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Research Letter

The Dyon Charge in Noncommutative Gauge Theories

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We construct a dyon solution for the noncommutative version of the Yang-Mills-Higgs model with a ϑ-term. Extending the Noether method to the case of a noncommutative gauge theory, we analyze the effect of CP violation induced both by the ϑ-term and by noncommutativity proving that the Witten effect formula for the dyon charge remains the same as in ordinary space.

Gauge theories coupled to Higgs scalars exhibit a remarkable phenomenon called Witten effect [1], related to the ϑ-angle. Indeed, if one adds to a Yang-Mills-Higgs Lagrangian a ϑ-term

$$\Delta L = \vartheta \frac{e^2}{32\pi^2} \varepsilon^{\mu\nu\alpha\beta} \operatorname{tr}(F_{\mu\nu}F_{\alpha\beta}), \tag{1}$$

which explicitly violates CP, the electric charge q_e of a dyon is modified. Instead of being quantized, as in the $\vartheta = 0$ case, one has, for $\vartheta \neq 0$,

$$q_e = \left(n + \frac{\vartheta}{2\pi}\right)e, \quad n \in Z. \tag{2}$$

This result corresponds to a Julia-Zee dyon [2] with magnetic charge $m = 4\pi/e$. There are also arguments leading to the conjecture that other CP violating interactions may also induce a shift of the dyon charge [1].

CP violation can be induced not just by adding, as in (1), new interactions to the Yang-Mills-Higgs Lagrangian but by radically changing the setting of the theory. This is the case of noncommutative gauge theories (NCGT) where the introduction of noncommutation in space-time coordinates has shown to affect the behavior under C, P, and T invariances [3–6]. More specifically, one can prove that when noncommutativity is restricted to space coordinates,

$$[x_i, x_j] = i\theta_{ij}, \quad [x_i, x_0] = 0, \quad i, j = 1, 2, 3. \tag{3}$$

As it is well known [4], NCGTs are not charge invariant. Only if the usual field transformations are accompanied by a change of sign in θ_{ij}, charge invariance is recovered. Hence, if one takes θ_{ij} as a fixed parameter, CP is violated, although CPT invariance is maintained since parity invariance is not affected by the introduction of θ and time reversal undergoes a change that compensates that in C.

It is then natural to pose the question whether the dyon charge in noncommutative gauge theories receives a contribution from a CP violating effect induced by noncommutativity even if the ϑ angle vanishes. Moreover, one could also ask how the addition of the noncommutative version of the term (1) modifies Q when both $\theta_{ij} \neq 0$ and $\vartheta \neq 0$.

We will analyze these questions in the present letter and, to this end, we first discuss the properties of the dyon in a noncommutative Yang-Mills-Higgs model with $U(2)$ gauge symmetry, calculate its charge at the quantum level and also extend the theory in order to include a ϑ-term. Some of these issues were briefly discussed in [7] starting from a monopole solution obtained generalizing Nahm's equations that describes BPS solitons [8] (some aspects of dyon solutions were also considered in [9]). Here, instead, we will extend the more explicit $U(2)$ monopole solution found in [10, 11] to the case of a dyon and then establish a noncommutative version of the Noether theorem. We then discuss the issue of the Witten effect in noncommutative space.

We start from the noncommutative $U(2)$ Yang-Mills-Higgs system

$$S_{\mathrm{NCYM}} = \operatorname{tr}\int d^4x \left(-\frac{1}{2} F_{\mu\nu} * F^{\mu\nu} + D_\mu\Phi * D^\mu\Phi \right). \tag{4}$$

Gauge fields $A_\mu = A_\mu^A t^A$ take values in the Lie algebra of $U(2)$ with generators $t^A, A = 0,1,2,3$ ($t^0 = I/2, t^a = \sigma^a/2, a = 1,2,3$). $\Phi = \Phi^A t^A$ is the Higgs multiplet and we consider the Prasad-Sommerfield limit [12] in which the symmetry breaking potential vanishes. Covariant derivatives and field strength are given by

$$\begin{aligned} D_\mu\Phi &= \partial_\mu\Phi - ie[A\mu, \Phi]_*, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A\mu, A\nu]_*, \end{aligned} \tag{5}$$

where the Moyal product and commutator are defined as

$$\begin{aligned} A(x)*B(x) &= \exp\left(\frac{i}{2}\theta_{\mu\nu}\partial_x^\mu\partial_y^\nu\right)A(x)B(y)\Big|_{y=x}, \\ [A(x), B(x)]_* &= A(x)*B(x) - B(x)*A(x). \end{aligned} \tag{6}$$

As in (3) we will take $\theta_{0i} = 0$. This ensures a well-defined Hamiltonian and unitarity at the quantum level.

