Quantum effects, anomalies and renormalization

in Electrodynamics, Cosmology and Black holes

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No hi ha camí. Més enllà de l'abast de la llum, més enllà dels confins de la foscor... I tot i això, el cerquem, insaciablement... Doncs eixe és el nostre destí.

Aldia

Abstract

Quantum Field Theory in Curved Spacetimes has proven to be a very useful semiclassical theory for studying physical phenomena that combine gravity and quantum effects. In particular, it predicts that the dynamics of a background gravitational field can spontaneously excite particles out of the quantum vacuum. The process of particle production is of particular importance in the study of the very early universe in Cosmology, and it is the basis of Hawking radiation in black hole physics. Physically, this quantum effect is analogous to the well-known Schwinger effect in quantum electrodynamics. The goal of this Thesis is to study this general phenomenon of particle production, as well as other related fundamental aspects, such as backreaction effects, quantum anomalies, and renormalization techniques.

One of the main contributions of this Thesis is the development and transfer of techniques typically used in QFT in curved spacetimes to quantum fields coupled to strong electrodynamics backgrounds. For instance, the study includes the exploration of whether the gravitational anomaly for Weyl fields is also present for electric backgrounds. Moreover, this Thesis also addresses if the fundamental property of the adiabatic invariance of the number of created particles in an expanding universe is maintained in the case of a pure electric background. Finally, the method of adiabatic renormalization, which is particularly useful for quantum fields in expanding

universes, is developed here for 4-dimensional Dirac fields that are coupled to a general, electric background.

This Thesis also provides relevant contributions in the area of Gravitation. On the one hand, we extend a successful regularization and renormalization method recently communicated in the literature, called pragmatic mode-sum regularization. This method was originally developed for black holes, and in this Thesis we adapted it for expanding universes. On the other hand, the Thesis includes a detailed study of quantum corrections to the Schwarzschild metric, originated from the back-reaction effects of quantum fields living in this black hole background. As we will see in more detail below, the driving argument in the analysis is the conformal anomaly and the assumption of staticity. The geometrical properties and applications of the new (horizonless) spacetime are also analyzed.

All these results improve considerably our understanding of the behavior of quantum fields coupled to external gravitational and electromagnetic backgrounds. The role of quantum anomalies has been fundamental to achieve this.

Agraïments

Hi ha moltes persones a qui voldria agrair moltes coses. Algunes que han contribuït al fet que puga estar en la situació que estic, altres que han col·laborat en tot el treball que hi ha darrere d'esta tesi, i altres que ho han fet amb suport emocional, o que simplement m'agradaria aprofitar esta oportunitat per a recordar-los el que signifiquen per a mi.

En primer lloc, als meus pares, Lupe i Vicent, perquè heu estat sempre ahí per a donar-me consell a l'hora de prendre les decisions més importants (sense ell ni tan sols crec que m'haguera atrevit a estudiar Física). Perquè mai heu tractat de forçar-me a escollir cap camí i m'heu donat sempre suport, independentment que el que decidira o pensara fora més o menys del vostre gust. I en definitiva, per haver-me ensenyat a interactuar amb el meu entorn amb ulls curiosos, ment crítica i cor obert. A la meua germana, Sara, perquè és igual com d'angoixant o depriment haja sigut la setmana, quan arriba el diumenge i ens trobem al maset sempre aconsegueixes traure'm un somriure. Ja siga fent el tonto, amb una conversa profunda al cotxe, o amb una de les teues "turres" inacabables (però normalment interessants). I al meu germà, Àngel, a.k.a. "ma broda". Perquè els caps de setmana que ens trobem a Castelló són com tornar a casa. Per aguantar les meues "xapes" queixant-me de com de mal està el món de la investigació, i del capitalisme en general. I perquè quan més desmotivat estava amb tot, em vas descobrir el que acabaria sent la meua autèntica passió (tu ja ho saps). En definitiva, no només sou els meus germans, sou dels millor amics que he pogut tindre, i no vos imagineu com d'orgullós em sent de vatros.

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Em deixo a molta gent que també m'ha acompanyat d'una manera o altra en esta etapa, però açò ja s'està fent massa llarg. A tots ells i elles, també, gràcies.

Amb molta estima,

Pau

Author's Declaration

I declare that the work presented in this Thesis was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Program.

The contents of this Thesis are based on the following papers, that have been added in the Part III. Credit copyright American Physical Society.

- *"Translational anomaly of chiral fermions in two dimensions"*; P. Beltrán-Palau, J. Navarro-Salas, and S. Pla, *Phys. Rev. D* **99**, 105008 (2019).
- *"Breaking of adiabatic invariance in the creation of particles by electromagnetic fields "*; P. Beltrán-Palau, A. Ferreiro, J. Navarro-Salas and S. Pla, *Phys. Rev. D* **100**, 085014 (2019).
- *"Adiabatic regularization for Dirac fields in time-varying electric backgrounds"*; P. Beltrán-Palau, J. Navarro-Salas, and S. Pla, *Phys. Rev. D* **101**, 105014 (2020).
- *"Note on the pragmatic mode-sum regularization method: Translational-splitting in a cosmological background "*; P. Beltrán-Palau, A. Del Río, S. Nadal-Gisbert and J. Navarro-Salas, *Phys. Rev. D* **103** 105002 (2021).

- *"Quantum corrections to the Schwarzschild metric from vacuum polarization"*; P. Beltran-Palau, A. Del Río and J. Navarro-Salas, *Phys. Rev. D* **107**, 085023 (2023).

The text presented here should be understood as a dissertation submitted to the University of Valencia as required to obtain the degree of Doctor of Philosophy in Physics.

Except where indicated by specific reference in the text, this is the candidate's own work, done in collaboration with, and/or with the assistance of, the candidate's supervisors and collaborators. Any views expressed in the Thesis are those of the author.

València, May 17, 2023
Pau Beltrán Palau

Resum de la Tesi

Introducció i motivació

Dos són les teories més fonamentals que vertebren la física moderna hui en dia, la Relativitat General (GR), que explica la física de l'univers, i la Teoria Quàntica de Camps (QFT), en la que es fonamenta la física de partícules. Les dos teories estan àmpliament acceptades per la comunitat científica, tot i que estan lluny de resoldre tots els problemes de la física actual. I el més fonamental d'estos problemes és precisament com encaixar estes dos teories per tal d'explicar els fenòmens físics que involucren física de partícules com efectes gravitatoris. Una teoria unificada podria donar llum a problemes que els físics tracten de resoldre des de fa dècades. Els dos més importants en este sentit són: D'una banda descobrir l'autèntica natura dels forats negres (i la física en les regions properes a estos), i d'altra banda, els processos físics que van tindre lloc en els orígens de l'univers i que van donar pas a la creació de la matèria.

