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Supporting mathematical modelling by upscaling real context in a sequence of tasks

Lluís Albarracín[†,*], Carlos Segura[‡], Irene Ferrando[‡] and Núria Gorgorió[†]

[†]*Departament de Didàctica de la Matemàtica i les Ciències Experimental, Universitat Autònoma de Barcelona, Campus de la UAB, Edifici G5, 08193 Bellaterra, Spain*

[‡]*Departament de Didàctica de la Matemàtica, Universitat de València, Avda. Tarongers, 4 46022 Valencia, Spain*

[*]*Corresponding author. Email: lluis.albarracin@uab.cat*

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Abstract

Creating and developing mathematical models to solve real-world problems is a complex task and students often have difficulties in tackling it successfully. The design and implementation of sequences that help students autonomously develop their ability to solve modelling tasks could be a useful scaffolding tool to foster modelling learning. In this paper we present a sequence of estimation tasks in a real context based on what we have called the upscaling technique, which consists of scaling the accessibility and knowledge of the context used. A study with two samples of grade 10 students, experimental and control, has allowed us to find out whether this sequence promotes success in solving complex contextualized estimation problems and whether it helps students to improve their mathematical models.

Mathematical modelling has been of interest to mathematics education since the seminal work of Pollak (1979), who started a movement to introduce activities to the classroom that evidence the powerful relationship between mathematics and the world around us. The research literature produced in recent decades has shown an increase in educational proposals that incorporate mathematical modelling at different educational levels (Vorhölter *et al.*, 2014). Following Blum (2015), the teaching of applications and modelling has a dual function. On the one hand, knowledge of mathematics and its applications is vital for understanding, interpreting and relating to the real world, mainly in terms of solving real problems and developing complex projects. On the other hand, the real world and the way it integrates mathematical knowledge are extremely important as a vehicle for making sense of the learning of mathematical concepts and, more generally, understanding mathematics as a discipline.

The focus of interest in this research is to explore teaching strategies to support secondary school students in creating, developing and improving their own mathematical models for real phenomena. The way we chose to encourage students to develop their own mathematical models is by studying a defined set of interesting situations that potentially promote the use of certain mathematical concepts. However, previous studies show us some difficulties for secondary school students to generate a mathematical model adjusted to the phenomena they are studying (Blum, 2011; Crouch & Haines, 2004; Ferrando & Albarracín, 2021; Jupri & Drijvers, 2016). In this way, we explore the didactical potential of an *upscaling sequence* of modelling activities where students start modelling simple and accessible situations and progress into studying more complex and inaccessible situations (Pla-Castells & Ferrando, 2019). The task sequence we pose to the students are a type of Fermi problems (Ärlebäck, 2009), requiring the estimation of large numbers of objects in a surface, initially situated in contexts close to their own reality that become progressively more distant. A sample of N = 99 students will face a non-accessible Fermi problem of unknown context, and another sample of N = 55 students will face the same problem after having solved three estimation problems with a scaling of context complexity according to their accessibility and knowledge, a technique we have called upscaling. This experience allows us to investigate two research questions regarding the didactic usefulness of this type of sequences:

R1: To what extent working an upscaling model development sequence does affect the success in solving a complex Fermi problem?

R2: Does having solved an upscaling model development sequence promote an enrichment of the mathematical model in solving a complex Fermi problem?

1. Theoretical framework

1.1. Mathematical modelling and modelling activity

There are various ways of understanding mathematical modelling in the literature (Abassian *et al.*, 2020; Blomhøj, 2009; Kaiser & Sriraman, 2006), but a widely shared conception is viewing mathematical modelling as a problem-solving process linking the real world and mathematics. It involves mathematizing real-world situations and elaborating mathematical models to describe the phenomena studied, often conceptualized as the result of having engaged in a complex modelling process (Blum, 2015). Throughout this document, we understand *mathematical modelling activities* as contextualized problem-solving tasks in which the mathematical representation of a phenomenon or a real-life fact is a key aspect of its solving process. To relate the characteristic elements of mathematical models to students' productions, in our study we rely upon Lesh and Harel's (2003) definition of mathematical models:

> Models are conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs or experience-based metaphors. Their purposes are to construct, describe or explain other system(s).
>
> Models include both: (a) a conceptual system for describing or explaining the relevant mathematical objects, relations, actions, patterns and regularities that are attributed to the problem-solving situation; and (b) accompanying procedures for generating useful constructions, manipulations or predictions for achieving clearly recognized goals. (p. 159).