To find noncommutative static dyon solutions we will minimize the energy

$$\begin{aligned} E &= \operatorname{tr}\int d^3x\left(E_i*E_i + B_i*B_i + D_i\Phi*D_i\Phi + D_0\Phi*D_0\Phi\right), \\ & E_i = -F_{0i}, \quad B_i = -\frac{1}{2}\varepsilon_{ijk}F^{jk}. \end{aligned} \tag{7}$$

Since we are working in the BPS limit, the vacuum expectation value enters as a boundary condition on the Higgs field, $\operatorname{tr}\Phi_{\text{vac}}^2 = v_0^2/2$. As in ordinary space, (7) can be written in the form

$$\begin{aligned} E = \operatorname{tr}\int d^3x\big(&(E_i - \sin\alpha D_i\Phi)*(E_i - \sin\alpha D_i\Phi) \\ &+ (B_i - \cos\alpha D_i\Phi)*(B_i - \cos\alpha D_i\Phi) \\ &+ D_0\Phi*D_0\Phi + 2\sin\alpha E_i*D_i\Phi \\ &+ 2\cos\alpha B_i*D_i\Phi\big), \end{aligned} \tag{8}$$

leading to a Bogomol'nyi bound on the energy

$$E \geq v_0\,\sin\alpha Q + v_0\,\cos\alpha M \tag{9}$$

with Q and M are the electric and magnetic charges defined as

$$Q = \frac{2}{v_0}\operatorname{tr}\int d^3x E_i*D_i\Phi, \qquad M = \frac{2}{v_0}\operatorname{tr}\int d^3x B_i*D_i\Phi. \tag{10}$$

The bound is saturated whenever the following BPS equations hold:

$$E_i = \sin\alpha D_i\Phi, \tag{11}$$

$$B_i = \cos\alpha D_i\Phi, \tag{12}$$

$$D_0\Phi = 0. \tag{13}$$

In order to find an explicit dyon solution, we will now consider an expansion of fields A_μ and Φ in powers of the noncommutative parameter in θ, thus extending to the dyon case the approach developed in [10] and [11] where a purely magnetically charged solution was considered. One starts from the exact Prasad-Sommerfield solution in ordinary space [12] as giving the zeroth order of an expansion in powers of θ for the monopole in noncommutative space. Plugging this expansion into the BPS equations, one obtains the noncommutative solution order by order in θ.

We take as zeroth-order approximation for the $SU(2)$ components the Prasad-Sommerfield dyon solution [12], which we call $\Phi^{a(0)}$, $A_i^{a(0)}$, and $A_0^{a(0)}$. Concerning the $U(1)$ components, absent in the original Julia-Zee dyon, we propose for simplicity $\Phi^{0(0)} = A_i^{0(0)} = A_0^{0(0)} = 0$. To first order in θ, an ansatz for the gauge potential and Higgs field components on $U(1)$, that obey covariance under the $SO(3)$ rotation corresponding to the diagonal subgroup of $SO(3)_{\text{gauge}} \times SO(3)_{\text{space}}$, is

$$\begin{aligned} \widetilde{A}_i^{0(1)} &= \theta_{ij}x_jA(r) + \varepsilon_{ijk}\theta_{jk}C(r) + x_i\varepsilon_{jkl}\theta_{jk}x_lD(r), \\ \widetilde{A}_0^{0(1)} &= \theta_{ij}\varepsilon_{ijk}x_kK(r), \\ \widetilde{\Phi}^{0(1)} &= \theta_{ij}\varepsilon_{ijk}x_kB(r), \end{aligned} \tag{14}$$

where $A(r), B(r), C(r), D(r)$, and $K(r)$ are radial functions to be determined. The component on $SU(2)$ to first order in θ of the Bogomol'nyi equation is not going to be analyze, because it is pure gauge.