Estem lluny de trobar una teoria unificada, però això no implica que estos problemes siguen una completa incògnita. Hi ha moltes maneres d'apropar-se a eixos problemes, com són teories que van més enllà del Model Estàndard o teories de Gravetat Modificada, les quals i proposen correccions a la Relativitat General clàssica. Hi ha també teories recents que tracten d'apropar-se a una teoria unificada de gravetat i mecànica Quàntica, com són teories de Loop Quantum Gravity o Teoria de Cordes. Però hi ha una teoria que destaca per la seua capacitat per a estudiar i donar resultats sobre els fenòmens físics que inclouen gravetat i camps quàntics, que és la Teoria Quàntica de Camps en espais corbats. Un dels principals descobriments d'esta teoria és el fet que partícules elementals poden ser creades per un camp gravitatori depenent del temps. Açò va ser descobert en [1, 2, 3, 4], donant lloc a un nou i fructífer camp d'investigació. Aquest va ser posteriorment analitzat i estés per molts autors, i recopilat en diversos monogràfics [5, 6, 7, 8]. Esta teoria parteix d'una idea ben coneguda en Teoria quàntica de Camps, l'aproximació semi-clàssica. Consisteix en acoblar un camp quàntic a un camp extern molt intens, que pot ser aproximat a un camp clàssic, sense necessitat de quantitzar-lo. Esta aproximació és àmpliament utilitzada, i ha donat interessants resultats per exemple en l'àmbit de l'òptica quàntica [9]. Gravetat semi-clàssica consisteix en traslladar esta idea al camp gravitatori. Si assumim que este camp actua com a un camp extern intens, que és el que ocorre per exemple en l'entorn d'un forat negres o en un univers en expansió, es pot tractar el camp gravitatori com un camp clàssic, sense necessitat de passar pel problema no resolt de quantitzar la gravetat. La teoria consisteix en acoblar este camp a camps de matèria quantitzats, com ara camps escalars, camps de Dirac o camps vectorials.

Així, esta és una teoria efectiva que no unifica la Gravetat i la Física Quàntica, ja que no proposa un mètode per a quantitzar la gravetat, però tot i això ha demostrat ser molt útil per a explicar fenòmens que abans no havia hagut manera d'abordar. Un d'ells és la ben coneguda radiació de Hawking [10, 11], que mostra com, a diferència del que durant dècades s'ha assumit, els forats negres no són simplement embornals de matèria que no deixen passar cap tipus de partícula més enllà de les seues fronteres. Els forats negres radien matèria, i açò és un efecte que només es pot explicar si es tenen en compte els efectes quàntics generats al voltant de l'horitzó. Altra important efecte que fusiona gravetat i quàntica i que la teoria semi-clàssica és capaç d'explicar està emmarcada en el context de Cosmologia. Consisteix en la creació espontània de partícules com a conseqüència de l'expansió de l'univers, un efecte que va ser per primera vegada descobert per L. Parker [1]. Este fenomen és conseqüència de que en un espai-temps no estàtic el buit quàntic no pot definir-se de manera unívoca, ja que un estat buit per a un observador pot no ser-ho per a un altre. Això provoca un resultat diferent en la mesura del nombre de partícules en diferents instants. Este efecte adquireix especial importància en l'estudi dels primers instants de l'univers posteriors al Big Bang i en la creació de la matèria. [12, 13] D'altra banda, el fenomen de creació espontània de partícules no és només propi d'espai-temps corbats, també pot donar-se en espai-temps plans. En particular, un camp elèctric intens pot generar també creació espontània de parells partícula-antipartícula a partir del buit quàntic. És el que es coneix com efecte Schwinger [14, 15], un efecte no pertorbatiu que només es pot obtenir mitjançant l'enfocament semi-clàssic (és a dir, la *one-loop effective action*).

En definitiva, en tots estos escenaris (Electrodinàmica, Cosmologia i Forats negres) veiem que els camps quàntics en presència de camps intensos externs tenen propietats interessants que s'han d'estudiar mitjançant teories semi-clàssiques. Este és el context teòric en què esta tesi està emmarcada. Els articles que conformen la tesi (mostrats en la part III) consisteixen en l'estudi, desenvolupament i aplicacions de la teoria semi-clàssica en estos escenaris físics (anomalies, renormalització, efectes de *backreaction*...).

Pel que fa al cas d'Electrodinàmica, la nostra aportació es pot resumir amb que hem estudiat com es traslladen al context d'un background electromagnètic certs fenòmens ben coneguts en el context de QFT en espais corbats. D'una banda, en l'Article 1 d'esta tesi, estudiem si la coneguda com *anomalia gravitatòria*, que es dona per a un fermió de Weyl 2-dimensionals acoblat a gravetat, es trasllada al cas d'un background elèctric. Veurem que efectivament en este segon cas també sorgeix una anomalia en la conservació del moment. D'altra banda, en l'Article 2, estudiem si el cas electromagnètic manté la *invariància adiabàtica* del número de partícules, que consisteix en la no creació de partícules en el límit d'un univers expandint-se infinitament lent. Veurem que, en certes condicions, esta invariància es trenca en este segon cas. Este fenomen està íntimament relacionat amb la ben coneguda *anomalia axial*. I per últim, en l'Article 3, estenem el mètode de *renormalització adiabàtica* (de gran utilitat en el context cosmològic) al cas d'un camp de Dirac 4-dimensional acoblat a un background elèctric.

Pel que fa a l'àmbit de Cosmologia, en l'Article 4, estenem al context cosmològic un recent mètode de renormalització (*pragmatic mode-sum*) que fins ara només s'havia aplicat al context de forats negres, on ha demostrat ser molt eficient. I per últim, ja en l'àmbit de Forats negres, estudiem correccions quàntiques de buit a la mètrica de Schwarzschild, concretament en l'Article 5. Estes correccions provenen dels efectes de *backreaction* que generen els camps quàntics sobre el propi background gravitatori. Veurem que la geometria del nou espaitemps generat presenta diferències significatives respecte de la dels forats negres. Este resultat pot contribuir a l'estudi de la formació d'objectes ultra-compactes que imiten la física dels forats negres.

Estructura i convencions

La tesi està organitzada de la següent manera. En la part I es fa un repàs sobre els principals conceptes que conformen el marc teòric d'esta tesi (situant-los històricament), els quals convé introduir per a facilitar la comprensió dels articles. En la part II es fa un resum dels resultats i conclusions que s'han obtingut al llarg del doctorat. I finalment, en la part

III incloem els articles que conformen la tesi.

Respecte a les convencions, al llarg de tota la tesi es treballa en unitats naturals, és a dir, $G = c = \hbar = 1$, excepte si resulta convenient introduir les constants. Per a la signatura de les mètriques s'utilitza $(+,-,-,-)$, excepte en l'Article 5 on s'utilitza la signatura contrària (i es mostra $\hbar$ explícitament). Per als tensors de curvatura es segueixen les convencions de [5].