From this definition, we understand that to create and develop mathematical models intended to abstractly describe or represent a certain phenomenon or reality is a complex task.

In the literature on mathematical modelling, there is a general consensus that modelling processes are cyclical in nature (Blum & Leiß, 2007; Carreira *et al.*, 2011; Doerr & English, 2003; Galbraith & Stillman, 2006; Greefrath, 2011; Schukajlow *et al.*, 2018). During a modelling activity, students are confronted with solving a problem by going through different stages in which they move from reality to the mathematical domain, each time re-evaluating the phenomenon studied. The whole process is repeated in several cycles, in which students progressively improve the models and the solutions found for the problem they are working on, adapting the models to the requirements of the problem statement (Blum & Borromeo 2009). Once they have achieved the desired answer, students have to communicate the result of this process. In order to successfully pass these stages, learners have to draw on a range of competences that go beyond adequately going through a set of stages and include aspects related to metacognition, motivation and their own ideas about the nature of mathematics (Maaß, 2006).

1.2. Fermi problems as modelling activities

Fermi problems are named after the Nobel Prize in Physics Enrico Fermi (1901–1954) who used them in his lessons to promote reasoning and to establish connections between theoretical aspects and laboratory practice. Following Ärlebäck (2009), Fermi problems are:

> Open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations. (p. 331).

One of the singularities of Fermi problems lies in their formulation: they always seem diffuse in their statement and offer little concrete information to approach the solving process (Efthimiou & Llewellyn, 2007). According to Sriraman and Knott (2009), Fermi problems are estimation problems that aim at getting students to make educated guesses. Only the detailed analysis of the situation enables decomposing the problem into simpler problems to arrive at the solution from reasoned conjectures (Carlson, 1997). This process of identifying the essential variables of the problem and their relationship involves synthesizing a model (Robinson, 2008).

In this study we use Fermi problems, since they are activities that are known to be useful for promoting mathematical modelling and for allowing students to generate new mathematical knowledge connected to content from other disciplines (Ärlebäck & Frejd 2013). Fermi problems have been used for different didactic purposes. With students in the first years of primary education they have been used as a general approach to a project in which the teacher poses the different sub-problems as parts of the project to be studied (Albarracín, 2021; Pla-Castells & Ferrando, 2019). Working with students in compulsory education, Fermi problems have been used as activities to encourage students to generate their own mathematical models, in primary school (Albarracín & Gorgorió, 2019; Haberzettl *et al.*, 2018; Henze & Fritzlar, 2010; Peter-Koop, 2009) and in secondary school (Albarracín & Gorgorió, 2014; Ärlebäck, 2009; Keune & Henning, 2003). At the university level, Fermi problems have been used to work on specific competencies of the modelling process, such as the validation of models in contrast to the phenomena studied (Czocher, 2016, 2018). Fermi problems have been also used in primary school preservice teachers' training as activities to promote flexible thinking in problem solving (Ferrando & Segura, 2020). This diversity of pedagogical uses of Fermi problems opens the way to explore their use for new pedagogical purposes.

In particular, in this study we will use a subset of Fermi problems, those that consist of estimating a large number of elements in a delimited surface, i.e. estimating how many people can fit in a square. We know from Albarracín and Gorgorió (2014) that secondary school students generate mathematical models to solve this type of real context estimation problems. We also know that when working with sequences of these contextualized estimation problems, students develop their flexibility as solvers, understood as the ability to choose and use the most appropriate strategy for each problem (Ferrando & Segura, 2020; Ferrando *et al.*, 2020). These studies show that using Fermi problem sequences is useful for understanding students' problem-solving proficiency and, in particular, for promoting mathematical modelling competence.

1.3. Upscaling task sequences to promote modelling

We address the research design from the models-and-modelling perspective, a research and teaching approach that has been applied in mathematics education research to investigate the nature and growth of problem solvers' conceptual systems across a wide variety of ages, disciplines and settings (Lesh, 2006). Following Doerr and English (2003), a model development sequence is an instructional sequence of modelling tasks that aims to support the development of students' models that can be applied in a range of contexts.