One can easily see that Bogomol'nyi (11) and (13) imply, to all orders in θ, the relation

$$A_0 = \sin\alpha\Phi. \tag{15}$$

We have then proven, using a θ-expansion, a relation that was proposed as an ansatz in [7] within Nahm's approach to the construction of monopole solutions. Now, (15) implies that the BPS (11) is automatically satisfied and then the study of the BPS system (11)-(12) is reduced to the analysis of the sole equation

$$\frac{1}{2}\varepsilon_{ijk}F_{jk} = -\cos\alpha D_i\Phi \tag{16}$$

which, except for the factor $\cos\alpha$, is nothing but the pure noncommutative monopole equation. Then, after making the appropriate rescaling, we have from [10, 11]

$$\begin{aligned} \widetilde{A}_i^{(n)}(x^i) &= \cos^{2n+1}\alpha A_i^{(n)}(\cos\alpha x^i), \\ \widetilde{\Phi}(x^i)(n) &= \cos^{2n}\alpha\Phi^{(n)}(\cos\alpha x^i), \quad n = 1,2,\ldots, \end{aligned} \tag{17}$$

where $A_i^{(n)}$ and $\Phi^{(n)}$ are the pure monopole solutions and $\widetilde{A}_0^{(n)}(x^i)$ is related to them through (15).

One can easily determine the asymptotic behavior of the proposed solution and then compute the electric and magnetic charges of the dyon solution. One finds that the $n > 0$ terms do not contribute at the surface at infinity so that one just recovers the $n = 0$ result, one finds

$$M = \frac{4\pi}{e}, \qquad Q = M\tan\alpha. \tag{18}$$

So that one can conclude that $n > 0$ orders in θ do not contribute to the charges that thus coincide with those in ordinary space. That is, CP violation induced by noncommutativity does not change, at least at the classical level, the charge of the dyon, no trace of θ_{ij}, appears in Q.

In order to analyze charge quantization at the quantum level, we use the Noether approach and canonically proceed as originally done in [1] regarding the unbroken symmetry which leaves the Higgs vacuum invariant and is associated to the electric charge. To do this, we will not just consider action (4) but the one including the noncommutative version of a ϑ-term,

$$S_\vartheta = S_{\text{NCYM}} + \frac{e^2\vartheta}{16\pi^2}\int d^4x F_{\mu\nu} * \widetilde{F}^{\mu\nu}. \tag{19}$$

In this way, we will be able to test possible modifications of the dyon charge because both the noncommutativity and the ϑ-term. As in ordinary space, the noncommutative version of the ϑ-term can be written as a surface term [13] and hence the equations of motion for action (19) and its dyon solutions remain the same for all values of ϑ.

After symmetry breaking through the condition $\Phi^2_{\text{vac}} = v_0^2/2$, the unbroken symmetry is related to rotations Λ in the direction of Φ leaving the Higgs vacuum invariant. Let us consider such gauge transformations along the Higgs field direction

$$\begin{gathered} \Lambda(x) = \frac{1}{v_0}\Phi^a t^a \epsilon(x), \qquad \epsilon(\infty) = 1, \\ \delta_\Lambda \Phi = 0, \qquad \delta_\Lambda A_\mu = \frac{1}{ev_0} D_\mu \Phi. \end{gathered} \tag{20}$$

As first noticed for a scalar theory in [14], in order to apply the Noether method in the noncommutative case, one has to take into account $*$-commutators that, once integrated, give a vanishing contribution. That is, from the well-honnored Noether formula

$$\delta_{\Lambda(x)} S = \int d^4x \partial_\mu J^\mu [\Phi(x), A_\mu(x)]\epsilon(x) \tag{21}$$

one can at most infer that

$$\delta_{\Lambda(x)} S = 0 \Longrightarrow (\partial_\mu J^\mu) = \text{tr}\big([O,P]_* + [B,C]_* * B + \cdots\big) \tag{22}$$

for some proper functionals $O, P, B, C, \ldots$ since, once integrated over space-time, the right-hand side in (22) vanishes due to the $*$-product cyclic properties under integration. Also when integrating (22) over 3 spaces to find the conserved charge commutators vanish since, as explained above, $\theta_{0i} = 0$. With all this, we finally have for the conserved charge, which we call N,

