $E/Z(E)$ is periodic and so even finite. Therefore the commutator subgroup E' of E is finite by the celebrated Schur's theorem, and hence E itself is finite. This contradiction completes the proof. □

Our next two auxiliary results are probably known, but we have not been able to find them in the literature.

Lemma 3.3 *Let V be a vector space over the field $\mathbb{Q}$ of rational numbers, and let a be an element of V such that the system $\{a^{\gamma^n} \mid n \in \mathbb{Z}\}$ is linearly dependent for some $\mathbb{Q}$-automorphism γ of V. Then the $\mathbb{Q}$-subspace $W = \langle a^{\gamma^n} \mid n \in \mathbb{Z}\rangle_{\mathbb{Q}}$ has finite dimension.*

Proof — As the system $\{v^{\gamma^n} \mid n \in \mathbb{Z}\}$ is linearly dependent, there exist pairwise different integers $k_1, \dots, k_t$ such that

$$\sum_{i=1}^{t} \varepsilon_i a^{\gamma^{k_i}} = 0$$

for suitable non-zero rational numbers $\varepsilon_1, \dots, \varepsilon_t$. Clearly,

$$\left(\sum_{i=1}^{t} \varepsilon_i a^{\gamma^{k_i}}\right)^{\gamma^n} = 0$$

for every integer n, and so we may suppose that $k_i \geqslant 0$ for all i. Let k be the largest element of the set $\{k_1, \dots, k_t\}$. Then $k > 0$, and a^{γ^k} belongs to the subspace

$$W^+ = \langle a^{\gamma^h} \mid 0 \leqslant h < k\rangle_{\mathbb{Q}}.$$

It follows that $a^{\gamma^n} \in W^+$ for all non-negative integers n. The same argument applied to the $\mathbb{Q}$-automorphism γ^{-1} yields that there exists also a negative integer s such that the $\mathbb{Q}$-subspace

$$W^- = \langle a^{\gamma^r} \mid s < r \leqslant 0\rangle_{\mathbb{Q}}$$

contains a^{γ^n} for each non-positive integer n. Therefore

$$W = \langle W^+, W^- \rangle_{\mathbb{Q}}$$

is a $\mathbb{Q}$-vector space of finite dimension. □

Corollary 3.4 *Let A be a torsion-free abelian group, and let γ be an automorphism of A. If a is an element of A such that the system $\{a^{\gamma^n} \mid n \in \mathbb{Z}\}$ is linearly dependent, then the subgroup $\langle a^{\gamma^n} \mid n \in \mathbb{Z}\rangle$ has finite rank.*

Proof — We may suppose, without loss of generality, that

$$A = \langle a^{\gamma^n} \mid n \in \mathbb{Z}\rangle.$$

As the tensor product

$$V = A \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a vector space over $\mathbb{Q}$, and γ can obviously be extended to a $\mathbb{Q}$-automorphism of V, it follows from Lemma 3.3 that V has finite dimension. Therefore the torsion-free abelian group A has finite rank. □

We can now prove the main resut of this section.

Theorem 3.5 *Let G be a weakly radical $\mathcal{W}$-group. Then the commutator subgroup G' of G is periodic.*

Proof — Suppose first that the group G is locally soluble, so that by Lemma 3.2 its elements of finite order form a characteristic subgroup T. Since G can obviously be replaced by the factor group G/T, it can be assumed without loss of generality that G is torsion-free, and of course we have to prove that G is abelian in this case.

Assume for a contradiction that G is not abelian. Clearly, G can be replaced by a suitable finitely generated subgroup, and so we may suppose that G is soluble and that its derived length is smallest possible among all soluble counterexamples. Then the commutator subgroup G' of G is abelian. Since G is not even nilpotent by Theorem 2.8, there exist elements a of G' and g of G such that $[a, g] \neq 1$. Again, G may be replaced by its non-abelian subgroup $\langle g, a\rangle$, so that $G = \langle g\rangle A$, where

$$A = \langle a\rangle^G = \langle a\rangle^{\langle g\rangle}$$

is an abelian normal subgroup. It follows from Lemma 3.1 that g induces on A an automorphism of infinite order, so that $C_{\langle g\rangle}(A) = \{1\}$ and hence $G = \langle g\rangle \ltimes A$. Put $a_i = a^{g^i}$ for all integers i, so that

$$A = \langle a_i \mid i \in \mathbb{Z}\rangle.$$

Assume that

$$A_k = \langle a_i \mid i \in k\mathbb{Z}\rangle$$

is a proper subgroup of A for some positive integer k. Then A_k is not normal in G and contains $a = a_0$, so that $\langle a \rangle$ is normal in the normalizer $N_G(A_k)$. On the other hand, $A_k^{g^k} = A_k$ and hence g^k normalizes $\langle a \rangle$, which is impossible because $[a, g^k] \neq 1$ and G cannot have infinite dihedral sections. Therefore $A_k = A$ for any positive integer k. In particular, the set

$$\{a_i \mid i \in \mathbb{Z}\}$$

is linearly dependent, and so A has finite rank by Corollary 3.4. Moreover, the counterexample G can be chosen in such a way that the rank r of A is smallest possible.