While students are faced with situations that challenge them, their initial models are often not very sophisticated or useful, but students can improve these initially elicited models by evaluating, revising and refining them through discussing their approaches with other students and sharing their interpretations and representations with the whole class. Zawojewski *et al.* (2003) suggest that students—working in small groups and facing a problem situation that is significant and relevant to them—must invent, expand and refine their own mathematical constructions to respond to the demands of the given problem. They also point out that when the results are communicated to other students, discussions and exchange of opinions can take place, which in turn can lead to a review of the process followed and the construction of new models. That is why an isolated modelling activity may not be sufficient to help students develop and generalize their own models so that they can use them in other contexts and, therefore, it is necessary to use other activities of a different nature. According to Ärlebäck and Doerr (2015) the proposal of model development sequences in the classroom should aim to get students to connect, coordinate and integrate different mathematical models since these would be the indispensable steps to create models that can be applied in other situations. These authors emphasize that special attention should be paid to the evolving learning space promoted by the initial activity. Ärlebäck *et al.* (2013) gave importance to the development of modelling task sequences so that through these sequences, students 'engage in multiple cycles of descriptions, interpretations, conjectures, and explanations that are iteratively refined while interacting with other students' (p. 316). In this way, by constructing a sequence of modelling tasks, it can be expected to support solving modelling tasks similar to those in the sequence, but even more complex.

This work presents the design of a model development sequence of estimation problems in real context, consisting of estimating a large number of elements in a delimited area. The design of the sequence is based on the *upscaling & downscaling* design activity technique (Pla-Castells & Ferrando, 2019), in which all the problems in the sequence are equivalent from the point of view of their mathematical structure, but which present different level of difficulty to the students because of the specific context chosen for each of them. The key aspect of this way of designing problem sequences is challenging students with a simple activity that can be easily solved (students generate solving strategies and mathematical models); once solved, they are presented with more complex tasks, which we hope will scaffold—without direct teacher intervention—their solving processes, until they reach the most difficult task through effectively adapting the previous models.

Thus, upscaling sequences consider the teacher-directed heuristics referred to in Stender and Kaiser (2017). Actually, these techniques are based on one of the heuristics described by Pólya (1945): the heuristic based on trying to solve a problem by generalizing a previously known strategy.

2. Goals of the study

Our aim is to study whether an upscaling model development sequence that proposes estimation tasks (Fermi problems) in progressively more distant contexts (from greater to lesser accessibility and from greater to lesser knowledge) allows 10th grade students (15–16 years old) working by teams of three to four students, to develop more successful and richer mathematical models to solve a more complex, non-accessible and context-unknown estimation task. Given that the problems used do not have a known specific numerical solution, we understand that a resolution is successful when it is based on a mathematical model that adequately describes the situation and allows a realistic estimate. In this way, we focus our study on the models generated and not on the accuracy of the numerical response generated. For this purpose, we organize the collection of data in two different experiences. On the one hand, a sample of $N = 99$ students, solve only the final task, acting as a control group; on the other hand, a sample of $N = 55$ students (experimental group) face an upscaling sequence whose last task is the same as the control group.

In order to answer the two questions posed in the introduction of this study, we defined the following specific research objectives:

a. To compare the success rate between students in the experimental group and students in the control group for the final task.
b. To compare the mathematical models developed by students in the experimental group and the control group for the final task.

These differentiating elements should be able to inform us about the role of the work developed in the problem sequence in order to promote learning in mathematical modelling.

3. Methodology

In order to achieve the goals, we carry out two different experiences. In the experience with the experimental group we used a model development sequence of Fermi problems (actually, an upscaling sequence) and analysed the resolutions developed by the students throughout the sequence. In the experience with the control group, we only proposed the last problem of the sequence to analyse the resolutions developed by these students when they were directly confronted with this problem without the background offered by the rest of the previous problems.

3.1. The upscaling sequence

For the purpose of the research, we designed a model development sequence that consisted of four Fermi problems, which addressed the same general problem—estimating the number of people or objects that can be placed on a given surface—though situated in different contexts. We know from previous studies (Albarracín & Gorgorió, 2015) that the problems we raised in the activities—that clearly required the construction or development of a model—made sense to the students, and the results could be contrasted among groups of students, and/or using external information (like news, wikis or official databases) to validate both the processes followed and the results obtained. We followed Wessels' (2014) recommendations, presenting activities that included different real-life contexts and that were complex tasks, far removed from conventional problems associated to already-defined problem-solving procedures.