Metodologia

Per a la realització dels articles s'ha consultat bibliografia actualitzada de les diferents àrees teòriques involucrades en el treball. Pel que fa a la realització dels càlculs necessaris per al desenvolupament dels articles, cal destacar principalment l'ús de *Mathematica* per als càlculs analítics (i en especial del paquet *x-Act* de càlcul tensorial), així com l'ús de *Matlab* per als càlculs numèrics més complicats.

Els articles s'han realitzat en col·laboració d'altres membres del grup d'investigació i científics externs, mitjançant reunions i repartiment de tasques. Per tant, l'autoria dels articles està repartida equitativament (els noms dels autors als articles estan ordenats alfabèticament, com és habitual en este les publicacions en este camp). I per últim, també s'ha assistit a congressos per a posar en comú els resultats obtinguts amb altres grups d'investigació, així com per a aprendre del treball d'altres autors en matèries semblants.

Anomalia translacional en backgrounds elèctrics

Les anomalies quàntiques són el trencament de simetries clàssiques produït al quantitzar el camp, com expliquem en detall en el capítol 6. És ben conegut que acoblar camps quàntics a backgrounds gravitacionals pot generar el que es coneixen com *anomalies gravitacionals* [16]. Estes anomalies consisteixen en el trencament de la covariància general, i per tant impliquen una no conservació del valor esperat del tensor energia-moment, és a dir $\langle \nabla_\mu T^{\mu\nu} \rangle \neq 0$. Estes anomalies són un tipus de gauge anomalies, les quals indiquen que la teoria no està ben construïda. En concret, les anomalies gravitacionals apareixen en teories amb fermions de Weyl (o quirals) acoblats a gravetat per a espaitemps de dimensió $2, 6, 10...$ En particular per al cas 2-dimensional s'obté l'anomalia

$$\langle \nabla_\mu T^\mu_\nu \rangle = \frac{1}{96\pi\sqrt{-g}} \epsilon^{\alpha\beta} \partial_\beta \partial_\rho \Gamma^\rho_{\nu\alpha} \,. \tag{1}$$

En l'Article 1 de la tesi (mostrat en la part III) demostrem que també es genera una anomalia de tipus "gravitacional" en este mateix cas però considerant un background elèctric en lloc d'un gravitacional. En concret ho demostrem per al cas d'un camp de Weyl en dos dimensions acoblat a un camp elèctric homogeni i depenent del temps $E(t)$. Des d'un punt de vista clàssic, este sistema es invariant baix translacions en la direcció espacial, la qual cosa implica la conservació de moment, és a dir, $\partial_\mu T^{\mu 1} = 0$. Però, com veurem en breu, en quantitzar el camp de Weyl trobem que esta simetria es trenca.

Per a obtenir esta anomalia cal trobar l'expressió renormalitzada del tensor energia-moment. Per a fer-ho hem aplicat la ben coneguda *renormalització adiabàtica* [5] que expliquem en detall en la secció 5.2. Amb ajuda d'este mètode arribem al següent resultat

$$\partial_\mu \left\langle T^{\mu 1}_{R,L} \right\rangle = \mp \frac{q^2 A \dot{A}}{2\pi} \,, \tag{2}$$

on R i L indiquen la quiralitat (dretana o esquerrana) del fermió considerat, i $A(t)$ és el potencial vector, definit com $E(t) = -\dot{A}(t)$. Este resultat no havia sigut indicat en la bibliografia previa. Com que esta anomalia trenca la simetria translacional, l'hem anomenada *anomalia translacional.*

L'aparició d'esta tipus d'anomalia és una senyal de que la teoria no està completa. De fet és ben conegut que este sistema físic presenta també una altra anomalia gauge en la conservació de la corrent elèctrica. Totes dues anomalies es cancel·len en considerar la teoria completa: el camp de Dirac. Per a un fermió de Dirac ($\Psi = \Psi_R + \Psi_L$) sense massa el tensor energia-moment és la suma de les dos components quirals, de manera que s'obté

$$\partial_\mu \left(\left\langle T_R^{\mu 1} \right\rangle_{\text{ren}} + \left\langle T_L^{\mu 1} \right\rangle_{\text{ren}} \right) = 0 \,. \tag{3}$$

D'altra banda, a l'article mostrem també la relació d'esta anomalia amb el fenomen de creació espontània de partícules per camps elèctrics intensos (que expliquem en el capítol 3). Per a un camp de Weyl les partícules creades es mouen totes en la mateixa direcció i sentit de la direcció espacial, generant una quantitat de moment total que coincideix amb el resultat de l'anomalia. En canvi si considerem un camp de Dirac, veiem que el que es creen són parells partícula-antipartícula que viatgen en direccions oposades, mantenint la conservació total del moment.

Finalment, a l'article resolem l'equació de Maxwell semi-clàssica del sistema per tal d'estudiar els efectes de backreaction que generen les partícules creades sobre el camp elèctric. Comprovem que la creació de moment en cada sector quiral oscil·la amb freqüència igual a la d'$E(t)$. Així mateix veiem que la suma de les oscil·lacions dels dos sectors quirals es cancel·len perfectament.

Trencament de la invariància adiabàtica en backgrounds electromagnètics

El fenomen de creació de partícules en un univers en expansió (que expliquem en la secció 2.2) posseeix una propietat interessant. En el límit d'una expansió de l'univers infinitament lenta (límit adiabàtic) no es produeix creació de partícules a partir del buit. És per això que es diu que el número de partícules és un invariant adiabàtic [17]. En l'Article 2 d'esta tesi (mostrat en la part III) estudiem si esta propietat es manté per al cas d'un background electromagnètic. Analitzem primerament el cas 2-dimensional per la seua simplicitat, i posteriorment estenem al cas 4-dimensional. Així mateix estudiem tant el cas d'un camp escalar carregat acoblat al camp electromagnètic (QED escalar) com el cas d'un camp de Dirac (QED).

Considerem un camp elèctric homogeni i depenent del temps actuant en la direcció espacial, $E(t)$. El seu potencial 2-vector associat serà $A_\mu = (0, -A(t))$, on $E(t) = \dot{A}(t)$. El potencial vector juga el rol anàleg al factor d'escala en el cas gravitatori, així que convé considerar una expansió asimptòtica per a $A(t)$ i així poder definir el número de partícules en $t \to \pm\infty$. Per tal de poder estudiar el problema analíticament, hem considerat una forma concreta per al camp elèctric que és ben coneguda: un pols elèctric de Sauter [18]. En este cas el potencial vector ve donat per

$$A(t) = \frac{1}{2} A_0 (\tanh(\rho t) + 1) \,, \tag{4}$$

on A_0 i ρ són constants. Es pot veure que el potencial tendeix a 0 en el límit $t \to \infty$ i a A_0 en $t \to \infty$. El paràmetre ρ estableix la velocitat amb que creix el potencial, de manera que es pot considerar com el paràmetre d'adiabaticitat. El límit adiabàtic (creixement extremadament lent) ve donat per $\rho \to 0$. L'objectiu en l'article és estudiar si en este límit el número de partícules tendeix o no a 0.

