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Effect of uncertain damping coefficient on the response of a SDOF system

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ABSTRACT

In this paper, a full probabilistic description of the response of a randomized SDOF system in both the time and the frequency domain is done. Considering that the damping of the structure does not simply relate to any single physical phenomenon, the sensitivity of the response to the randomness of the damping parameter is investigated. The stochastic analysis is conducted via the Probability Transformation Method therefore the first probability density function of the response is evaluated. The effect of the uncertain damping coefficient on the response of the SDOF system has been investigated through several numerical examples. From the response probability density function as well as from some response statistical indexes, it has been observed that the randomness in the damping parameter significantly affects the output functions of the system. Moreover, from the conducted analyses for different scenarios of damping mean value, it is possible to appreciate that more variability of the response occurs for the smaller damping value. This aspect has been found both, in the time and in the frequency domain analyses.

1. Introduction

The dynamic analysis of structural systems strongly depends on the external actions as well as on the physical-geometrical properties of the structures. In many cases, these quantities have such a high level of uncertainty as to produce random variations in the dynamic response [1–4]. In these circumstances, stochastic analyses must be performed to study the probabilistic behavior of the response functions. The harmonic oscillator is certainly the simplest model to examine the characteristics of dynamical systems in engineering and applied sciences when uncertainties are involved. In the last 50 years, many authors providing studies on the oscillator's response due to random loadings [5–7]. Moreover, significant attention is paid to the effect of the uncertain structural parameters on the response functions, such as randomness in the mass, in the stiffness and in the damping [8–10]. Among these parameters, the estimation of the damping value of a structural system is the most difficult task for a design engineer. Given the fact that there is almost always uncertainty about the energy dissipation during dynamic motion of a mechanical or structural system, the damping of a system may vary significantly from its "design" value. In fact, unlike mass and stiffness values, damping is not related to a unique and well-defined physical phenomenon. For this reason, damping is surely the most uncertain parameter influencing the dynamic responses of structures, and its incorrect estimation results in a large error in the response [11–18].

In this paper, the analysis of a Single Degree-of-Freedom (SDOF) system with uncertain damping parameter under deterministic excitation is addressed. The main objective of this contribution is to analyze how the damping parameter affects the response function when it is assumed to be a random variable (RV). It is common knowledge that structural damping is a measure of energy dissipation and that energy dissipation in a vibrating structure is dependent on complex mechanisms among which an indefinite number of uncertain factors. Moreover, the damping values of structures depend on many aspects, for instance, the site geological conditions, the high of the structures, the natural frequency and the vibration amplitude. Considering that the damping of the structure does not simply relate to any single physical phenomenon, in order to investigate the damping uncertainty associated with the structure construction materials, SDOF systems with random damping under deterministic excitation have been investigated. Therefore, several analyses, both in the time and in the frequency domain, have been performed. In particular, a fully stochastic characterization of the system response in terms of probability density function (PDF) has been performed by applying the Probability Transformation Method (PTM). In the last decade, a growing literature demonstrates how this method is a convenient tool for the estimation of the response PDFs of uncertain systems. In particular, the authors have recently analyzed some problems with uncertainty, from a probabilistic point of view, by successfully applying such a technique [19–25]. Furthermore, in contrast to various literature's preliminary investigations, often based on the first and second-order statistics, the PTM allows great accuracy and a low computational effort thorough analysis. In this regard, it is worth emphasizing that, in other research work, the authors have tested

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that the use of the PTM allows a computer time saving of around 80% compared with classical Monte Carlo simulations [26]. This work is organized as follows. At first, some preliminary concepts, including the fundamental Theorem of the PTM, are summarized. Then, in Sections 2 and 3, the dynamics of the harmonic oscillator with a random damping parameter will be investigated for different loads, both in the time and in the frequency domain, respectively. At last, in Section 4, the main findings of all numerical investigations are summarized. That is, some conclusions about the influence of the uncertainty of the damping parameter in the response of the SDOF system are drawn.

2. Preliminary concepts

Consider a rigid girder with mass m which is supported by columns with combined stiffness k being the internal friction between the girder and the columns described by a viscous dashpot damper with damping coefficient c. Suppose that the girder is also subjected to an externally applied force $f(t)$. Therefore, denoting by $k/m=\omega_0^2$ and $c/m=2\xi\omega_0$, the equation of motion of an SDOF, written in the standard form in the theory of vibration, is

$$\ddot{x}(t)+2\xi\omega_0\dot{x}(t)+\omega_0^2x(t)=\frac{f(t)}{m}, \tag{1}$$

ω_0 being the natural circular frequency and ξ the non-dimensional damping coefficient. Let the following initial conditions

$$x(0)=x_0, \qquad \dot{x}(0)=x_1, \tag{2}$$

where, without loss of generality, the initial time is equal to zero.

Assuming that the damping coefficient is random rather than deterministic, we analyze the behavior of the response of the system (1) which, in this case, is a stochastic process (SP). In the initial value problem (IVP) (1)–(2), the damping parameter $\xi(\theta)$ and the initial conditions, $x_0(\theta)$ and $x_1(\theta)$, are assumed to be an absolutely continuous RVs defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. We are going to carry out the analysis of the randomization of the initial value problem (IVP) (1)–(2) in the time domain and in the frequency domain. The corresponding random IVP is

$$\begin{cases}\ddot{x}(t,\theta)+2\xi(\theta)\omega_0\dot{x}(t,\theta)+\omega_0^2x(t,\theta)=\dfrac{f(t)}{m},\\ x(0,\theta)=x_0(\theta), \qquad \dot{x}(0,\theta)=x_1(\theta).\end{cases} \tag{3}$$

It is important to note at this point that, in most structural and mechanical vibration problems, values of the parameter ξ smaller than 0.05 are needed. Furthermore, in some particular cases, such as those for which the energy of the input is located at frequencies lower than the natural frequency, if $\xi\geq 2^{-1/2}$, the power spectral density of the response $S_{XX}(\omega)$, if it exists, simply decays monotonically from $S_{XX}(0)$, see [2]. Thus, in this paper we assume the following domain to the random damping coefficient $\mathcal{D}(\xi(\theta))=[0,1)$ and the system described in the IVP (3) is an underdamped system with probability one (w.p.1). Regarding the distribution of the initial conditions, no restrictions are assumed. In addition, for sake of generality, we assume that all RVs are dependent with a known joint PDF $f_{x_0,x_1,\xi}(x_0,x_1,\xi)$.

Notice that, the theoretical part of this paper is done in the most general case when all the RVs in the IVP (3) are dependent with a given joint PDF. But, in order to analyze the effect of the randomness of the damping coefficient on the response of the system, the uncertainty in the initial conditions can affect the conclusions. That is, assuming randomness in the initial conditions the response will contain uncertainty inherited from the initial conditions and not only from the damping parameter. So we would not be able to analyze how the parameter of interest, no other influences, affects the system. In order to avoid this problem, we can consider independence between the RVs, therefore the joint PDF is the product of the marginals, $f_{x_0,x_1,\xi}(x_0,x_1,\xi)=f_{x_0}(x_0)f_{x_1}(x_1)f_\xi(\xi)$. In addition, we should assume that the initial conditions take a fixed value w.p. 1, i.e., $x_0(\theta)=a$ and $x_1(\theta)=b$ w.p.1. Thus,

$$f_{x_0}(x_0)=\delta(x_0-a), \quad \text{and} \quad f_{x_1}(x_1)=\delta(x_1-a), \quad -\infty<x_0,x_1<+\infty, \tag{4}$$

being $\delta(\cdot)$ the Dirac Delta function defined by

$$\delta(v-v_0)=\begin{cases}\infty & v=v_0,\\ 0 & \text{otherwise}.\end{cases} \tag{5}$$

In this case, the joint PDF is

$$f_{x_0,x_1,\xi}(x_0,x_1,\xi)=\delta(x_0-a)\delta(x_1-b)f_\xi(\xi). \tag{6}$$

The main objective in the classical deterministic theory of differential equations is to determine an expression for the response of a given IVP. In the random scenario, solving a random differential equation does not only consist of determining its solution. It is also interesting to calculate as much statistical information as possible. The computation of the 1-PDF of the solution SP, say $f_1(x,t)$, is worthwhile, since from it one has a full probabilistic description of the response at every time instant. From the 1-PDF the mean, variance, skewness and the kurtosis, among other statistical quantities of interest, can be derived, provided these exist. Integrating the 1-PDF all the one-dimensional statistical moments can be computed, in particular the mean and the variance,

$$\mathbb{E}\left[x(t,\theta)^k\right]=\int_{\mathbb{R}}x^kf_1(x,t)\,\mathrm{d}p,$$
$$k=1,2,\dots \quad \begin{cases}\mathbb{E}\left[x(t,\theta)\right]= & \int_{\mathbb{R}}xf_1(x,t)\,\mathrm{d}x,\\ \mathbb{V}\left[x(t,\theta)\right]= & \int_{\mathbb{R}}x^2f_1(x,t)\,\mathrm{d}x-\mathbb{E}\left[x(t,\theta)\right]^2.\end{cases} \tag{7}$$

As indicated in the introduction, the PTM is a useful technique to compute the PDF of an RV which is given by the transformation of another variable whose PDF is known. We are going to apply this technique to determine an expression for the 1-PDF of the response in terms of the joint PDF of the random input parameters (in particular, the damping coefficient). Below, the PTM method is introduced in its multidimensional version.