Applying the upscaling design, a task related to a phenomenon in a nearby environment, in the same educational centre, is presented first. The following problems are contextualized in places that are not accessible but known. The final problem (which will be the one we will use for the contrast of models in our study) is contextualized in an inaccessible place in an unknown location. So that students can

TABLE 1. *Statements and main characteristics of the problems conforming the sequence*

Problem	Context
A: How many people fit in the schoolyard?	Accessible, in the educational centre itself. Schoolyard is rectangular.
B1: How many people can fit into the 'Fonteta de Sant Lluis' (a sports pavilion located in the students' city) for a concert?	Inaccessible but known, in the same city of the students.
B2: How many people fit in the Town Hall Square of your city in a demonstration?	Inaccessible but known, in the same city of the students.
Activity C: How many trees are there in Central Park?	Inaccessible and unknown, in a city far away

TABLE 2. *Descriptive parameters for mathematics grades of experimental and control sample*

	N	Mean	Median	SD	Kurtosis	K-S value	p-value
Experimental	55	6,479	6,67	2.32	1.27	0.19	0.02
Control	99	6,587	6,67	2.09	0.65	0.13	0.04

TABLE 3. *Rate of completeness in problem C*

	Experimental sample(N = 55 students in 14 teams)	Control sample(N = 99 students in 27 teams)
Complete resolutions	13 (93%)	19 (70%)
Incomplete resolutions	1 (7%)	8 (30%)

get a precise idea of the places studied and can make the measurements they consider necessary for the resolution, the use of Google Maps is recommended and allowed. With all these considerations in mind, the upscaling model development sequence is as follows:

3.2. *Data collection*

Our data come from the implementation of the aforementioned sequence of activities in a school in Valencia (Spain). In the experience, a convenience sample of N = 154 students from five natural groups in grade 10 (15–16 years old) have participated. Two natural groups acting as experimental sample (N = 55 students, 26 girls and 29 boys), and three natural groups (N = 99 students, 52 girls and 47 boys) as control sample that have only solved problem C. Regarding the academic profile of the students, we have information on the mathematics grades of each student. Table 2 shows the values of the descriptive parameters and the values that confirm that, based on the K-S Goodness of Fit test, the data are not normally distributed.

However, when comparing both samples, we observe that the value of the Kolmogorov–Smirnov statistic is 0.312, with p = 0.99, so the null hypothesis is accepted and it is inferred that the two samples follow the same grade distribution. We therefore consider that the experimental and control groups are equivalent on the basis of math grades. None of the participating students had previous experience in solving estimation activities such as the ones in the study.

Following Zawojewski *et al.* (2003), we decided to organize students of the sample in teams formed by three to four students. Group work enables more students to participate in the modelling processes

TABLE 4. *Proportion of number of teams of experimental and control group organized by strategy and complexity factor used in problem C*

	Experimental sample		Control sample	
	No obstacle removal	Area with obstacle removal	No obstacle removal	Area with obstacle removal
Average density			1 (5%)	3 (16%)
Average iteration of base unit				2 (10%)
Simple density		2 (15%)	3 (16%)	3 (16%)
Simple iteration of base unit	3 (23%)	8 (62%)		6 (32%)
Linearization			1 (5%)	

when they face this type of activities, since not all students develop mathematical models when working individually (Albarracín & Gorgorió, 2014).

The N = 55 students from the experimental sample, grouped in 14 teams, worked on the problem A, B1 and B2 during a 50 min session. In a second session, they had to solve problem C for 30 min.

The N = 99 students from the control sample, grouped in 27 teams, worked on the problem C in one session for 30 min.

The experiment was carried out over two academic years, with data from the experimental group being collected in the 2017–18 academic year and data from the control group in the 2018–19 academic year. In all cases, the same regular mathematics teacher was in charge of the class. The teacher had previous experience implementing similar experiences based on Fermi problems. The organization of the teams was previously done by the regular teacher attending to preserve heterogeneity of the teams based on the mathematics performance of the students.