En l'article obtenim que per a bosons (b) i per a fermions (f) el valor esperat del número de partícules creades per un camp elèctric extern ve donat per

$$\langle N_{b/f} \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \frac{\cosh\left(2\pi \frac{\omega_-}{\rho}\right) \pm \cosh\left(2\pi \frac{\kappa_{b/f}}{\rho}\right)}{2 \sinh\left(\pi \frac{\omega_{\text{in}}}{\rho}\right) \sinh\left(\pi \frac{\omega_{\text{out}}}{\rho}\right)} \,, \tag{5}$$

on $\omega_{\text{in}} = \sqrt{k^2+m^2}$, $\omega_{\text{out}} = \sqrt{(k-qA_0)^2+m^2}$, $\omega_{\pm} = \frac{1}{2}(\omega_{\text{out}} \pm \omega_{\text{in}})$, $\kappa_b = \frac{1}{2}\sqrt{(qA_0)^2-\rho^2}$, i $\kappa_f = qA_0/2$.

Estudiant el límit $\rho \to 0$ en estes expressions extraiem les següents conclusions. Per al cas amb massa ($m \neq 0$) $\langle N_{b/f} \rangle \to 0$ en el límit adiabàtic, és a dir per a un creixement infinitament lent d'$A(t)$ no es creen bosons ni fermions massius, mantenint-se així la invariància adiabàtica del número de partícules. Però la situació és diferent en el cas sense massa ($m = 0$). En este cas s'obté que $\langle N_{b/f} \rangle \neq 0$ quan $\rho \to 0$, en concret

$$\langle N_{b/f} \rangle = \frac{|qA_0|}{\pi}. \tag{6}$$

Per tant concloem que per a un potencial vector creixent infinitament lent sí que es creen partícules sense massa, i la invariància adiabàtica es trenca. També hem comprovat que l'espectre de moments d'estes partícules sense massa creades es troba en l'interval $k \in [-|qA_0|, |qA_0|]$. Cal remarcar que hi ha una diferencia clara entre el cas de bosons i el de fermions que es pot extraure de l'expressió (5). Els bosons sense massa creats tendeixen a acumular-se en els valors $k = 0$ i $k = \pm qA_0$, mentre que els fermions sense massa es creen en la mateixa proporció per a tot k. Açò és pot interpretar en termes del principi d'exclusió de Pauli, que no permet que els fermions s'acumulen en un mateix estat. A més a més, a diferència del cas escalar, el número de fermions sense massa creats (així com el seu espectre de moments) no depèn del paràmetre ρ, és a dir no depèn de la història d'$A(t)$, sinó només del seu valor inicial i final.

D'altra banda, per tal de donar consistència a este resultat, hem calculat també mitjançant el mètode de renormalització adiabàtica (explicat en la secció 5.2) el valor esperat de la corrent elèctrica i de la densitat d'energia del camp quàntic. De manera anàloga al número de partícules, s'obté que estos observables tendeixen a 0 en el límit adiabàtic, excepte en el cas sense massa. Eixe romanent d'energia i corrent correspon al generat per les partícules sense massa creades. A més a més, la simplicitat del cas de fermions sense massa permet donar una expressió analítica de la corrent elèctrica renormalitzada en funció del temps, que ve donada per $\langle j^x \rangle_{\rm ren} = -\frac{q^2 A(t)}{\pi}$. Aquesta expressió permet obtenir l'equació semi-clàssica de Maxwell, que ve donada per $\ddot{A} + \frac{q^2}{\pi} A$. Esta equació de tipus oscil·lador armònic te en compte els efectes de backreaction de les partícules creades sobre el camp elèctric. En concret s'obté que el camp elèctric oscil·la amb freqüència $|q|/\sqrt{\pi}$, així com ho fa el número de partícules. Es pot veure fàcilment que l'energia associada al camp elèctric i l'energia de les partícules creades es cancel·len per a tot t, mantenint-se la conservació d'energia. El valor obtingut per a la freqüència és consistent amb el fet ben conegut de que les correccions radiatives al model de Schwinger indueixen una massa al fotó de valor $m_\gamma^2 = q^2/\pi$ [19].

Per últim hem estes el càlcul al cas 4-dimensional, considerant un camp elèctric $\vec{E}(t)$ en la direcció z per conveniència. En este cas trobem que $\langle N_{b/f} \rangle \to 0$ en el límit adiabàtic (independentment de m). Així, en 4 dimensions es manté la invariància adiabàtica per a un background elèctric. Però la situació canvia si afegim un camp magnètic. Hem considerat per simplicitat un camp magnètic constant $\vec{B}$ en direcció paral·lela a $\vec{E}(t)$. La presència del camp magnètic genera una discretització del moment en la direcció perpendicular als camps, en els coneguts com nivells de Landau, la qual cosa canvia dràsticament el resultat. Trobem que, mentre per a bosons amb qualsevol massa la invariància adiabàtica es respecta, per a fermions sense massa en presència de camps elèctric i magnètic la invariància adiabàtica es perd. Este resultat es manté per a altres direccions de $\vec{B}$, excepte quan és perpendicular al camp elèctric. En eixe cas la invariància adiabàtica és preservada.

En resum, hem obtingut que la invariància adiabàtica de les partícules creades es manté per al cas d'un background electromagnètic exceptuant alguns casos concrets. Eixos casos són: bosons i fermions sense massa en 2 dimensions, i fermions sense massa en 4 dimensions en presència de camps elèctric i magnètic no perpendiculars. Açò indica que hi ha una relació entre el fenomen del trencament de la invariància adiabàtica i la ben coneguda anomalia axial [14], ja que aquesta està present precisament en els casos esmentats. Esta anomalia consisteix en el trencament de la simetria axial, pròpia del camp de Dirac sense massa clàssic, que es produeix al quantitzar el camp (a la secció 6.1 expliquem amb més detall esta anomalia). En 2 dimensions l'anomalia axial ve donada per l'expressió

$$\langle \partial_\mu j_5^\mu \rangle_{\rm ren} = -\frac{q}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}, \tag{7}$$

que en el cas d'un camp homogeni és equivalent a dir que la densitat de càrrega quiral j_A^0 no és conservada. En 2 dimensions esta càrrega és proporcional a la corrent elèctrica. En l'article hem comparat l'expressió de l'anomalia amb la corrent generada per les partícules sense massa creades en el límit adiabàtic, i hem comprovat que efectivament la creació de càrrega quiral coincideix amb la causada per l'anomalia axial. Esta idea es pot visualitzar fàcilment en el cas 2-dimensional. El camp elèctric crea les partícules en parells partícula-antipartícula de carrega elèctrica i moment oposats. La quiralitat per a partícules sense massa en 2 dimensions està relacionada amb el sentit de moviment i canvia el criteri entre partícules i antipartícules. De manera que, per exemple, una partícula sense massa movent-se cap a la dreta tindria quiralitat dretana, i una antipartícula sense massa movent-se cap a l'esquerra també. Així, la creació de parells sense massa implica una creació de càrrega quiral. Eixa no conservació de la càrrega quiral és consistent amb l'anomalia axial. Esta anomalia es manté independentment de la velocitat a la que canvie el camp background, fins i tot en el límit adiabàtic, i per això en eixe límit ha de quedar sempre un romanent de creació de parells sense massa.