$$N = -\frac{1}{e}Q + \frac{\vartheta e}{8\pi^2} M. \tag{23}$$

At large distances, $\exp(2\pi i N)$ should implement a 2π rotation about the direction of Φ and hence the identity. Then the eigenvalues of N have to be quantized in integer units n. If we call q_e and q_g, the eigenvalues of the electric and magnetic charge operators, one then has

$$q_e = \left(ne + \frac{\vartheta e^2}{8\pi^2} q_g\right). \tag{24}$$

That is, we have obtained for the nonconmmutative dyon the same formula that holds for the case of ordinary space, (2) ($q_g = 4\pi/e$).

In summary, we have constructed an explicit noncommutative dyon solution showing that the relation (18) between classical electric and magnetic charge also holds in noncommutative space. Moreover, after extending the Noether approach to the case of a noncommutative gauge theory, we have proven that CP violation introduced by the commutation rule (3) does not change the Witten effect formula; indeed, the dyon's charge shift is θ_{ij}-independent. In this respect, it should be interesting to consider other type of noncommutativity and in particular, to investigate the case of the dyon in the fuzzy sphere along the lines developed in [15] where monopole solutions were constructed for the case in which $\theta_{ij} = \theta r \varepsilon_{ijk} x_k$ since in that case the coordinate dependence of θ_{ij} may introduce definite changes in (24). We hope to discuss this issue in the future.

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REFERENCES

[1] E. Witten, "Dyons of charge eϑ/2Π," *Physics Letters B*, vol. 86, no. 3-4, pp. 283–287, 1979.
[2] B. Julia and A. Zee, "Poles with both magnetic and electric charges in non-Abelian gauge theory," *Physical Review D*, vol. 11, no. 8, pp. 2227–2232, 1975.
[3] M. Hayakawa, "Perturbative analysis on infrared aspects of noncommutative QED on R^4," *Physics Letters B*, vol. 478, no. 1–3, pp. 394–400, 2000.
[4] M. M. Sheikh-Jabbari, "*C*, *P*, and T invariance of noncommutative gauge theories," *Physical Review Letters*, vol. 84, no. 23, pp. 5265–5268, 2000.
[5] P. Aschieri, B. Jurčo, P. Schupp, and J. Wess, "Noncummutative GUTs, standard model and *C*, *P*, *T*," *Nuclear Physics B*, vol. 651, no. 3, pp. 45–70, 2003.
[6] M. Chaichian, K. Nishijima, and A. Tureanu, "Spin-statistics and CPT theorems in noncommutative field theory," *Physics Letters B*, vol. 568, no. 1-2, pp. 146–152, 2003.
[7] D. J. Gross and N. A. Nekrasov, "Dynamics of strings in noncommutative gauge theory," *Journal of High Energy Physics*, vol. 10, no. 4, p. 021, 2000.
[8] D. J. Gross and N. A. Nekrasov, "Monopoles and strings in noncommutative gauge theory," *Journal of High Energy Physics*, vol. 7, p. 034, 2000.
[9] D. Bak and K. Lee, "Elongation of moving noncommutative solitons," *Physics Letters B*, vol. 495, no. 1-2, pp. 231–236, 2000.
[10] K. Hashimoto, H. Hata, and S. Moriyama, "Brane configuration from monopole solution in non-commutative super

Yang-Mills theory," *Journal of High Energy Physics*, vol. 12, no. 3, p. 021, 1999.