During the solving, students had access to internet-connected devices, but their use was limited to searching for information useful for solving the activity. The teacher emphasized in all groups that answers deduced directly from web searches were not allowed and not accepted. The teacher set the task, observed the work of the students, and moderated the group discussion and exchange. He did not in any case suggest ways of solving the tasks, they merely clarified the meaning of a task statement when necessary. He answered students' questions with other questions, but did not hint at problem-solving processes or models, nor did they suggest that there may be more appropriate models than others.

In summary, our primary source of data consists of the written reports of 14 working teams when solving the model development sequence that include problems A, B1, B2 and C. We also have the 27 written reports of the teams that only worked on problem C. A combination of quantitative and qualitative analysis of their productions was carried out by the researchers to address the goals of this work.

3.3. Analysis

To analyse student productions, we have based on previous research. Gallart *et al.*, (2017) present an analysis tool based on the model definition established by Lesh and Harel (2003). During the analysis, all researchers use this tool independently in the first instance and pool the categorization generated. As explained in Albarracín *et al.* (2021), this tool is key to describe, with a tree-shaped scheme, the solution space (in the sense of Leikin, 2007, and Leikin & Levav-Waynberg, 2008) of the general problem of estimating the number of elements on a surface. This scheme is fundamental in the analysis because it allows us to discriminate, firstly, the main strategies (linearization, density and iteration of a base unit) and, secondly, the complexity factors that students include in their resolutions (obstacle removal

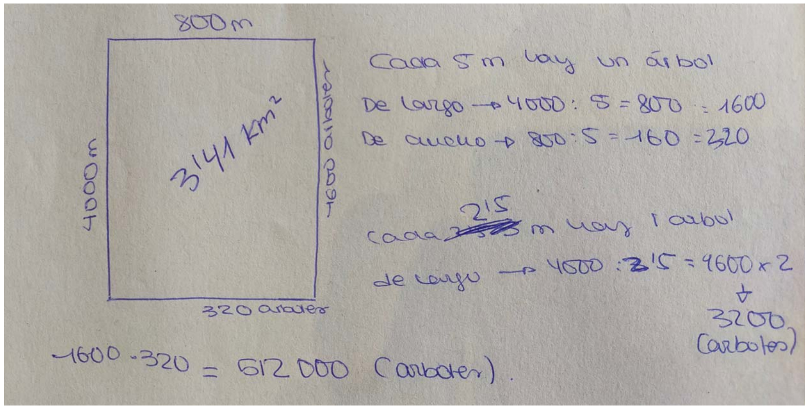

FIG. 1. Resolution categorized as linearization.

and averages). These are described in detail below and illustrated with examples so that the elements considered in the analysis of the experience developed in this study remain clear.

Based on the solution space described in Albarracín *et al.*, (2021) three *main strategies* have been categorized:

- *Linearization*: The students consider that objects can be distributed following a regular pattern (on a grid) and then, they estimate their number for each dimension (height and width in the case of surfaces), and they use the product rule to establish a result. In Fig. 1 we show the resolution of problem C by the students' team. The students first made an estimation of the number of trees in length and width, they wrote: 'a tree each 5 meters', then they obtained an estimation of the number of trees in each direction. Next, they find that the estimation was not good and they re-estimate a tree each 2,5 metres, that is why they multiply by 2. Finally, they multiply the numbers of trees in length and in width and they get the final solution.
- *Density:* The students consider that objects can be distributed following an irregular pattern, then the resolution is based on determining the amount of people or objects arranged across a portion of a given surface area that they have determined themselves, as in the case of population density. In Fig. 2 we show the resolution of a students' team to problem C. The starting point is the density (indeed, a density average), then the students pointed out that they need to compute the area (actually the area that is covered by trees) and then they have to multiply these quantities.
- *Iteration of a base unit*: The students consider that objects can be distributed following an irregular pattern, then, they determine the total surface area of the location where the people or objects to be estimated should be placed and they divide it by the surface area occupied by one single object, which acts as a base unit (see for instance the study from Joram *et al.*, 2005). In Fig. 3 we show the resolution of problem C by a team of students. In this resolution we observe that the starting point is the surface of the park, then they claim that they need to subtract the surface of the zoo and the lakes and, finally, they claim that they will assume that each tree occupies 5 square metres.