Pel que fa al cas 4-dimensional, l'anomalia només sorgeix per a fermions sense massa, i ve donada per

$$\left\langle \partial_\mu j_A^\mu \right\rangle = -\frac{q^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,. \tag{8}$$

En el cas d'un camp elèctric depenent del temps i un camp magnètic constant està expressió és equivalent a

$$\left\langle j_5^0 \right\rangle_{\rm ren} = -\frac{q^2}{2\pi^2}\int_{-\infty}^{t} dt' \vec{E}\left(t'\right)\vec{B}\,. \tag{9}$$

Podem veure que només es crea càrrega axial quan els camps $\vec{E}$ i $\vec{B}$ no són perpendiculars, el mateix cas en què es dona el trencament de la invariància adiabàtica. A més a més, també en este cas hem comprovat que la creació de càrrega quiral dels fermions sense massa en el límit adiabàtic coincideix amb l'expressió de l'anomalia.

En definitiva concloem que el trencament de la invariància adiabàtica es dona en els casos on sorgeix l'anomalia axial, de manera que estos dos fenòmens estan íntimament relacionats. En altres paraules, el trencament de la invariància adiabàtica és una condició necessària per al compliment de l'anomalia axial.

Mètode de renormalització adiabàtica per a camps de Dirac en un background elèctric

El mètode de renormalització adiabàtica va ser introduït per L. Parker i S. A. Fulling per a renormalitzar observables físics, com és el tensor energia-moment, en el context de QFT en espais corbats [20, 21]. En la secció 5.2 fem un repàs d'este mètode per al cas d'un camp escalar en un univers en expansió. Tot i que este mètode s'aplica habitualment en el context d'un background cosmològic, es pot estendre també a altres teories semi-clàssiques on el background clàssic és un camp homogeni que depèn del temps. El cas concret d'un background elèctric va ser estudiat primerament per Cooper et.al. [22, 23, 24], qui van proposar una extensió del mètode adiabàtic a este cas. Tot i això, estudis recents han demostrat que hi ha un inconvenient en estos treballs [25, 26]. El potencial vector A^μ es considera d'ordre adiabàtic 0, de manera anàloga al factor d'escala $a(t)$ en el cas cosmològic. Açò és consistent en el cas de (només) un background elèctric, però si afegim la presència d'un background gravitatori s'obtenen expressions renormalitzades que són inconsistents amb la conservació covariant del tensor energia-moment, així com amb l'anomalia axial i de traça. En estos treballs s'ha demostrat que per a recuperar la consistència del mètode cal imposar que A^μ siga d'ordre adiabàtic 1 (la primera derivada seria d'ordre 2, la segona d'ordre 3...). Així mateix, es proposa una nova reformulació del mètode amb aquesta assumpció per al cas de camps escalars carregats i per a camps de Dirac en 2 dimensions. L'extensió de 2 a 4 dimensions per camps de Dirac (amb la nova assumpció) resulta no ser trivial, i requerix d'un anàlisi en profunditat. Eixe és l'objectiu de l'Article 3 d'esta tesi (mostrat en la secció III).

El primer resultat que obtenim en l'article és un nou argument que fonamente l'elecció d'A^μ com a ordre adiabàtic 1. És sabut que el mètode de renormalització adiabàtica per a un background gravitatori és consistent amb el mètode de *DeWitt-Schwinger point-splitting* [27, 28] (el qual expliquem breument en la secció 5.1). En l'article comprovem que en presència de backgrounds elèctric i gravitatori esta consistència només es manté amb l'assumpció de que A^μ és d'ordre adiabàtic 1. En concret provem que, tant per al camp escalar com per al de Dirac en 2 dimensions, l'expansió adiabàtica de la funció de dos punts $\langle\phi^2\rangle$ coincideix exactament amb la de DeWitt-Schwinger si considerem esta assumpció (ho comprovem explícitament fins a ordre adiabàtic 6). Este argument, junt amb els explicats prèviament, motiven a una reformulació del mètode adiabàtic, aplicant la nova assumpció, per a camps de Dirac en 4 dimensions acoblats a un background elèctric. El principal problema que sorgeix en este cas és que l'ansatz habitual de tipus WKB per als modes del camp no funciona a l'hora de desenvolupar l'expansió adiabàtica. És per això que en este article proposem un nou ansatz, el qual comprovem que és consistent i permet procedir amb la regularització adiabàtica.

Vegem breument en què consisteix el nostre mètode. Considerem un camp de Dirac ψ en 4 dimensions, de massa m i càrrega q, acoblat a un background elèctric amb potencial vector de la forma $A_\mu = (0, 0, 0, -A(t))$. L'equació de Dirac d'este sistema ve donada per

$$(i\gamma^\mu D_\mu - m)\,\psi = 0\,, \tag{10}$$

on $D_\mu \equiv \partial_\mu - iqA_\mu$ i γ^μ són les matrius de Dirac. Per tal de poder construir l'ansatz convé aplicar una transformació unitària al camp de la forma $\psi' = U\psi$, on

$$U = \frac{1}{\sqrt{2}}\gamma^0 \left(I - \gamma^3\right)\,. \tag{11}$$

Açò ens ha permés expressar el camp de Dirac en termes de només dos funcions dependents del temps, $h^I_{\vec{k}}(t)$ i $h^{II}_{\vec{k}}(t)$, que es poden considerat com els modes del camp amb moment $\vec{k} = (k_1, k_2, k_3)$. Cal destacar que la idea d'aplicar esta transformació ha sigut crucial per a poder desenvolupar el mètode, i considerem que cal remarcar-la. Finalment obtenim que l'equació de Dirac es redueix a les següents equacions diferencials per als modes del camp

$$\dot{h}^I_{\vec{k}} - i\,(k_3 + qA)\,h^I_{\vec{k}} - i\kappa h^{II}_{\vec{k}} = 0\,, \tag{12}$$

$$\dot{h}^{II}_{\vec{k}} + i\,(k_3 + qA)\,h^{II}_{\vec{k}} - i\kappa h^I_{\vec{k}} = 0\,, \tag{13}$$

on $\kappa \equiv \sqrt{k_1^2 + k_2^2 + m^2}$. La principal avantatja d'este procediment és que ens ha permès escriure l'equació de Dirac en termes de dos equacions diferencials molt semblants a les del mateix cas en 2 dimensions (veure [26]). L'única diferència és que ara κ juga el paper de m. Finalment, a parir d'estes expressions, es pot quantitzar el camp en termes dels operadors creació i destrucció (veure l'article per a més detalls). Les relacions d'anticommutació dels operadors estan garantides si es compleix la condició de normalització

$$\left|h^{I}_{\vec{k}}\right|^2 + \left|h^{II}_{\vec{k}}\right|^2 = 1 \ . \tag{14}$$