Theorem 1 (*Probability Transformation Method (PTM) [27]*)**.** *Let $\mathbf{u}(\theta)=(u_1(\theta),\dots,u_n(\theta))$ and $\mathbf{v}(\theta)=(v_1(\theta),\dots,v_n(\theta))$ be two n-dimensional absolutely continuous random vectors. Let $\mathbf{r}:\mathbb{R}^n\to\mathbb{R}^n$ be a one-to-one deterministic transformation of $\mathbf{u}$ into $\mathbf{v}$, i.e., $\mathbf{v}=\mathbf{r}(\mathbf{u})$. Assume that $\mathbf{r}$ is continuous in $\mathbf{u}$ and has continuous partial derivatives with respect to $\mathbf{u}$. Then, if $f_{\mathbf{U}}(\mathbf{u})$ denotes the joint PDF of vector $\mathbf{u}(\omega)$, and $\mathbf{s}=\mathbf{r}^{-1}=(s_1(v_1,\dots,v_n),\dots,s_n(v_1,\dots,v_n))$ represents the inverse mapping of $\mathbf{r}=(r_1(u_1,\dots,u_n),\dots,r_n(u_1,\dots,u_n))$, the joint PDF of vector $\mathbf{v}(\omega)$ is given by*

$$f_{\mathbf{V}}(\mathbf{v})=f_{\mathbf{U}}(\mathbf{s}(\mathbf{v}))\left|J_n\right|, \tag{8}$$

where $\left|J_n\right|$ is the absolute value of the Jacobian, which is defined by

$$J_n=\det\left(\frac{\partial\mathbf{s}}{\partial\mathbf{v}}\right)=\det\begin{pmatrix}\dfrac{\partial s_1(v_1,\dots,v_n)}{\partial v_1} & \cdots & \dfrac{\partial s_n(v_1,\dots,v_n)}{\partial v_1}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial s_1(v_1,\dots,v_n)}{\partial v_n} & \cdots & \dfrac{\partial s_n(v_1,\dots,v_n)}{\partial v_n}\end{pmatrix}. \tag{9}$$

3. Analysis in the time domain

In this section, a closed expression for the 1-PDF of the response, in the time domain, is determined. Two numerical examples are performed in order to analyze the effect of the random damping coefficient in the response of the IVP (3). In the first example, a sinusoidal force is considered, being it the classical choice in the literature. A polynomial force is assumed in the second example. We select this function given its applicability, for example when the force is approximated via interpolation techniques.

3.1. Computing the 1-PDF of the response

In the deterministic theory, the response of the non-homogeneous second-order differential equation (1) is calculated as the sum of the solution of the corresponding homogeneous differential equation, also called complementary solution, and a particular solution, assuming that it exists. That is $x(t)=x_c(t)+x_p(t)$, being $x_c(t)$ the solution of the free system, when the right-hand side of Eq. (1) is set zero, $f(t) = 0$, and $x_p(t)$ a particular solution of the complete differential equation. Now, we shall calculate the response of IVP (1)–(2). First, to calculate the complementary solution, we define the associate homogeneous second-order differential equation

$$\ddot{x}_c(t) + 2\xi\omega_0\dot{x}_c(t) + \omega_0^2 x_c(t) = 0. \tag{10}$$

Assuming that $0 \le \xi < 1$, the solution of Eq. (10) is

$$x_c(t) = \mathrm{e}^{-\xi\omega_0 t}\left(A\cos\left(\omega_0\sqrt{1-\xi^2}\,t\right) + B\sin\left(\omega_0\sqrt{1-\xi^2}\,t\right)\right), \tag{11}$$

where the constants A and B are calculated from the initial conditions (2). Let $x_p(t)$ a particular solution of the complete system (1), therefore the coefficients A and B in (11) are calculated, from the initial conditions given in (2), as follows

$$\left.\begin{aligned} x_0 &= x(0) = x_c(0) + x_p(0) = A + x_p(0),\\ x_1 &= \dot{x}(0) = -\xi\omega_0 A + \omega_0\sqrt{1-\xi^2}B + \dot{x}_p(0), \end{aligned}\right\}$$
$$\Longrightarrow A = x_0 - x_p(0),\ B = \frac{x_1 + \xi\omega_0(x_0 - x_p(0)) - \dot{x}_p(0)}{\omega_0\sqrt{1-\xi^2}}. \tag{12}$$

In conclusion, the response of the SDOF system (1) with initial conditions (2) is

$$x(t) = h_0(t)x_0 + h(t)x_1 + h_p(t), \tag{13}$$

being

$$\begin{aligned} h_0(t) &= \mathrm{e}^{-\xi\omega_0 t}\left(\cos\left(\omega_0\sqrt{1-\xi^2}t\right) + \frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_0\sqrt{1-\xi^2}t\right)\right),\\ h(t) &= \frac{\mathrm{e}^{-\xi\omega_0 t}}{\omega_0\sqrt{1-\xi^2}}\sin\left(\omega_0\sqrt{1-\xi^2}t\right),\\ h_p(t) &= x_p(t) - \mathrm{e}^{-\xi\omega_0 t}\left(x_p(0)\cos\left(\omega_0\sqrt{1-\xi^2}t\right)\right.\\ &\quad\left. + \frac{\xi\omega_0 x_p(0) + \dot{x}_p(0)}{\omega_0\sqrt{1-\xi^2}}\sin\left(\omega_0\sqrt{1-\xi^2}t\right)\right). \end{aligned} \tag{14}$$

where $h_0(t)$ indicates the solution function related to the initial condition x_0, $h(t)$ is the well-known impulse response of the system while $h_p(t)$ is the solution function related to the force $f(t)$.

In this manner, assuming the damping parameter $\xi(\theta)$ and the initial conditions $x_0(\theta)$ and $x_1(\theta)$ absolutely continuous RVs defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the response of the randomized IVP (1)–(2) is

$$x(t,\theta) = h_0(t,\xi(\theta))x_0(\theta) + h(t,\xi(\theta))x_1(\theta) + h_p(t,\xi(\theta)), \tag{15}$$

where the function $h_0(t)$, $h(t)$ and $h_p(t)$ are stochastic processes (SP) calculated from expressions (14) with $\xi(\theta)$ a RV. Notice that, for clarity in the following development, the dependence of functions h_0, h and h_p on $\xi(\theta)$ is indicated in (15) rather than directly write it as $h_0(t,\theta)$, $h(t,\theta)$ and $h_p(t,\theta)$, respectively.

The response is a stochastic process and it is interesting not only to calculate it but also to compute the mean, the variance and other statistical quantities of interest, such as the symmetry and the kurtosis, for each time instant $t > 0$. A major goal is the computation of the 1-PDF, since from it all these quantities can be easily derived. To determine the expression of the 1-PDF of the response SP we apply the PTM stated in Theorem 1. Given a fixed time instant $\hat{t} > 0$, we apply the PTM to determine the PDF of the random vector $\mathbf{v}(\theta) = (v_1(\theta), v_2(\theta), v_3(\theta)) = (x_0(\theta), x(\hat{t},\theta), \xi(\theta))$, in terms of the PDF of the input parameters $\mathbf{u}(\theta) = (x_0(\theta), x_1(\theta), \xi(\theta))$, being $\theta \in \Omega$. Let us to define the deterministic transformation $\mathbf{r} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{aligned} v_1 &= r_1(x_0,x_1,\xi) = x_0,\\ v_2 &= r_2(x_0,x_1,\xi) = x(\hat{t}) = h_0(\hat{t},\xi)x_0 + h(\hat{t},\xi)x_1 + h_p(\hat{t},\xi),\\ v_3 &= r_3(x_0,x_1,\xi) = \xi. \end{aligned} \tag{16}$$

The inverse mapping $\mathbf{s} = \mathbf{r}^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ and the Jacobian are

$$\begin{aligned} x_0 &= s_1(\mathbf{v}) = v_1,\\ x_1 &= s_2(\mathbf{v}) = \frac{v_2 - h_0(\hat{t},v_3)v_1 - h_p(\hat{t},v_3)}{h(\hat{t},v_3)}, \qquad J(\mathbf{v}) = \left|\frac{1}{h(\hat{t},v_3)}\right| \neq 0.\\ \xi &= s_3(\mathbf{v}) = v_3, \end{aligned} \tag{17}$$

Substituting in Eq. (8), the PDF of the random vector $\mathbf{v}(\theta)$ is

$$f_{\mathbf{v}}(v_1,v_2,v_3) = f_{x_0,x_1,\xi}\left(v_1, \frac{v_2 - h_0(\hat{t},v_3)v_1 - h_p(\hat{t},v_3)}{h(\hat{t},v_3)}, v_3\right)\left|\frac{1}{h(\hat{t},v_3)}\right|. \tag{18}$$

Finally, marginalizing with respect to the RVs $v_1(\theta) = x_0(\theta)$ and $v_3(\theta) = \xi(\theta)$ and taking the time t arbitrary, the 1-PDF of the response SP $x(t,\theta)$ is

$$f_1(x,t) = \int_{\mathbb{R}^2} f_{x_0,x_1,\xi}\left(x_0, \frac{x - h_0(t,\xi)x_0 - h_p(t,\xi)}{h(t,\xi)}, \xi\right)\left|\frac{1}{h(t,\xi)}\right| \mathrm{d}\,\xi\,\mathrm{d}\,x_0, \quad t > 0. \tag{19}$$

Remark 1. Highlight that $h(\hat{t},\xi(\theta)) \neq 0$ w.p.1, since $\xi(\theta)$ is assumed to be a continuous RV. As the transformation defined in the application of the PTM is deterministic, we can have some computational problems when $h(\hat{t},\xi(\hat{\theta})) = 0$, for some $\theta \in \Omega$. In this case, an alternative transformation can be defined isolating the initial condition x_0. In this alternative transformation the Jacobian is given by $J(\mathbf{v}) = |1/h_0(\hat{t},\xi)|$, and $h_0(\hat{t},\xi(\hat{\omega})) \neq 0$, given that

$$\begin{aligned} &h(\hat{t}) = 0 \Longleftrightarrow \sin\left(\omega_0\sqrt{1-\xi^2}\hat{t}\right) = 0 \Longleftrightarrow \cos\left(\omega_0\sqrt{1-\xi^2}\hat{t}\right)\\ &\quad = \pm 1 \Longleftrightarrow h_0(\hat{t}) = \pm\,\mathrm{e}^{-\xi\omega_0\hat{t}} \neq 0. \end{aligned} \tag{20}$$

3.2. Numerical examples

In this subsection, we show the capability of the theoretical results established in Section 3.1 throughout two numerical examples. In the first example, a sinusoidal force is considered, being this representation of the force one of the most typical in the literature. On the other hand, in the second example, a polynomial force is assumed. This is due to the fact that there are mathematical techniques that allow us to obtain approximations of a given force via a polynomial expression, as interpolation methods. The statements of both examples are shown below. As the conclusions obtained inspecting the results in both numerical experiments are similar, in an additional subsection our findings are drawn.