Those productions that did not provide sufficient detail to obtain an estimate were categorized as *Incomplete*.

As we have already shown in Figs 2 and 3, students incorporate quantifiable elements from the real context into the constructed model, which provide greater precision in the values obtained (Albarracín

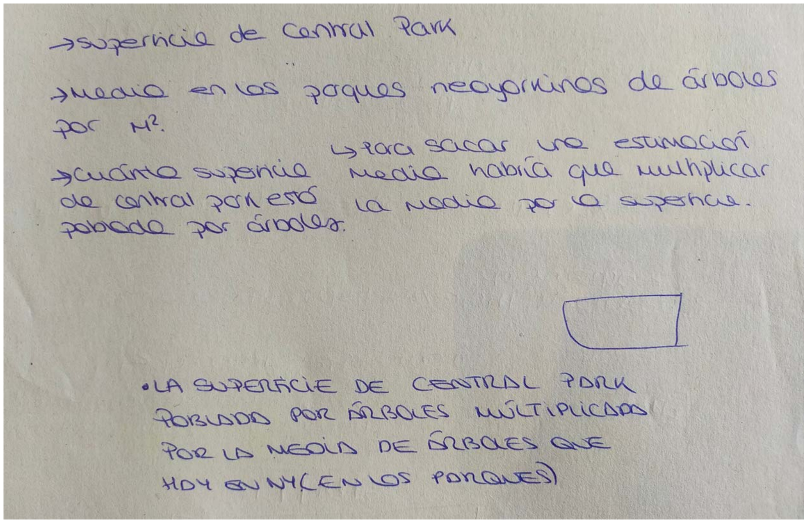

FIG. 2. Resolution categorized as density.

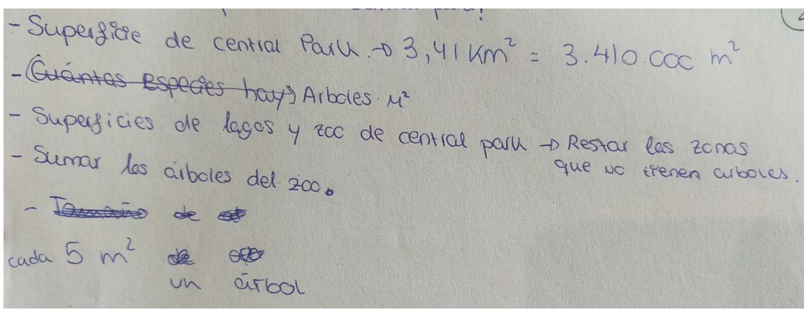

FIG. 3. Resolution categorized as base unit.

et al., 2021). These elements are what we call complexity factors. In the students' productions we have found the following complexity factor categories, which coincide with those of the Segura *et al.* (2021):

- *Obstacle removal:* areas in the region that prevent the placement of elements are identified and the area they occupy is deducted from the total area in order not to include these areas in the estimation of the total number of elements. This complexity factor appears in the resolutions shown in Figs 2 and 3.
- *Average density/size:* For density-based resolutions, different population densities are recognized and an average density is calculated. For resolutions based on iteration of a base unit strategy, different sizes of the elements to be estimated are recognized and an average size is calculated for the elements. In both variants (density or size), this complexity factor expresses a heterogeneous model in the arrangement of the elements and uses the mean to perform the mathematical work of estimation on a simple, homogeneous model, which tries to 'capture' the possible deviations.

4. Results and discussion

In this section we show the characterization of the students' resolutions produced by each of the teams when working on problem C. We will analyse the differences between the teams of the experimental sample, with which the upscaling sequence has been used, and between the teams of the control sample, which have faced problem C directly.

4.1. Comparison of the resolution success rate of the experimental and control samples

The following table (Table 3) shows the proportion of *complete* (resolutions that succeed to provide an estimate that solves the problem, irrespective of the type of model adopted) and *incomplete* (resolutions that do not develop the necessary procedures to find an estimate that solves the problem) resolutions for the 14 teams in the experimental sample ($N = 55$ students) and the 27 teams in the control sample ($N = 99$ students) when working on problem C.