Amb tots estos ingredients ja es pot desenvolupar l'expansió adiabàtica. Ací és on entra el nostre ansatz. Seguint la idea de l'analogia amb el cas 2-dimensional, construïm el mateix ansatz proposat en eixe cas (veure [26]) però aplicant el canvi $m \to \kappa$, és a dir

$$h^{I}_{\vec{k}} = \sqrt{\frac{\omega - k_3}{2\omega}} F(t) e^{-i\int^t \Omega(t')dt'}, \tag{15}$$

$$h^{II}_{\vec{k}} = -\sqrt{\frac{\omega + k_3}{2\omega}} G(t) e^{-i\int^t \Omega(t')dt'}\ , \tag{16}$$

on $\omega = \sqrt{k_3^2 + \kappa^2}$, F i G són funcions complexes i Ω és una funció real. Expressant estes funcions com una expansió adiabàtica i resolent les equacions diferencials ordre a ordre s'obté l'expansió adiabàtica dels modes, de manera anàloga a com es fa en el cas d'un background gravitatori [5]. En l'article donem expressions de recurrència que serveixen com a algoritme per a obtenir els ordres adiabàtics d'estes funcions fins a qualsevol ordre. A partir d'esta expansió es poden expandir també els valors esperats d'observables en ordres adiabàtics. Així es poden identificar i sostraure els ordres que generen les divergències, és a dir, aplicar la renormalització adiabàtica. En l'article hem aplicat este mètode per a calcular el valor esperat renormalitzat de la corrent elèctrica, definida per $\langle j^\mu \rangle = -q \left\langle \bar{\psi}\gamma^\mu\psi \right\rangle$. Per a la component rellevant (j^3) s'obté l'expressió

$$\left\langle j^3 \right\rangle_{\rm ren} = \frac{q}{2\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3 \Bigg[\left(\left|h^{II}_{\vec{k}}\right|^2 - \left|h^{I}_{\vec{k}}\right|^2 \right) - \frac{k_3}{\omega}$$

$$- \quad \frac{\kappa^2 q A}{\omega^3} + \frac{3\kappa^2 k_3 q^2 A^2}{2\omega^5} + \frac{\left(\kappa^2 - 4k_3^2\right)\kappa^2 q^3 A^3}{2\omega^7} + \frac{\kappa^2 q \ddot{A}}{4\omega^5}\Bigg], \quad (17)$$

on $k_\perp = \sqrt{k_1^2 + k_2^2}$.

Així mateix, a l'article incloem dos tests d'este mètode. D'una banda calculem l'expressió renormalitzada de la traça del tensor energia-moment, que ve donada per $\langle T_\mu^\mu \rangle = m\langle \bar{\psi}\psi \rangle$, i comprovem que en el límit $m \to 0$ s'obté l'expressió de l'anomalia de traça. En el cas de camps de Dirac sense massa en presència d'un background electromagnètic esta anomalia ve donada per $\langle T_\mu^\mu \rangle_{\text{ren}} = \frac{q^2}{24\pi^2} F_{\mu\nu}F^{\mu\nu}$ [29]. [En la secció 6.2 expliquem en detall esta anomalia]. D'altra banda, comprovem també que, de la mateixa manera que ocorre en tota la resta de casos on s'aplica el mètode adiabàtic, l'expansió adiabàtica coincideix amb l'expansió de DeWitt-Schwinger, provant així l'equivalència entre els dos mètodes. Finalment comprovem també, en un apèndix, l'equivalència amb el mètode de renormalització de Hadamard [30].

El formalisme adiabàtic usual assumeix implícitament que l'escala de renormalizació μ és igual a la massa del camp. En el nostre treball estenem el mètode per a una escala de renormalització arbitrària. Per a fer-ho fem servir, com ja s'ha fet en treballs previs [31], l'ambigüitat intrínseca que hi ha en el mètode adiabàtic en l'elecció de l'ordre adiabàtic zero quan es solucionen les equacions de recurrència. En lloc de $\sqrt{\vec{k}^2 + m^2}$ és possible definir $\omega^{(0)} \equiv \omega = \sqrt{\vec{k}^2 + \mu^2}$, on μ és una escala de massa arbitrària. Així, obtenim una nova expansió dels modes en termes de l'escala de massa. A més a més, l'apliquem per a renormalitzar la corrent elèctrica amb esta extensió, obtenint una expressió que depèn de μ. Este tipus d'ambigüitats en la renormalització poden ser absorbides en la renormalització de la constant d'acoblament, en este cas q. Seguint esta idea obtenim l'expressió de la càrrega efectiva en funció de l'escala: $q^{-2}(\mu) - q^{-2}(\mu_0) = -\left(12\pi^2\right)^{-1} \ln \frac{\mu^2}{\mu_0^2}$. Esta expressió coincideix amb l'obtinguda en QED pertorbativa per mig de regularització dimensional [19].

Per últim, per tal de provar la utilitat pràctica del mètode, l'hem aplicat a un background elèctric concret. Hem considerat un pols de tipus Sauter donat per $E(t) = E_0 \cosh^{-2}(t/\tau)$, on E_0 indica l'altura del pols i τ l'amplària. Això ens ha permès també estudiar propietats físiques del fenomen de creació de partícules. Hem calculat numèricament la corrent renormalitzada en funció del temps per a este cas a partir de l'expressió obtinguda amb el nostre mètode (17). A l'article es poden trobar representacions del resultat per a diferents valors dels paràmetres. Comprovem que la corrent tendeix a fer-se constant en el límit $t \to \infty$, com és esperat per a este background. Este límit es pot calcular analíticament, en concret obtenim

$$\langle j^3 \rangle_{\rm ren} \sim -\frac{q}{\pi^2} \int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3 \frac{k_3 + qA_0}{\omega_{\rm out}} \left|\beta_{\vec{k}}\right|^2 , \tag{18}$$

on $\omega_{\rm out} = \sqrt{(k_3 + qA_0)^2 + \kappa^2}$ i $\left|\beta_{\vec{k}}\right|^2$ és el coeficient de Bogoliubov que dona la densitat de partícules creades amb moment k en $t \to \infty$. Noteu que esta expressió és vàlida per a qualsevol background que tendisca a un valor constant. Aplicant l'expressió de $\left|\beta_{\vec{k}}\right|^2$ corresponent al pols de Sauter (obtinguda en la secció anterior) obtenim el resultat per a eixe cas. A més a més, hem utilitzat este resultat per a fer una estimació del valor de la corrent elèctrica en el límit d'un camp elèctric molt intens ($E_0 >> 0$). En este límit obtenim l'expressió

$$\langle j^3 \rangle_{\rm ren} \sim \frac{2}{3\pi^3} q^3 E_0^2 \tau \,. \tag{19}$$

Així mateix, hem obtingut l'expressió de la densitat de partícules en este mateix límit, obtenint $\langle N \rangle \sim \frac{2}{3\pi^3} q^2 E_0^2 \tau$.