Example 1. In the particular case where the force is periodic and of the form $f(t) = F_0\sin(\phi t)$, we are able to determine an exact expression to the particular solution

$$x_p(t) = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \phi^2\right) + \left(2\xi\omega_0\phi\right)^2}}\sin(\phi t - \alpha), \tag{21}$$

where α is calculated as

$$\cos(\alpha) = \frac{\omega_0^2 - \phi^2}{\sqrt{\left(\omega_0^2 - \phi^2\right) + \left(2\xi\omega_0\phi\right)^2}}, \qquad \sin(\alpha) = \frac{2\xi\omega_0\phi}{\sqrt{\left(\omega_0^2 - \phi^2\right) + \left(2\xi\omega_0\phi\right)^2}}. \tag{22}$$

Then, substituting the particular solution obtained in (13) a closed expression for the response is determined. We choose the following distributions for the random parameters which, for sake of simplicity in the calculations, will be assumed independent:

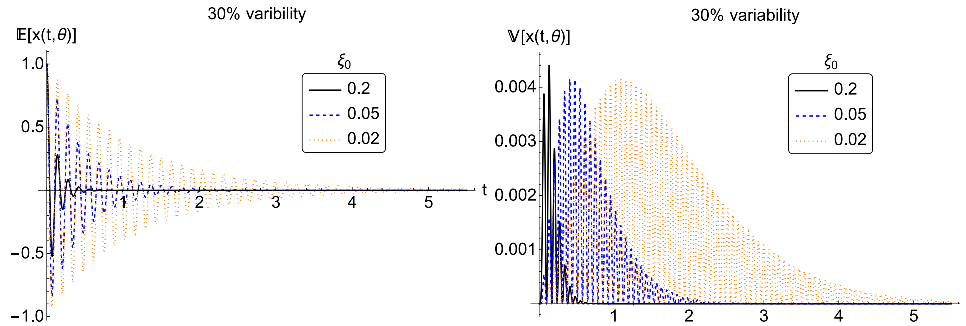

Fig. 1. Mean (left) and variance (right) of the response of the system (3), $x(t,\theta)$, when the damping parameter has a fixed percentage of variability, 30%, and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$. Numerical Example 1.

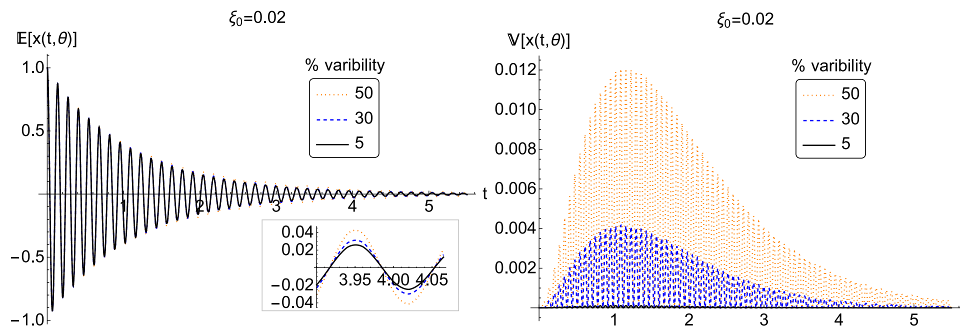

Fig. 2. Mean (left) and variance (right) of the response of the system (3), $x(t,\theta)$, when the damping parameter has a fixed mean $\xi_0 = 0.02$, and to different values of the percentage of variability, $p \in \{0.05, 0.3, 0.5\}$. Numerical Example 1.

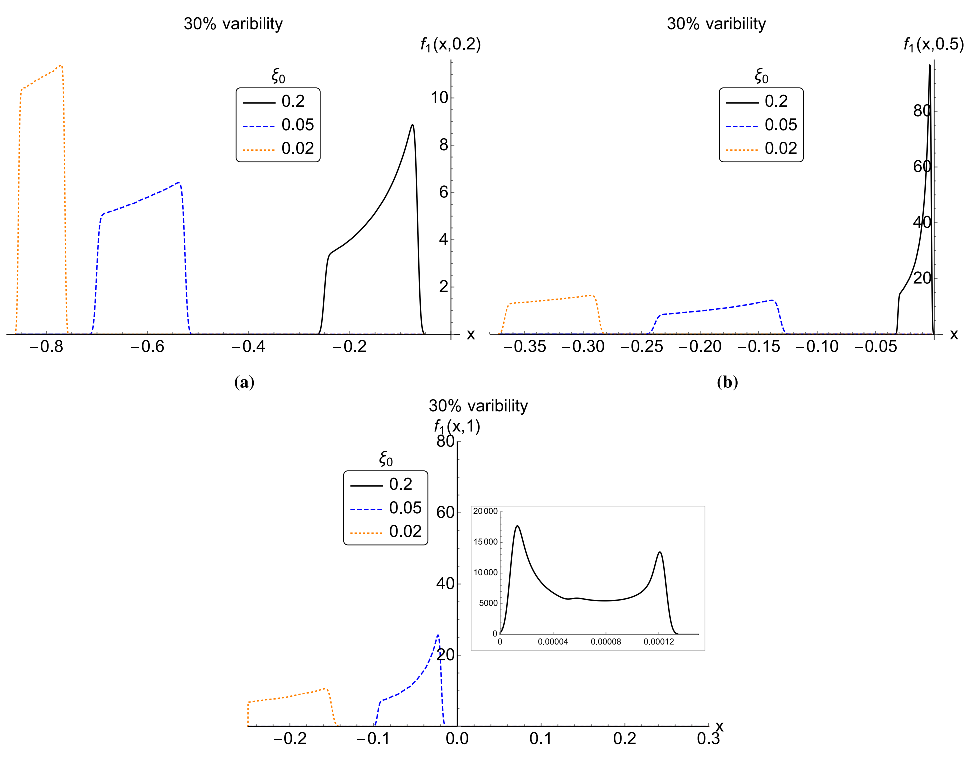

Fig. 3. 1-PDF of the response of the system $x(t,\theta)$, $f_1(x,t)$, when the damping parameter has a fixed percentage of variability, 30%, comparing different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$ at the time instants $t \in \{0.2, 0.5, 1\}$ ((a), (b) and (c), respectively). Numerical Example 1.

- The initial conditions $x_0(\theta)$ and $x_1(\theta)$ have an Uniform distribution in the interval $[1 - 10^{-10}, 1 + 10^{-10}]$, i.e., $x_i(\theta) \sim \mathrm{U}([1-10^{-10}, 1+10^{-10}])$, $i = 1, 2$.
- We compare different scenarios to the parameter $\xi(\theta)$ depending on its mean, ξ_0, and the proportion of variability respect to the mean, p. We consider that the random parameter $\xi(\theta)$ follows an

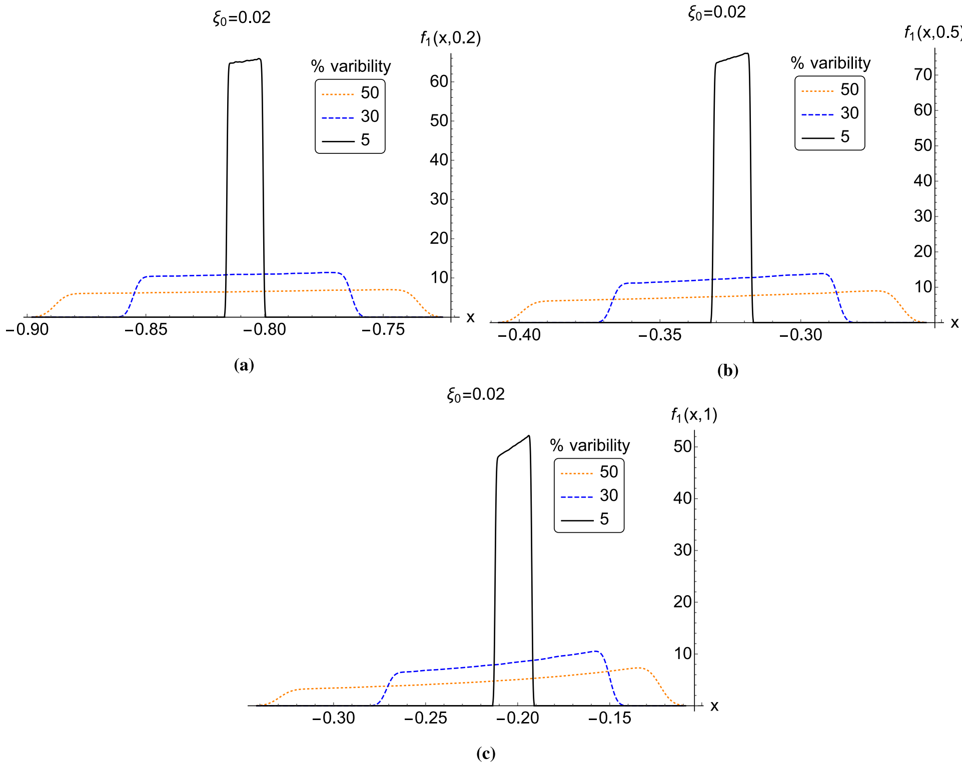

Fig. 4. 1-PDF of the response of the system $x(t,\theta)$, $f_1(x,t)$, when the damping parameter has a fixed mean $\xi_0 = 0.02$ comparing different values of the percentage of variability, $p \in \{0.05, 0.3, 0.5\}$ at the time instants $t \in \{0.2, 0.5, 1\}$ ((a), (b) and (c), respectively). Numerical Example 1.