Since G is a finitely generated metabelian group, it is residually finite (see [11] Part 2, Theorem 9.51), and so there is a prime number p such that $A^p \neq A$. Let m be the order of the automorphism induced by g on the finite group A/A^p. As $A_m = A$, we have that g acts trivially on A/A^p, so that A/A^p has order p and hence A/A^{p^n} is cyclic of order p^n, for each positive integer n. Then the order of the automorphism induced by g on A/A^{p^n} has the form rp^h, where r divides $p-1$ and $0 \leqslant h < n$. It follows that G has a homomorphic image of the form

$$\overline{G} = \langle \overline{x}_n \rangle \ltimes \langle \overline{a}_n \rangle,$$

where $\overline{a}_n$ has order p^n and $\overline{x}_n$ is an automorphism of order p^h of $\langle \overline{a}_n \rangle$. Suppose $h \geqslant 2$, so that

$$\overline{a}_n^{\overline{x}_n} = \overline{a}_n^{1+ep^s},$$

where p does not divide e and $0 < s \leqslant n-2$. Put $t = n-s-1$. Then

$$\overline{a}_n^{p^t} \overline{x}_n \overline{a}_n^{-p^t} = \overline{x}_n \overline{a}_n^{ep^{n-1}},$$

and hence $\overline{a}_n^{p^t}$ normalizes the subgroup

$$\overline{X}_n = \langle \overline{x}_n, \overline{a}_n^{p^{n-1}} \rangle = \langle \overline{x}_n \rangle \times \langle \overline{a}_n^{p^{n-1}} \rangle.$$

On the other hand, $\overline{X}_n$ is not normal in $\overline{G}$, and so the $\mathcal{W}$-property yields that $\overline{a}_n^{p^t}$ normalizes $\langle \overline{x}_n \rangle$, which is impossible as $t \leqslant n-2$. This contradiction shows that $h \leqslant 1$, and hence the automorphism induced by g on A/A^{p^n} has order dividing $p(p-1)$. It follows that $g^{p(p-1)}$ acts trivially on A/A^{p^n} for each positive integer n, so that it acts trivially also on A/A^*, where

$$A^* = \bigcap_{n \geqslant 1} A^{p^n}.$$

Obviously, A/A^* does not satisfy the minimal condition on subgroups, and so it cannot be periodic. Thus the characteristic subgroup A^* has rank strictly smaller than r, so that $\langle g, A^* \rangle$ is abelian by the minimal choice of r, and hence $\langle g^{p(p-1)}, A \rangle$ is a nilpotent normal subgroup of G, which is even abelian by Theorem 2.8. An application of Lemma 3.1 yields now that G itself is abelian, and this contradiction completes the proof of the statement when G is locally soluble.

Assume now that G is an arbitrary weakly radical $\mathcal{W}$-group, and let

$$\{1\} = G_0 < G_1 < \ldots < G_\alpha < G_{\alpha+1} < \ldots < G_\tau = G$$

be an ascending normal series of G whose factors are either locally soluble or locally finite. If T is the largest periodic normal subgroup of G, we may replace G by the factor group G/T, and so suppose without loss of generality that G has no periodic non-trivial normal subgroups. Assume for a contradiction that the statement is false, and let $\mu \leqslant \tau$ be the first ordinal such that the commutator subgroup G'_μ of G_μ is not periodic. Obviously, μ cannot be a limit ordinal, and the subgroup $G_{\mu-1}$ is torsion-free abelian. Moreover, it follows from the first part of the proof that G_μ is not locally soluble, and so $G_\mu/G_{\mu-1}$ must be locally finite. As $G_{\mu-1}$ is contained in the centre of G_μ by Lemma 3.1, an application of Schur's theorem yields that G'_μ is locally finite. This last contradiction completes the proof of the theorem. □

Corollary 3.6 *Let* G *be a weakly radical* $\mathcal{W}$*-group. Then* G *is soluble and has derived length at most* 5.

Proof — The commutator subgroup G' of G is locally finite by Theorem 3.5, and so G' is soluble of derived length at most 4 by Theorem 2.13. Therefore G itself is soluble and has derived length at most 5. □

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