Mètode de regularització *pragmatic mode-sum* en un background cosmològic

El mètode de renormalització de DeWitt-Schwinger point-splitting (que expliquem en la secció 5.1) proposa un procediment per a renormalitzar observables físic en el context de QFT en espais corbats. Este mètode està àmpliament acceptat, però no és fàcilment aplicable en molts escenaris on els modes dels camps es tenen només en forma numèrica, com és el cas dels forats negres. Recentment, A. Levi i A. Or han proposat un mètode que ha demostrat ser molt eficient per a implementar numèricament el procediment de point-splitting, conegut com mètode de regularització *pragmatic mode-sum* [32, 33, 34]. Pot aplicar-se en mètriques que posseeixen algun tipus de simetria (com ara forats negres estàtics, estacionaris) i es pot entendre com un mètode que completa l'inicialment proposat per Candelas en els anys 80 [35]. [En la secció 5.3 fem un repàs històric dels mètodes proposats per a tractar d'implementar el point-splitting en forats negres.] En l'Article 4 d'esta tesi (mostrat en la part III) fem un repàs d'este mètode i l'estenem al cas d'espai-temps amb simetria respecte de translacions en les 3 direccions espacials (homogeni), en concret al context cosmològic. Així mateix demostrem que en este context el mètode de Levi i Ori és consistent amb mètode de renormalització adiabàtica.

En particular hem considerat un camp escalar acoblat a una mètrica FLRW, $ds^2 = dt^2 - a^2(t)d\vec{x}^2$, i ens hem centrat en la renormalització de $\langle\phi^2\rangle$. Seguint el mètode de point-splitting, l'expressió renormalitzada d'este observable ve donada per

$$\left\langle\phi^2(x)\right\rangle_{\text{ren}} = \lim_{x'\to x}\left[\left\langle\left\{\phi(x), \phi\left(x'\right)\right\}\right\rangle - G_{\text{DS}}^{(1)}\left(x, x'\right)\right] . \qquad (20)$$

$G_{\text{DS}}^{(1)}$ és el terme de sostracció de DeWitt-Schwinger per a la funció de dos punts, que ve donat per

$$
\begin{aligned}
G_{\mathrm{DS}}^{(1)}\left(x, x'\right) &= \frac{1}{8\pi^2}\left[-\frac{1}{\sigma}+\left(m^2+(\xi-1/6)R\right)\left(\gamma+\frac{1}{2}\log\left(\frac{m^2|\sigma|}{2}\right)\right)\right. \\
&- \left.\frac{m^2}{2}+\frac{1}{12}R_{\alpha\beta}\frac{\sigma^{;\alpha}\sigma^{;\beta}}{\sigma}\right], \qquad (21)
\end{aligned}
$$

on R és l'escalar de curvatura, $R_{\alpha\beta}$ el tensor de Ricci, γ la constant d'Euler i σ un mig del quadrat de la distància geodèsica que connecta x i x'. Seguint la guia del mètode de Candelas, posteriorment completat per Levi i Ori, escollim els punts en base a la simetria del sistema. En este cas la simetria translacional de l'espaitemps ens indica que convé escollir punts separats espacialment, és a dir, $x \equiv (t, \vec{x})$ i $x' \equiv (t, \vec{x}+\vec{\epsilon})$. Així, obtenim que el valor esperat de la funció de dos punts per a estos punts ve donat per

$$
\left\langle\left\{\phi(x), \phi\left(x'\right)\right\}\right\rangle = \frac{1}{4\pi^2 a(t)^3}\int_0^\infty dk k^2\left|h_k(t)\right|^2\frac{\sin k\epsilon}{k\epsilon}\,. \qquad (22)
$$

on $\epsilon = |\vec{\epsilon}|$ i $h_k(t)$ són els modes del camp.

El terme $G_{\mathrm{DS}}^{(1)}$ es pot expandir en potències d'ϵ com

$$
\begin{aligned}
G_{\mathrm{DS}}^{(1)}\left(x, x'\right) &= \frac{1}{4\pi^2}\left[\frac{1}{a^2\epsilon^2}+\frac{1}{2}\left(m^2+(\xi-1/6)R\right)\left(\gamma+\log\left(\frac{ma}{2}\epsilon\right)\right)\right. \\
&- \left.\frac{m^2}{4}+\frac{R}{72}\right]+\mathcal{O}(\epsilon)\,. \qquad (23)
\end{aligned}
$$

Aplicant identitats integrals del tipus $\int_0^\infty dk k\frac{\sin k\epsilon}{k\epsilon} = \frac{1}{\epsilon^2}$, podem expressar (23) com una integral en k i sostraure-la en (22). La divergència en $\epsilon \to 0$ és cancel·lada, de manera que podem prendre el límit de punts coincidents abans de la integració. L'expressió final obtinguda és

$$
\left\langle\phi^2\right\rangle_{\mathrm{ren}} = \frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2\left[|h_k|^2-\frac{1}{\omega}-\frac{\left(\frac{1}{6}-\xi\right)R}{2\omega^3}\right]-\frac{R}{288\pi^2}\,. \qquad (24)
$$

Esta expressió coincideix exactament amb l'obtinguda mitjançant el mètode de renormalització adiabàtica (veure l'equació (5.24) de la secció 5.2). Així, concloem que l'extensió del mètode de Levi i Ori per a un background homogeni depenent del temps és compatible amb el mètode adiabàtic. En l'article donem també un argument per a provar que l'equivalència entre el mètode *pragmatic mode-sum* i l'adiabàtic es manté també per al cas del tensor energia-moment.

Per últim, també en este cas, hem estès el mètode incloent una escala de massa arbitrària μ. Esta extensió és necessària per tal de poder aplicar el mètode al cas $m = 0$, ja que el terme de sostracció (21) no està ben definit en eixe cas. Seguint la tècnica proposada en [36], apliquem un canvi del tipus $m^2 \to m^2 + \mu^2$ en un punt específic del mètode de point-splitting. Així, arribem a la següent expressió per al terme de sostracció que s'ha d'aplicar a la integral de la funció de dos punts

$$
\begin{aligned}
G^{(1)}_{\mathrm{DS}}\left(x,x'\right) &= \frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \frac{\sin(k\epsilon)}{k\epsilon}\left[\frac{1}{\omega_{\mathrm{eff}}} + \frac{\left(\frac{1}{6}-\xi\right)R}{2\omega_{\mathrm{eff}}^3} + \frac{\mu^2}{2\omega_{\mathrm{eff}}^3}\right] \\
&+ \frac{R}{288\pi^2} + \mathcal{O}(\epsilon)\,, \qquad (25)
\end{aligned}
$$

on $\omega_{\mathrm{eff}}^2 = \frac{k^2}{a^2} + m^2 + \mu^2$. Així mateix, obtenim el terme de sostracció d'ordre 2 de la renormalització adiabàtica, afegint el mateix tipus d'escala μ, i comprovem que efectivament coincideix amb (25), reforçant la consistència entre els dos mètodes.