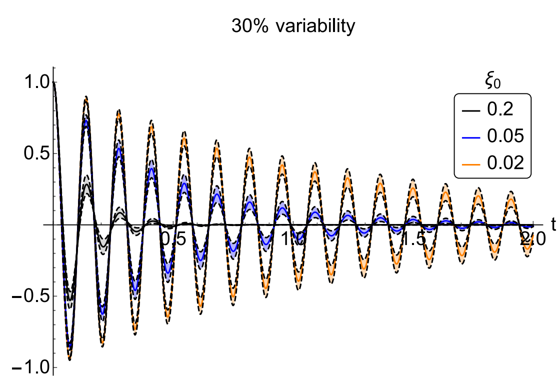

Fig. 5. Confidence interval of the response of the system (3), $x(t,\theta)$, when the damping parameter has a fixed percentage of variability, 30%, and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$. Numerical Example 1.

Uniform distribution in a given interval, i.e., $\xi(\theta) \sim \mathrm{U}([\xi_0(1-p), \xi_0(1+p)])$ with $\xi_0 \in \{0.02, 0.05, 0.2\}$ and $p \in \{0.05, 0.3, 0.5\}$.

Remark 2. It is important to remark here that we consider that the initial conditions have a uniform distribution with a low variance because, as previously indicated in the introduction, our main goal is to analyze the effect of the randomness of the damping parameter in the response. As previously indicated, in this case we can assume that $x_0(\theta) = x_1(\theta) = 1$ w.p.1, then its PDFs are the Dirac Delta functions. From a computational point of view, it is infeasible to consider the Dirac Delta function. Let $\delta(v - v_0)$ the Dirac Delta function defined in Eq. (5). It is well known that it can be approximated by many functions, see [28–30]. In particular, let $\delta_k(v - v_0)$ be a rectangular function centered in v_0, with the rectangle surface equal to 1, i.e.,

$$\delta_k(v - v_0) = \begin{cases} \frac{1}{k} & v \in [v_0 - k/2, v_0 + k/2], \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

It can be proven that

$$\lim_{k \to 0} \delta_k(v - v_0) = \delta(v - v_0). \tag{24}$$

Therefore, the Uniform distribution is used as an approximation of the Dirac Delta function.

Regarding the deterministic parameters, the undamped angular frequency is $\omega_0 = 46.13$ rad/s, the force $f(t) = 200\sin(t)$ and the mass $m = 36\,000$ kg.

Example 2. Now, consider the scenario where the force is a polynomial of degree n, i.e., $f(t) = \sum_{i=0}^{n} c_i t^i$. It is interesting to analyze this scenario, since any function for which we know the image of n + 1 values can be approximated by a polynomial, of degree less than or equal to n, applying interpolation techniques. With a polynomial force, the particular solution of the system (1) shall also be polynomial $x_p(t) = \sum_{i=0}^{n} a_i t^i$, where the coefficients a_i are calculated substituting it in (1) and equation term by term in both parts of the system. After some algebra we obtain

$$a_n = \frac{1}{\omega_0^2}\frac{c_n}{m}, \qquad a_{n-1} = \frac{1}{\omega_0^2}\left(\frac{c_{n-1}}{m} - 2\xi\omega_0 n a_n\right), \tag{25}$$

$$a_i = \frac{1}{\omega_0^2}\left(\frac{c_i}{m} - 2\xi\omega_0(i+1)a_{i+1} - (i+2)(i+1)a_{i+2}\right), \qquad \forall i = 0, 1, \dots, n-2. \tag{26}$$

To perform the numerical experiments the same distributions for the random initial conditions are considered. Regarding the damping coefficient, we assume that it follows a Beta distribution with mean $\xi_0 \in \{0.02, 0.05, 0.2\}$ and standard deviation proportional to the mean, $\sigma = \xi_0 p$, being σ the standard deviation and p a given proportion $p \in \{0.15, 0.3, 0.5\}$. Then, the random damping parameters has a Beta distribution defined as follows

$$\xi(\theta) \sim \mathrm{Be}\left(\frac{1-(1+p^2)\xi_0}{p^2}, \frac{(\xi_0 - 1)(p^2\xi_0 + \xi_0 - 1)}{p^2\xi_0}\right). \tag{27}$$

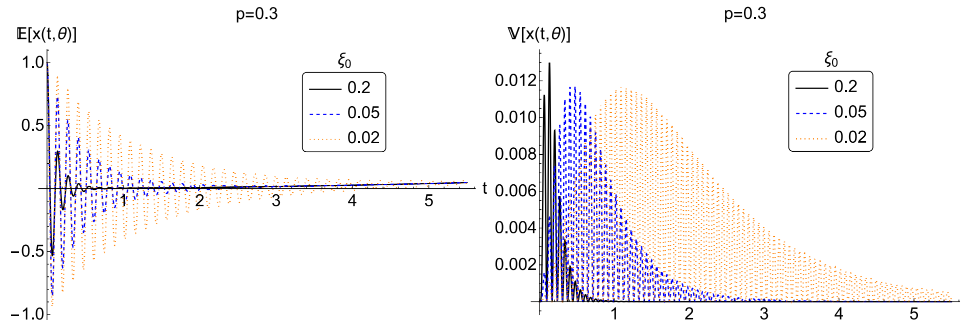

Fig. 6. Mean (left) and variance (right) of the response of the system (3), $x(t,\theta)$, when the damping parameter has a fixed proportion of the standard deviation, $p = 0.3$, and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$. Numerical Example 2.

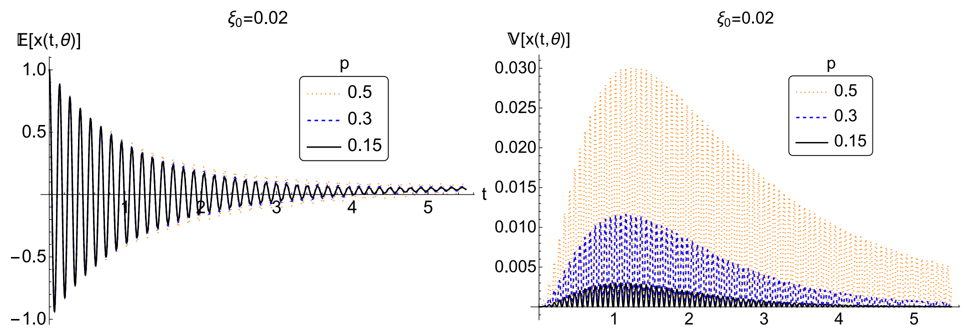

Fig. 7. Mean (left) and variance (right) of the response of the system (3), $x(t,\theta)$, when the damping parameter has a fixed mean $\xi_0 = 0.02$ and to different values of the proportion of the standard deviation $p \in \{0.15, 0.3, 0.5\}$. Numerical Example 2.

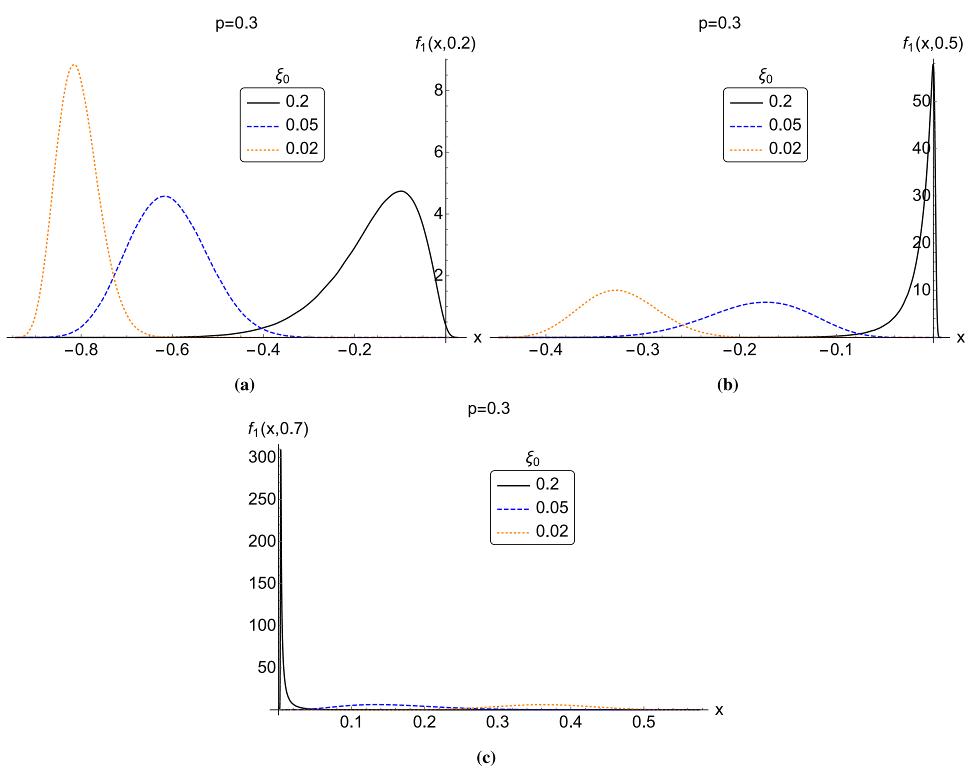

Fig. 8. 1-PDF of the response of the system $x(t,\theta)$, $f_1(x,t)$, when the damping parameter has a fixed proportion of the standard deviation, $p = 0.3$, comparing different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$ at the time instants $t \in \{0.2, 0.5, 0.7\}$ ((a), (b) and (c), respectively). Numerical Example 2.