In order to confirm that there is a significant relationship between the sample variable (experimental, control) and the success variable (complete resolution, incomplete resolution), we performed an inferential analysis based on the chi-square test ($DF = 1, N = 41$). As in one cell of Table 3 the frequency is less than 5, the result may or may not be reliable, and therefore we must base our results on the likelihood ratio chi-square test (LR), which allows for cells with frequencies less than 5 (Özdemir and Eyduran, 2005; McHugh, 2013). Although this study is based on a sample of $N = 154$ students, we are aware that having selected teams ($N = 41$) as units of analysis, we have a low number of data to draw statistical conclusions with a large significance level. Thus, setting a significance level of $\alpha = 0.1$, the test gives $\chi 2 = 2.721$ and the likelihood ratio chi-square test is $LR = 3.135$ with asymptotic (bilateral) significance of 0.077. This confirms that there is a statistically significant relationship between the two variables. In addition, we have measured the strength of the correlation with Cramer's V, obtaining $V = 0.26$ with a significance of 0.099. The correlation is significant although weak (close to moderate), but we should keep in mind that Cramer's V tends to produce relatively low correlation measures, even for highly significant results. These results are confirmed by Pearson's contingency coefficient, with a value of 0.25 (in a range of 0 to 0.707), which is also weak to close to moderate. Fisher's exact test, which is usual for analysing 2×2 contingency tables, has also been passed. However, in this case, it gives significance values (bilateral 0.131 and unilateral 0.102) slightly above the established values. However, it is well known that this is a conservative test (Routledge, 1992; Crans & Shuster, 2008).

Despite limitations in the significance of the analysis results, these results allow us to answer the first research question (R1). Indeed, this upscaling model development sequence does foster students' ability to solve more complex modelling tasks, in this case, Fermi problems with non-accessible and unknown contexts. In fact, Activity C presents some difficulties for the students even after working during the sequence. The key aspects that make it difficult for students to solve problem C is the fact that it is a problem in which trees must be estimated and that they do not have direct access to them. Trees make different use of space than humans do and students cannot experiment directly with them in the classroom.

4.2. Comparison of strategies and complexity factors identified on the experimental and control samples complete resolutions of problem C

In order to compare the mathematical models produced by control and experimental groups when they face problem C, we will identify the main strategies and the complexity factors considered by those students that are able to provide a complete resolution in both groups (a total of 13 in the experimental group and 19 in the control group). We will consider, on the one hand, the three strategies previously described (linearization, density and iteration of a base unit) and, on the other hand, the complexity factors (obstacle removal, average density/size).

Table 4 shows the results of the analysis of the strategies and complexity factors on the resolutions of problem C developed by the experimental and control student teams.

Regarding the strategies, as we can see in Table 4, a large majority of the teams in the experimental group (11 out of 13) opted for resolutions based on the iteration of a base unit strategy. Whereas in the control group we find 10 resolutions based on density, 8 on iteration of a base unit and 1 on linearization. Actually, when we analyse the resolutions proposed by the teams in the experimental group to the three problems in the sequence (A, B1 and B2), we can see that many groups that initially proposed models based on the density strategy switch to the iteration of base unit strategy in problem C. Indeed, it does not seem easy to reason from the number of trees per unit area; it is more intuitive and simpler to start from the estimation of the area occupied by a tree. When we look at the variety of strategies proposed by the teams in the control group, we can deduce that this suggests a certain confusion, which is confirmed by the high proportion of dropouts (8 of the 27 teams are not able to give a complete resolution).

Concerning the complexity factors introduced in the resolutions of problem C, we observe that 10 teams in the experimental group (77%) include the elimination of obstacles, this proportion being slightly lower in the control group (74%). However, in this group, there are six teams (21%) that take into account the heterogeneous distribution of trees and therefore include the complexity factor average density/unit, while no team in the experimental group considers this.

So, in the experimental group, there is no complete resolution to problem C that includes more than one complexity factor, while in the control group there are five resolutions (26%) that include both complexity factors.

The results of the comparative analysis between the control sample and the experimental sample lead to a negative answer to the second research question (R2). It does not seem that the upscaling model development sequence promotes the enrichment of mathematical models in students' solving, in the sense of adding a large number of complexity factors to the model. Indeed, although the proportion of teams without enriching their mathematical model is similar (21% in the control sample and 23% in the experimental sample), in the control sample there is a significant proportion of teams (26%) that enrich their model with two complexity factors. However, no team in the experimental sample considers heterogeneity as a factor that can enrich their model and improve estimation, and therefore no team introduces two complexity factors. In this way it would seem that upscaling sequence allows to identify more efficient factors of complexity to solve the problem.