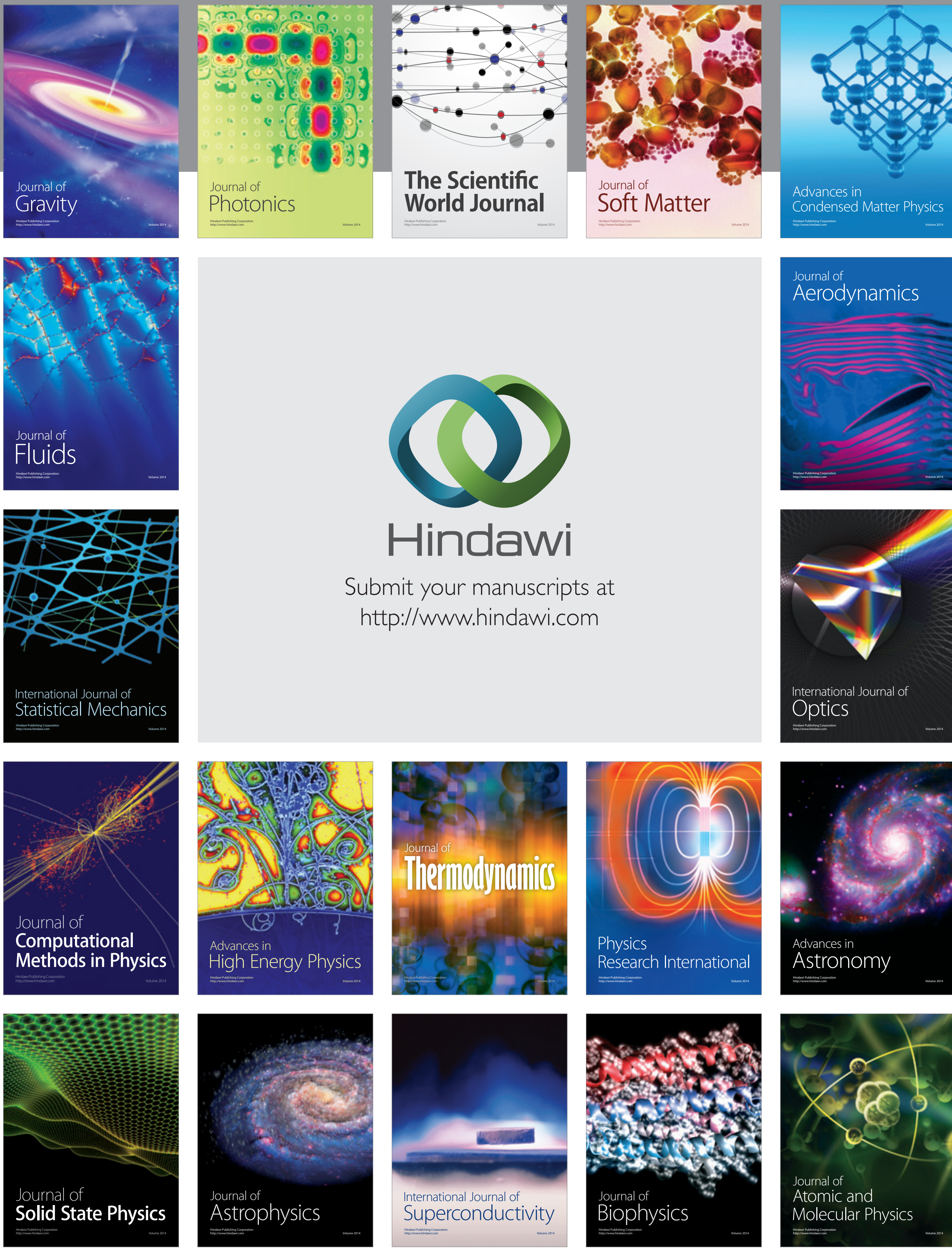
[11] S. Goto and H. Hata, "Noncommutative monopole at the second order in ϑ," *Physical Review D*, vol. 62, no. 8, Article ID 085022, 10 pages, 2000.

[12] M. K. Prasad and C. M. Sommerfield, "Exact classical solution for the 't hooft monopole and the Julia-Zee Dyon," *Physical Review Letters*, vol. 35, no. 12, pp. 760–762, 1975.

[13] K. Furuuchi, "Equivalence of projections as gauge equivalence on noncommutative space," *Communications in Mathematical Physics*, vol. 217, no. 3, pp. 579–593, 2001.

[14] A. Micu and M. M. Sheikh-Jabbari, "Noncommutative Φ^4 theory at two loops," *Journal of High Energy Physics*, vol. 5, no. 1, p. 025, 2001.

[15] E. F. Moreno, "Spherically symmetric monopoles in noncommutative space," *Physical Review D*, vol. 72, no. 4, Article ID 045001, 9 pages, 2005.



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