Thus, we compare different scenarios depending on the mean and the dispersion of the randomized damping coefficient. The deterministic quantities are also the same, being the force $f(t) = 10^5(1 + t + t^2)$.

3.2.1. *Comments on the results obtained in Examples 1 and 2*

In Figs. 1 and 6 the mean and variance of the response of the system (3), $x(t,\theta)$, have been plotted when the damping parameter has a fixed

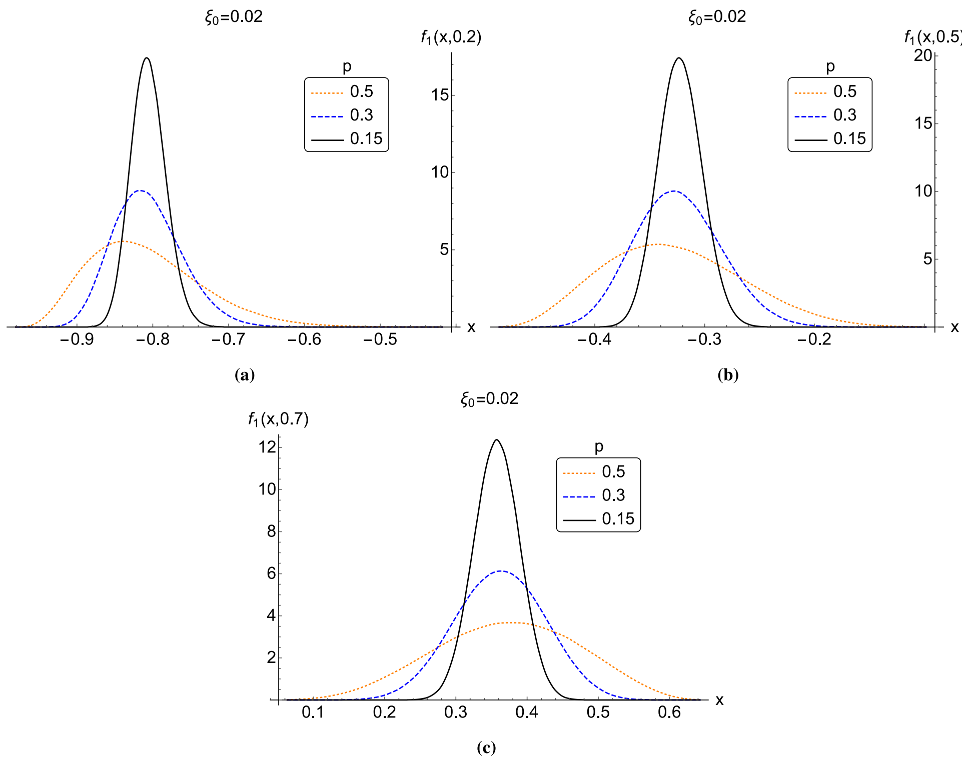

Fig. 9. 1-PDF of the response of the system $x(t,\theta)$, $f_1(x,t)$, when the damping parameter has a fixed mean $\xi_0 = 0.02$ comparing different values of the proportion of the standard deviation, $p \in \{0.15, 0.3, 0.5\}$ at the time instants $t \in \{0.2, 0.5, 0.7\}$ ((a), (b) and (c), respectively). Numerical Example 2.

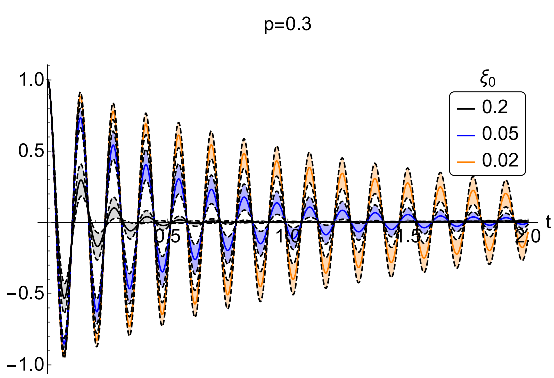

Fig. 10. Confidence interval of the response of the system (3), $x(t,\theta)$, when the damping parameter has a fixed proportion of the standard deviation, $p = 0.3$, and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$. Numerical Example 2.

variability and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$, in the context of the numerical examples corresponding to the analysis in the time domain, Examples 1 and 2, respectively. In Figs. 2 and 7 a similar graphical representation has been done, but in this case fixing the mean of the damping parameter, $\xi_0 = 0.02$, and to different values of the variability. From these graphical representations, it can be appreciated that the damping has a significant effect on the variance of the response. In particular, from Figs. 1 and 6 it can be observed that the less variability of the response occurs for the higher damping value. On the contrary, the more noticeable dispersion of the response is in the case of a smaller damping value ($\xi_0 = 0.02$). While in Figs. 2 and 7 it can be noted a significant difference in the mean and in the variance of the system response for the higher percentage of the variability of the damping parameter ($p = 0.5$).

Moreover, in Figs. 3–4 and 8–9 the 1-PDF of the response of the system at the time instants $t \in \{0.2, 0.5, 0.7\}$, related to Examples 1 and 2, has been plotted. From these graphs it is possible to appreciate the non-linear relation between the input random parameter and the response of the system. Therefore the choice of using a stochastic tool, as the PTM, that allows the definition of the PDF of the output functions system, is certainly recommended because the first two order statistics are not sufficient to describe the output probabilistically. In addition, the graphs of the PDF corroborate the above considerations, in fact, for a fixed value of the percentage of the damping variation, when the mean coefficient ξ_0 decreases, the response PDFs are more dispersed.

Table 1
The area enclosed between the extremes of the confidence interval at the time interval [0,2]. In the context of Examples 1 and 2.

	Confidence area		
	$\xi_0 = 0.02$	$\xi_0 = 0.05$	$\xi_0 = 0.2$
Example 1	0.134226	0.095728	0.027319
Example 2	0.226964	0.176279	0.059159

Finally, Figs. 5 and 10 report the confidence interval of the response of the system for a fixed percentage of variability, $p = 0.3$, and to different values of the mean of the random variable $\xi(\theta)$, for Examples 1 and 2, respectively. From this last inspection, in agreement with the above results, the value of the area enclosed between the extremes of the confidence interval at the time interval [0,2] is greater in the case of $\xi_0 = 0.02$ (see Table 1).

4. Analysis in the frequency domain

In this section, we analyze the effect of the randomness in the damping parameter $\xi(\theta)$ in the system (1) in the frequency domain. In this scenario and for the sake of completeness we assume that the initial conditions are deterministic and zero. Then, the density of the response can be computed directly from the PDF of the damping parameter.

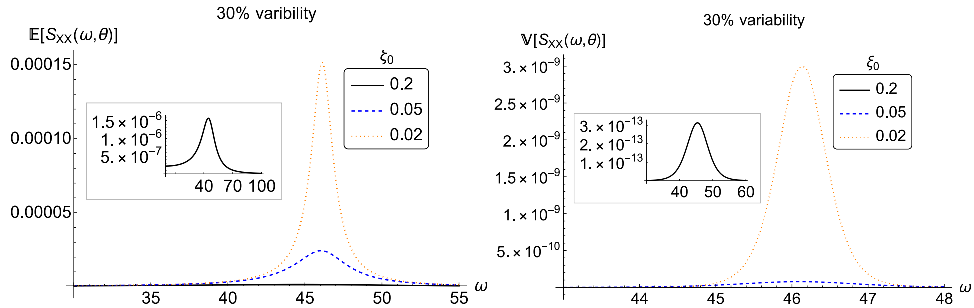

Fig. 11. Mean (left) and variance (right) of the power spectral density of the response of the system (3), $S_{XX}(\omega,\theta)$, when the damping parameter has a fixed percentage of variability, 30%, and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$. Numerical Example 3.

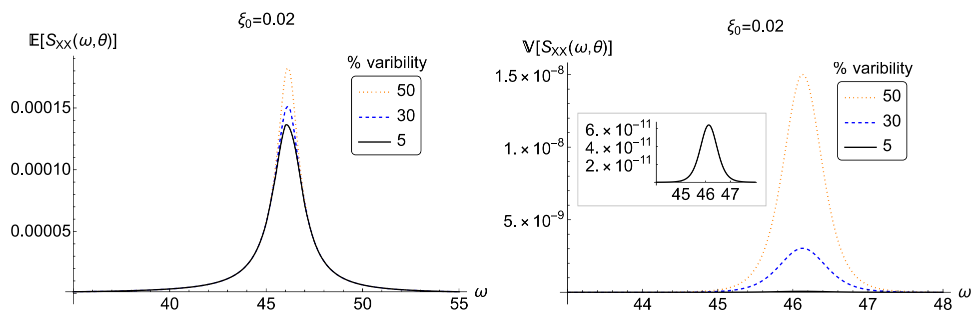

Fig. 12. Mean (left) and variance (right) of the power spectral density of the response of the system (3), $S_{XX}(\omega,\theta)$, when the damping parameter has a fixed mean $\xi_0 = 0.02$ and to different values of the percentage of variability, $p \in \{0.05, 0.3, 0.5\}$. Numerical Example 3.