5. Conclusions

The results of the comparative analysis of the experimental and control sample, combining qualitative and quantitative techniques, have allowed the two research questions to be answered and the study goals to be achieved.

Thus, analysing the correlation between the success variable (complete production, incomplete production) in solving a difficult Fermi problem (problem C: unavailable and unknown context) and the sample variable (experimental, control), it is confirmed that there is a statistically significant difference in favour of the experimental sample, although with a low level of significance. That is, those students who performed the upscaling model development sequence (which were graded on accessibility and context knowledge by means of four tasks, of which the last one was problem C) were more successful in solving the difficult Fermi problem. We are aware that this is a small study and this conditions the level of significance of the results. It should also be noted that the time devoted by the two groups of students to this type of activity is quite unequal: the students in the control group only devoted a 30-min session to solving a single activity, while the experimental group also had a previous 50-min session. This limitation should therefore be considered, as it is possible that the difference in the results of the two groups is also due to this factor. But we understand that this work opens the door for further studies to explore how upscaling and downscaling sequences help learners to deal with modelling tasks. Despite these limitations, this result allows further development of the idea of upscaling sequence as a scaffolding tool to enable students to autonomously improve their skills in modelling problem solving.

In contrast, a comparative analysis of the resolutions produced by the two samples leads us to reject that the use of an upscaling sequence influences the model enrichment in a difficult Fermi problem. This fact has implications from the perspective of interpreting the role of each activity in a model development sequence. Although it seems that using an upscaling sequence provides students with a detailed knowledge of the mathematical models that can be useful to solve the situation posed in the final activity so they focus their performance, and encourages them to go directly to the resolution that seems most optimal (the use of the tree as the base unit), it does not lead them to consider complexity factors that would enrich the model, such as the heterogeneity of the tree sizes. The students' work in the first part of the model development sequences (Ärlebäck & Doerr, 2015) provides them with knowledge about different mathematical models that can be useful in the resolution, emphasizing those elements that best fit each aspect of the studied phenomenon. Thus, although each of the Fermi problems is an open-ended activity (Ärlebäck, 2009), in the sense that it does not involve the use of a specific procedure, working in the sequence allows students to have a range of elements that they can include in the model and experience on how each of these elements allows them to develop a more accurate model.

However, the fact that the control sample tested more resolution strategies, some of them not very efficient (linearization), may denote a certain confusion in the face of the novelty of a task of this type. Paradoxically, this confusion and variability in resolution types seems to have led the teams to consider more complexity factors (not only areas of the park without trees, but also heterogeneity in tree densities or sizes). This leads us to consider the possibility that other contextual features, rather than accessibility or remoteness, influence the promotion of complexity factors. This fact also opens the door to the need to explore new ways of implementing bottom-up model development sequences with the ultimate goal of enabling students to enrich their mathematical models. In future research it would be interesting to explore whether there are features of the context, such as the size of the elements to be estimated, the irregularity of their arrangement or the total area size, that influence the decisions students make to enrich their model. Knowing these characteristics would make it possible to design other types of sequences that help the progressive enrichment of the students' mathematical models.

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Lluís Albarracín is a Serra Húnter Fellow in Mathematics Education at the Universitat Autònoma de Barcelona (Spain). His current research interests centre on mathematics modelling and estimation. He is also interested in the use of video games in Mathematics Education.

Carlos Segura is a PhD student and a part-time instructor at the Universitat de València (Spain). His current research interests centre on mathematics modelling and estimation processes, particularly on the promotion of flexibility through problem solving.

Irene Ferrando is a Lecturer at the Universitat de València (Spain). Her current research interests centre on mathematics modelling and estimation processes. She is also interested on the research of affective aspects related to mathematics learning and problem solving processes.

Núria Gorgorió is a Professor in Mathematics Education at the Universitat Autònoma de Barcelona (Spain). Her research has focused on spatial abilities and the learning of geometry, teaching mathematics to immigrant students in mainstream schools and teaching mathematics content knowledge and pedagogical knowledge to prospective primary teachers.