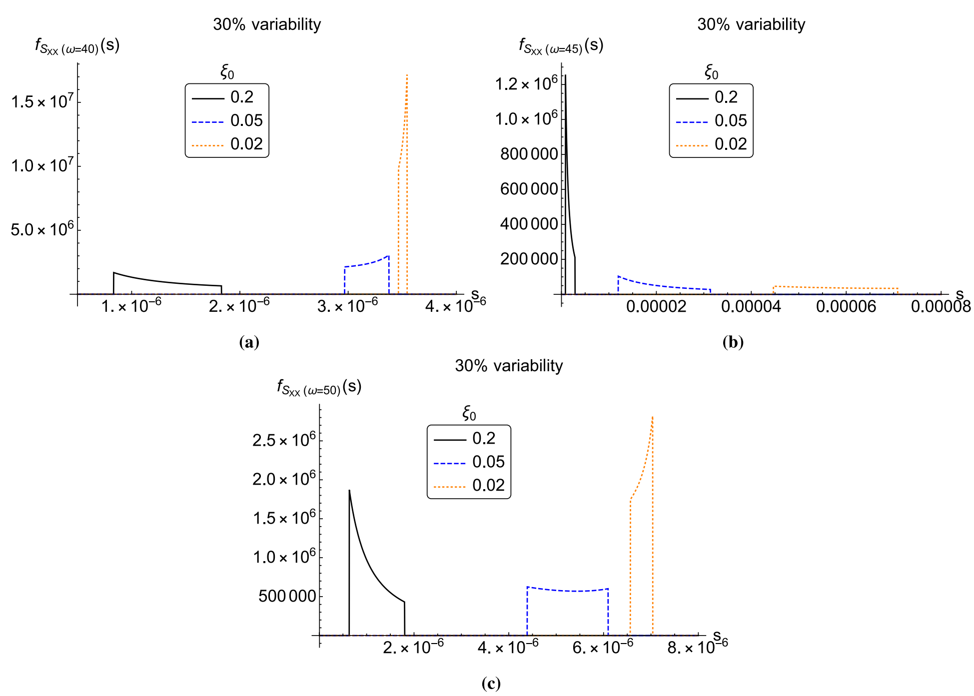

Fig. 13. 1-PDF of the power spectral density of the response of the system $S_{XX}(\omega,\theta)$, $f_{S_{XX}(\omega)}(s)$, when the damping parameter has a fixed percentage of variability, 30%, comparing different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$ at the fixed frequencies $\omega \in \{40, 45, 50\}$ ((a), (b) and (c), respectively). Numerical Example 3.

4.1. Computing the 1-PDF of the response

Applying the Fourier transformation in each member of the differential equation (1) and taking into account the properties of the derivative of the Fourier transformation, we get

$$X(\omega) = H(\omega)F(\omega), \quad H(\omega) = \frac{(\omega_0^2 - \omega^2) - \mathrm{i}\, 2\xi\omega_0\omega}{(\omega_0^2 - \omega^2)^2 + (2\xi\omega_0\omega)^2}. \tag{28}$$

where $X(\omega)$ and $F(\omega)$ are the Fourier transformations of the response and the force normalized with respect to the mass, respectively, and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. If the force $f(t)$ is a stationary SP with an expression for the power spectral density, we can apply the following relationship between the power spectral density of the response and the force

$$S_{XX}(\omega) = |H(\omega)|^2\, S_{FF}(\omega) = \frac{S_{FF}(\omega)}{(\omega_0^2 - \omega^2)^2 + (2\xi\omega_0\omega)^2}, \tag{29}$$

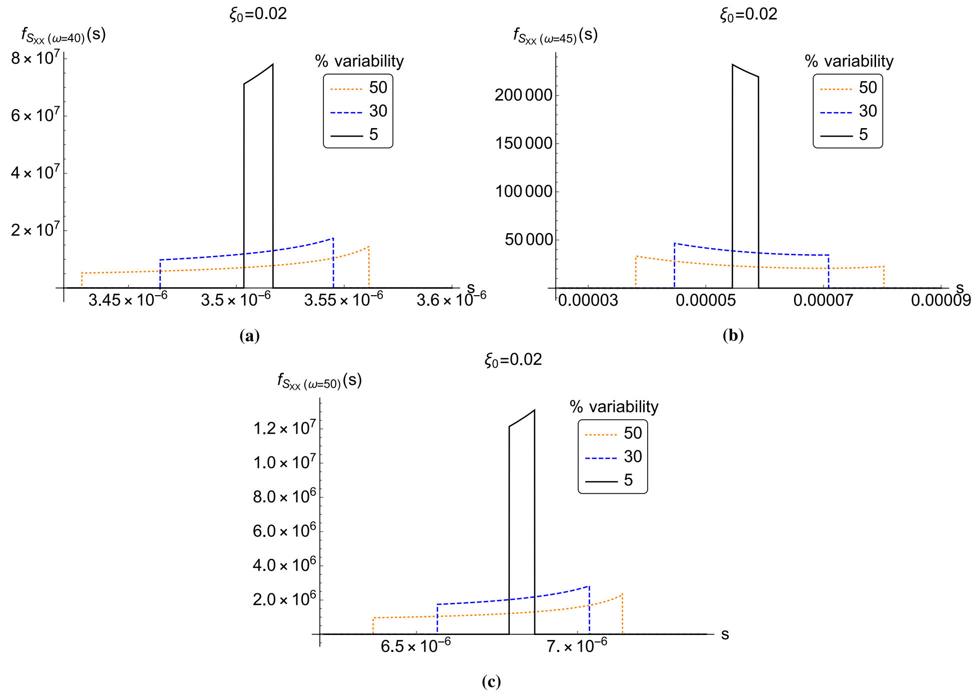

Fig. 14. 1-PDF of the power spectral density of the response of the system $S_{XX}(\omega,\theta)$, $f_{S_{XX}(\omega)}(s)$, when the damping parameter has a fixed mean $\xi_0 = 0.02$ and to different values of the percentage of variability, $p \in \{0.05, 0.3, 0.5\}$, at the fixed frequencies $\omega \in \{40, 45, 50\}$ ((a), (b) and (c), respectively). Numerical Example 3.

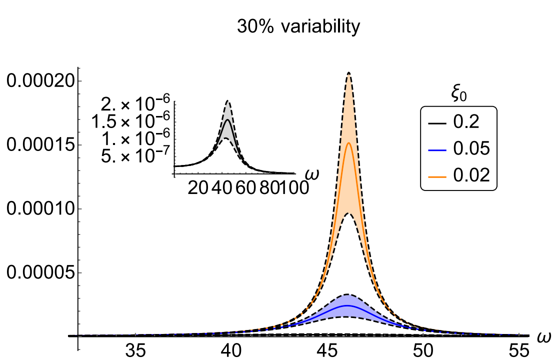

Fig. 15. Confidence interval of the power spectral density of the response of the system (3), $S_{XX}(\omega,\theta)$, when the damping parameter has a fixed percentage of variability, 30%, and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$. Numerical Example 3.

where $H(\omega)$ is the frequency-domain transfer function of the system while $|\cdot|$ denotes the modulus of a complex number. In this case, we can obtain the PDF of $S_{XX}(\omega,\theta)$ in terms of the PDF of $\xi = \xi(\theta)$, for each ω in the frequency domain, applying directly the PTM. Given a frequency $\omega > 0$, fixed, we apply the PTM to the following deterministic transformation $\mathbf{r} : \mathbb{R} \to \mathbb{R}$

$$v = r(\xi) = \frac{S_{FF}(\omega)}{(\omega_0^2 - \omega^2)^2 + (2\xi\omega_0\omega)^2}. \tag{30}$$

The inverse mapping $\mathbf{s} = \mathbf{r}^{-1} : \mathbb{R} \to \mathbb{R}$ and the Jacobian are

$$\xi = s(v) = \frac{\sqrt{S_{FF}(\omega) - \left(\omega^2 - \omega_0^2\right)^2 v}}{2\omega\omega_0\sqrt{v}},$$
$$J(v) = \left| \frac{S_{FF}(\omega)}{4\omega\omega_0 v^{3/2}\sqrt{S_{FF}(\omega) - \left(\omega^2 - \omega_0^2\right)^2 v}} \right| \neq 0. \tag{31}$$

Therefore, the PDF of the RV $v(\theta) = S_{XX}(\omega,\theta)$, for every ω in the frequency domain, is

$$f_{S_{XX}(\omega)}(s) = f_\xi\left(\frac{\sqrt{S_{FF}(\omega) - \left(\omega^2 - \omega_0^2\right)^2 s}}{2\omega\omega_0\sqrt{s}}\right) \times \left| \frac{S_{FF}(\omega)}{4\omega\omega_0 s^{3/2}\sqrt{S_{FF}(\omega) - \left(\omega^2 - \omega_0^2\right)^2 s}} \right|. \tag{32}$$

Remark 3. It should be noted that, for a fixed ω, the PDF of Eq. (32) is well defined if and only if $S_{FF}(\omega) - \left(\omega^2 - \omega_0^2\right)^2 s > 0$, s in the domain of the RV $S_{XX}(\omega,\theta)$. Otherwise the damping parameter would be a complex RV and this does not make sense in the context of the problem we are dealing with. Therefore, we must check this inequality. By Eq. (29), given a fixed frequency ω, we have for each $\theta \in \Omega$

$$s = \frac{S_{FF}(\omega)}{(\omega_0^2 - \omega^2)^2 + (2\xi(\theta)\omega_0\omega)^2} < \frac{S_{FF}(\omega)}{(\omega_0^2 - \omega^2)^2} \tag{33}$$

because of $(2\xi(\theta)\omega_0\omega)^2 > 0$. As $(\omega_0^2 - \omega^2)^2$ is also positive, then $s(\omega_0^2 - \omega^2)^2 < S_{FF}(\omega)$. In other words, $0 < S_{FF}(\omega) - s(\omega_0^2 - \omega^2)^2$, and the inequality is proved for all s in the domain of $S_{XX}(\omega,\theta)$ and for every selection of the involved parameters.

4.2. Numerical examples

In this subsection we show the capability of the theoretical results established in Section 4.1 throughout two numerical examples. In the first example, the simple case of the system under white noise has been investigated. Then, a typical model, widely used by people working in seismic areas, the Clough–Penzien spectra model, is used. For both cases, the effect of the random damping parameter, in the frequency domain, has been inspected. The statements of both examples are shown below. In the end, the outcomes of this last investigation will be summarized.

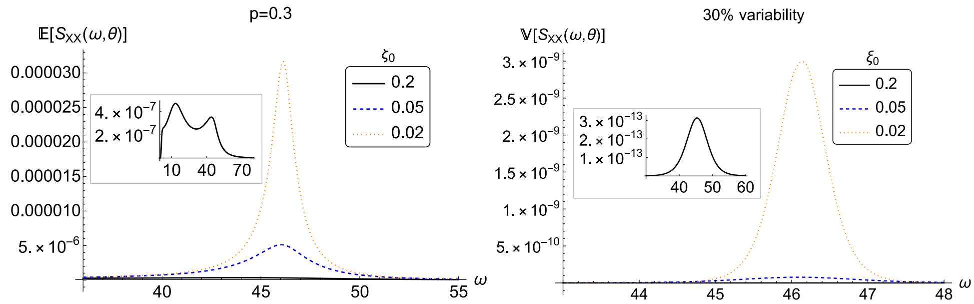

Fig. 16. Mean (left) and variance (right) of the power spectral density of the response of the system (3), $S_{XX}(\omega,\theta)$, when the damping parameter has a fixed proportion of the standard deviation, $p = 0.3$, and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$. Numerical Example 4.

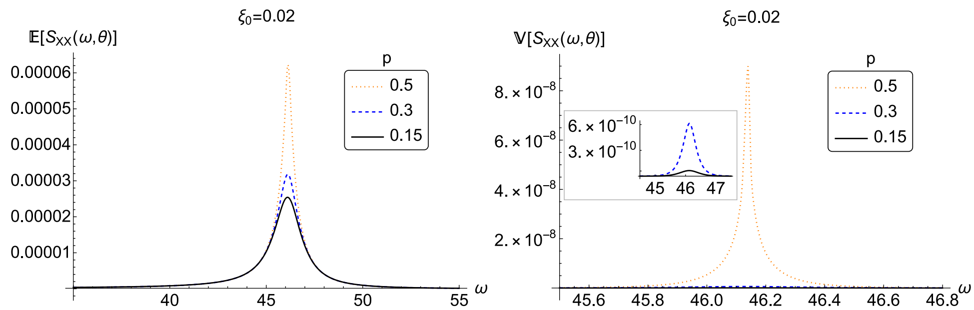

Fig. 17. Mean (left) and variance (right) of the power spectral density of the response of the system (3), $S_{XX}(\omega,\theta)$, when the damping parameter has a fixed mean $\xi_0 = 0.02$ and to different values of the proportion of the standard deviation, $p \in \{0.15, 0.3, 0.5\}$. Numerical Example 4.

Example 3. Suppose that the force is a white noise which the simplest and one of the most interesting examples of a stationary process. The power spectral density of a white noise is a constant value $S_{FF}(\omega) = F_0$. In this example we take, without loss of generality, $F_0 = 1$. In addition, we fix the undamped angular frequency as $\omega_0 = 46.136765$. As in the time domain analysis, we compare different scenarios to the parameter $\xi(\theta)$ depending on its mean, ξ_0, and the proportion of variability respect to the mean, p. We consider that the random parameter $\xi(\theta)$ follows a Uniform distribution in a given interval, i.e., $\xi(\theta) \sim \mathrm{U}([\xi_0(1-p), \xi_0(1+p)])$ with $\xi_0 \in \{0.02, 0.05, 0.2\}$ and $p \in \{0.05, 0.3, 0.5\}$ (see Fig. 14).

Example 4. In this last example, the power spectral density of the force, S_{FF}, is assumed as the Clough–Penzien spectra, given by the following relationship:

$$S_{FF}(\omega) = \frac{\omega_k^4 + 4\xi_k^2\omega_k^2\omega}{\left(\omega_k^2-\omega^2\right)^2 + 4\xi_k^2\omega_k^2\omega^2} \frac{\omega^4}{\left(\omega_p^2-\omega^2\right)^2 + 4\xi_p^2\omega_p^2\omega^2} S_0, \tag{34}$$

where ω_k is the ground frequency, $\xi_k = \omega_k/25$, while $\omega_p = \omega_k/10$ and ξ_p are the parameters of the additional filter. Then, S_0 is the constant PSD of the bedrock acceleration that controls the ground acceleration peaks. The following filtering coefficients are assumed $\omega_k = 15$, $\xi_p = 0.6$ and $S_0 = 1$. With respect the undamped angular frequency, we have choose the same than in Example 3, $\omega_0 = 46.136765$. We also compare different scenarios to the parameter $\xi(\theta)$ depending on its mean, ξ_0, and the standard deviation, calculated as a proportion of the mean $\xi_0 p$. In this way, the parameter $\xi(\theta)$ follows the Beta distribution defined in Example 2 with $\xi_0 \in \{0.02, 0.05, 0.2\}$ and $p \in \{0.15, 0.3, 0.5\}$ (see Fig. 19).

4.2.1. *Comments on the results obtained in Examples 3 and 4*

About the sensitivity of the stochastic response to the random damping in the frequency domain, the power spectral density function of the response has been inspected. Overall it can be appreciated that the mean and the variability of the damping coefficient has a significant effect on the response functions also in the context of the frequency domain. In particular, from the mean value of the power spectral response for both Examples 3 and 4, that means from Figs. 11 and 16, it can be notice that as the mean damping parameter increases the expectation function becomes more spread. Moreover, it is interesting to see the behavior of spectral peak which assume bigger value as the damping parameter decrease. In addition, from the outline of the variance of the response, also plotted in Figs. 11 and 16, it can be observed that the response is more dispersed in the case of a smaller mean of the damping value also in the frequency domain.

Additionally, in Figs. 12 and 17 the mean and variance of the power spectral response have been plotted when the expectation of damping parameter is 0.02 and to different values of the variability. We observe how when the variability of the damping coefficient increases the uncertainty of the response significantly grows.

The trend of PDFs of the power spectral density function is similar to the previous results in the time domain. Namely, in the frequency values for which the peak of the power spectral function occurs, the PDF of the response are more dispersed for the case of $\xi_0 = 0.02$, (see Figs. 13 and 18).

At last, Figs. 15 and 20 show the confidence interval of the power spectral response of the system for Examples 3 and 4, respectively. From Table 2, it is possible to appreciate that, also for these analyses, the value of the area enclosed between the extremes of the confidence interval is greater for the smallest damping value.

Table 2
The area enclosed between the extremes of the confidence interval at the frequency interval [0,100]. In the context of Examples 3 and 4.

	Confidence area		
	$\xi_0 = 0.02$	$\xi_0 = 0.05$	$\xi_0 = 0.2$
Example 3	0.000149	0.000059	0.000015
Example 4	0.000052	0.000021	0.000006

5. Conclusions

Overall, both in the time and in the frequency domain analyses, the uncertainty in damping has a significant influence on system response.

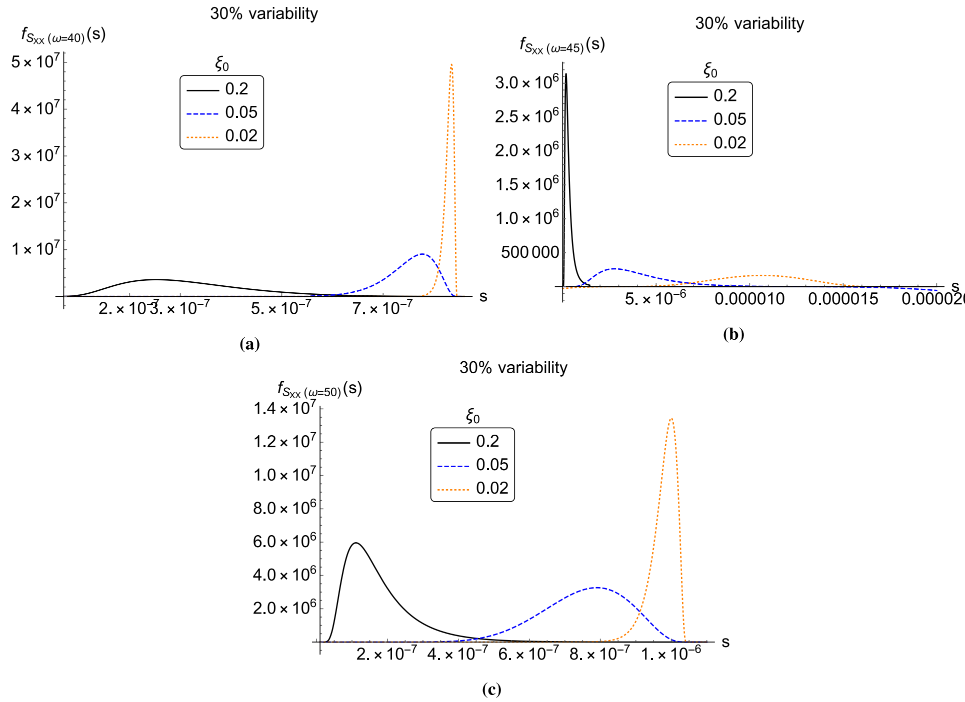

Fig. 18. 1-PDF of the power spectral density of the response of the system $S_{XX}(\omega,\theta)$, $f_{S_{XX}(\omega)}(s)$, when the damping parameter has a proportion of the standard deviation, $p = 0.3$, comparing different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$ at the fixed frequencies $\omega \in \{40, 45, 50\}$ ((a), (b) and (c), respectively). Numerical Example 4.

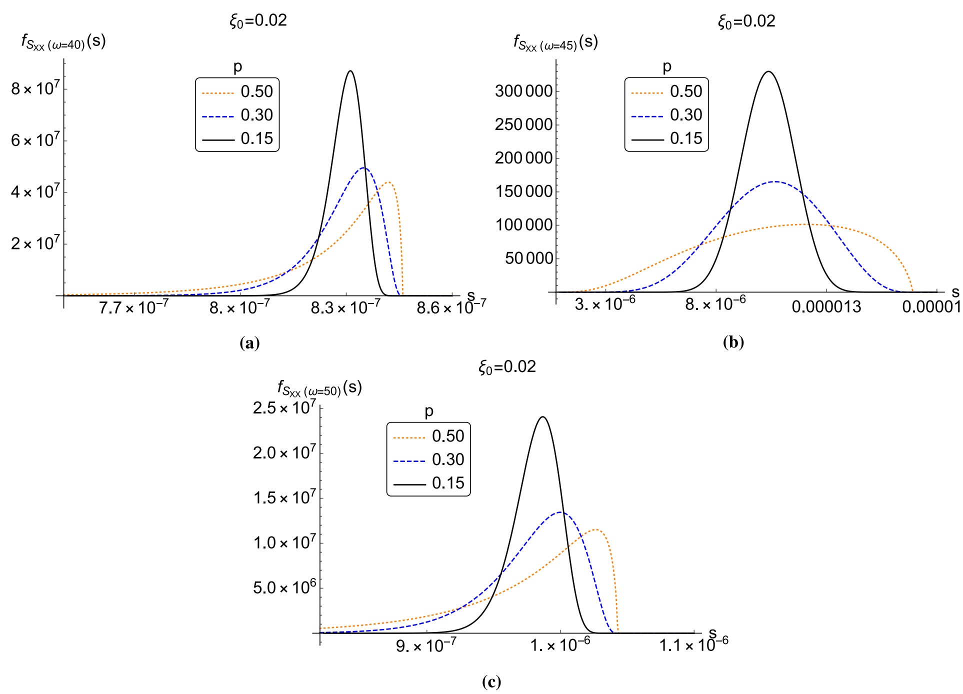

Fig. 19. 1-PDF of the power spectral density of the response of the system $S_{XX}(\omega,\theta)$, $f_{S_{XX}(\omega)}(s)$, when the damping parameter has a fixed mean $\xi_0 = 0.02$ and to different proportions of the standard deviation, $p \in \{0.15, 0.3, 0.5\}$, at the fixed frequencies $\omega \in \{40, 45, 50\}$ ((a), (b) and (c), respectively). Numerical Example 4.

In general, the effect of the randomness in damping in the response has the same trend for all different cases of study. Moreover, it is observed the level of sensitivity to the random damping of the response function depends on the mean value of the parameter. In particular, the most significant dispersion of the response is evident in the case of a smaller damping value.

The linear dynamic analysis of a harmonic oscillator with uncertain damping parameter subjected to deterministic excitation has been investigated. The obtained results, albeit for some benchmark examples, give interesting food for thought for a design engineer. In general, from both analyses, in the time and frequency domain, it can be observed that the role of random damping plays a significant effect on the response functions. That is, the uncertainty in the damping coefficient has a significant influence on system response. In general, the effect of the randomness in damping in the response has the same trend for all different cases of study. From the different studied scenarios, it is possible to appreciate that more variability of the response occurs for the smaller damping value. Moreover, the variability of the random

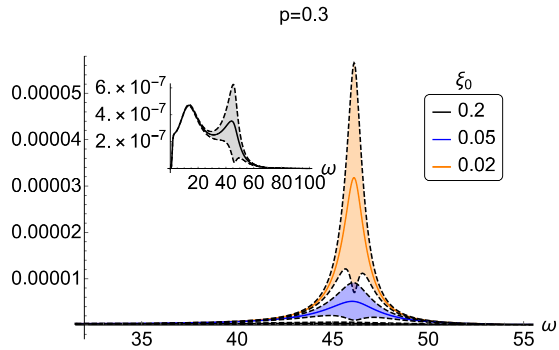

Fig. 20. Confidence interval of the power spectral density of the response of the system (3), $S_{XX}(\omega, \theta)$, when the damping parameter has a fixed proportion of the standard deviation, $p = 0.3$, and to different values of the mean, $\xi_0 \in \{0.02, 0.05, 0.2\}$. Numerical Example 4.

parameter strongly affects the variability of the response functions. It is also observed that the level of sensitivity to the random damping of the response function depends on the mean value of the parameter. In particular, the most significant dispersion of the response is evident in the case of a smaller damping value. Therefore, the main conclusion from this study is that for dynamic analysis of systems, a design engineer should be taken into consideration the possibility to do a stochastic analysis instead of a deterministic one especially for systems characterized by a low value of damping. Finally, in the authors' opinion, cause a non-linear relation between the random input and the output of the system occurs, a complete stochastic characterization of the response in terms of PDF is recommended, and the use of the PTM confirms again how the stochastic method is an adequate and competitive tool for the probabilistic characterization of the structural response.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] L. Arnold, Random dynamical systems, in: Dynamical Systems, Springer, 1995, pp. 1–43.
[2] L. Lutes, S. Sarkani, Random Vibrations: Analysis of Structural and Mechanical Systems, Elsevier, Oxford, 2004.
[3] J.B. Roberts, P.D. Spanos, Random Vibration and Statistical Linearization, Courier Corporation, 2003.
[4] A. Preumont, Random Vibration and Spectral Analysis/Vibrations Aléatoires et Analyse Spectral, Vol. 33, Springer Science & Business Media, 1994.
[5] L. Shen, M. Ostoja-Starzewski, E. Porcu, Harmonic oscillator driven by random processes having fractal and hurst effects, Acta Mech. 226 (11) (2015) 3653–3672.
[6] T. Caughey, H. Stumpf, Transient response of a dynamic system under random excitation, 1961.
[7] M. Gitterman, Classical harmonic oscillator with multiplicative noise, Phys. A 352 (2-4) (2005) 309–334.
[8] S. Adhikari, B. Pascual, The 'damping effect' in the dynamic response of stochastic oscillators, Probab. Eng. Mech. 44 (2016) 2–17.
[9] R. Singh, C. Lee, Frequency response of linear systems with parameter uncertainties, J. Sound Vib. 168 (1) (1993) 71–92.
[10] M. Gitterman, Stochastic oscillator with random mass: new type of Brownian motion, Phys. A 395 (2014) 11–21.
[11] A. Kareem, K. Gurley, Damping in structures: its evaluation and treatment of uncertainty, J. Wind Eng. Ind. Aerodyn. 59 (2-3) (1996) 131–157.
[12] B. Tiliouine, B. Chemali, On the sensitivity of dynamic response of structures with random damping, in: Proc. 21st. French Congress on Mechanics, Citeseer, Bordeaux, France, 2013.
[13] Y. Tamura, Amplitude dependency of damping in buildings and critical tip drift ratio, Int. J. High-Rise Build. 1 (1) (2012) 1–13.
[14] G.C. Hart, Random damping in buildings, J. Wind Eng. Ind. Aerodyn. 59 (2-3) (1996) 233–246.
[15] P. Gueguen, M. Langlais, P. Roux, J. Schinkmann, I. Douste-Bacqué, Frequency and damping wandering in existing buildings using the random decrement technique, in: EWSHM-7th European Workshop on Structural Health Monitoring, 2014.
[16] J. Fang, A. Jeary, Q. Li, C. Wong, Random damping in buildings and its AR model, J. Wind Eng. Ind. Aerodyn. 79 (1-2) (1999) 159–167.
[17] K. Sepahvand, F.S. Khosroshahi, C. Geweth, S. Marburg, Stochastic structural dynamic analysis with random damping parameters, in: Proceedings of the 46th International Congress and Exposition on Noise Control Engineering, 2017.
[18] K. Sepahvand, Stochastic finite element method for random harmonic analysis of composite plates with uncertain modal damping parameters, J. Sound Vib. 400 (2017) 1–12.
[19] R. Laudani, G. Falsone, Use of the probability transformation method in some random mechanic problems, ASCE-ASME J. Risk Uncertain. Eng. Syst. A 7 (1) (2021) 04020054.
[20] G. Falsone, R. Laudani, Closed-form solutions of redundantly constrained stochastic frames, Probab. Eng. Mech. 61 (2020) 103084.
[21] R. Laudani, G. Falsone, Response probability density function for multi-cracked beams with uncertain amplitude and position of cracks, Appl. Math. Modell. 99 (2021) 14–26.
[22] M. Casabán, J. Cortés, A. Navarro-Quiles, J. Romero, M. Roselló, R. Villanueva, Probabilistic solution of the homogeneous riccati differential equation: a case-study by using linearization and transformation techniques, J. Comput. Appl. Math. 291 (2016) 20–35.
[23] J. Cortés, A. Navarro-Quiles, J. Romero, M. Roselló, Probabilistic solution of random autonomous first-order linear systems of ordinary differential equations, Rom. Rep. Phys. 68 (2016) 1397–1406.
[24] J. Cortés, A. Navarro-Quiles, J. Romero, M. Roselló, Probabilistic solution of random autonomous first-order linear systems of ordinary differential equations, Appl. Math. Lett. 68 (2017) 150–156.
[25] J. Cortés, A. Navarro-Quiles, J. Romero, M. Roselló, E. Zuazua, Full probabilistic solution of a finite dimensional linear control system with random initial and final conditions, J. Franklin Inst. 357 (2020) 8156–8180.
[26] G. Falsone, R. Laudani, Matching the principal deformation mode method with the probability transformation method for the analysis of uncertain systems, Int. J. Numer. Methods Eng. 118 (7) (2019) 395–410.
[27] T.T. Soong, Random Differential Equations in Science and Engineering, Academic Press, New York, 1973.
[28] R. Hoskings, Delta Functions: Introduction to Generalised Functions, Woodhead Publishing, 2009.
[29] L. Piela, Ideas of Quantum Chemistry, Elsevier, 2007.
[30] B.E. Rapp, Microfluidics: Modeling, Mechanics and Mathematics, Elsevier, 2017.