Correccions quàntiques de buit a la mètrica de Schwarzschild

Els recents progressos en la detecció d'ones gravitacionals [37] així com en interferometria de molt llarga base [38] han obert la porta a la possibilitat de demostrar experimentalment l'existència d'horitzons de forats negres. Això ha despertat en els últims anys un creixent interés en l'estudi d'objectes exòtics compactes (ECOs) que imiten la física dels forats negres, així com dels processos físics que permetrien diferenciar-los dels forats negres [39]. [En la secció 4.2 donem una breu explicació sobre els tipus d'ECOs proposats fins ara.] La Relativitat General clàssica no permet l'existència d'este tipus d'objectes a causa del teorema de Buchdahl, però la introducció d'efectes quàntics pot permetre la violació d'este teorema, obrint la porta a la possible formació d'ECOs. Existeixen diverses maneres de construïr estos objectes, i una d'elles és considerant els efectes semi-clàssics generats pels camps quàntics. Eixa és la via que explorem en l'Article 5 d'esta tesi (mostrat en la part III). En particular, estudiem els efectes de backreaction produïts per la polarització del buit quàntic al voltant d'un forat negres estàtic i sense rotació, obtenint així correccions quàntiques a la mètrica de Schwarzschild.

Per a tal objectiu, busquem solucions de les equacions semi-clàssiques d'einstein sense matèria

$$G_{ab} = 8\pi \langle T_{ab} \rangle \ . \tag{26}$$

El principal problema a l'hora d'afrontar este problema és que en 4 dimensions no tenim una expressió analítica renormalizada de $\langle T_{ab} \rangle$ per a una mètrica general. Però en el cas de dimensions $1+1$ sí que es coneix l'expressió exacta del tensor energia-moment renormalitzat. És per això que en [40, 41, 42] es va proposar una aproximació per a resoldre les equacions semi-clàssiques d'Einsten mitjançant la integració de les components angulars, traslladant el problema a un espai 2-dimensional. Açò va ser posteriorment analitzat en més detall i estudiat per a diferents casos per altres autors [43, 44, 45, 46, 47]. En canvi, en este article proposem una via alternativa per a resoldre este problema directament en 4 dimensions. En particular, considerem només els efectes quàntics generats per camps conformes (en concret un camp escalar conforme), ja que per la seua simetria es simplifica considerablement el problema. És raonable pensar que els resultats per a altres tipus de camps seran qualitativament similars. Per a camps conformes la ben coneguda anomalia de traça (que expliquem en detall en la secció 6.2) defineix unívocament una relació entre les components del tensor energia-moment, que ve donada per

$$-\langle \rho \rangle + \langle p_r \rangle + 2\, \langle p_t \rangle = \langle T^a_a \rangle \ , \tag{27}$$

on $\langle \rho \rangle$ és la densitat del buit quàntic, $\langle p_r \rangle$ i $\langle p_t \rangle$ les pressions radial i tangencials, i $\langle T^a_a \rangle$ és l'expressió de l'anomalia de traça, que depèn de la mètrica. [Com que busquem solucions estàtiques i esfèricament simètriques, hem escollit també un estat de buit amb estes simetries, la qual cosa dona lloc a un tensor energia-moment diagonal i independent del temps.] Així, la nostra proposta consisteix en resoldre les equacions semi-clàssiques d'Einstein afegint (27) com a equació d'estat. Noteu que amb este procediment no és necessari donar una expressió del tensor energia-moment en termes d'una mètrica general (que era el principal problema), ja que ara les seues components s'introdueixen com a incògnites del sistema d'equacions diferencials.

Per últim fem una última assumpció per tal de fer el sistema resoluble, que consisteix en considerar la pressió radial igual a la tangencial ($\langle p_r \rangle = \langle p_t \rangle$). Esta simplificació està inspirada en el resultat del tensor energia-moment per a un background de Schwarzschild fixat [35], on s'obté que prop de l'horitzó les pressions tendeixen a igualar-se. És raonable esperar que la solució exacta, afegint backreaction, es comporte de manera semblant ($\langle p_r \rangle \approx \langle p_t \rangle$) prop de $r = 2M$. En qualsevol cas, posteriorment hem comprovat que els resultats per a altres assumpcions per a les pressions són qualitativament similars.

Busquem solucions estàtiques i amb simmetria esfèrica, per tant el sistema d'equacions a resoldre és anàleg a les equacions de TOV (amb la densitat i la pressió quàntiques), afegint l'equació d'estat esmentada abans.

Com a primera aproximació a la solució resolem el sistema pertorbativament en $\hbar$. Restringint-nos a la regió propera a l'horitzó (on més pes tenen els efectes quàntics), obtenim la següent correcció a primer ordre en $\hbar$ de la mètrica de Schwarzschild

$$\begin{aligned} ds^2 &= -\left(f(r) - \hbar\left(\frac{1}{13440\pi M^2 f(r)} + \mathcal{O}(\log f(r))\right) + \mathcal{O}\left(\hbar^2\right)\right) dt^2 \\ &+ \frac{dr^2}{f(r) - \hbar\left(\frac{1}{4480\pi M^2 f(r)} + \mathcal{O}(\log f(r))\right) + \mathcal{O}\left(\hbar^2\right)} + r^2 d\Omega^2 \,, \end{aligned} \tag{28}$$

on $f(r) = 1 - 2M/r$. D'este resultat podem extraure una conclusió principal: l'horitzó clàssic de la mètrica de Schwarzschild desapareix. Per al valor de r per al qual $g_{rr}^{-1}(r) = 0$, que ve donat per

$$r_0 = 2M + \frac{\sqrt{\hbar}}{4\sqrt{70\pi}} + \mathcal{O}(\hbar) \,, \tag{29}$$

la component $g_{tt}(r)$ no s'anul·la ($g_{tt}(r_0) \neq 0$), a diferència del que ocorre en la mètrica de Schwarzschild clàssica. S'obté així una mètrica de tipus forat de cuc (veure secció 4.2 per a més detalls sobre estos objectes). Tot i això, este resultat no és totalment fiable, ja que la densitat i pressió quàntiques resulten ser són d'ordre $\hbar/f^2$, que prop de la gola del forat de cuc ($r = r_0$) tendeixen a ser d'ordre $\hbar^0$. Per tant en la regió propera a la gola la hipòtesi pertorbativa falla, i cal estudiar el problema de manera exacta numèricament. A l'article mostrem la representació obtinguda de la solució exacta, i obtenim que és qualitativament semblant a la pertorbativa, llevat de factors numèrics. En concret obtenim que la gola està situada en $r_0 \approx 2M + 0.01947\sqrt{\hbar}$, que difereix lleugerament del resultat anterior.

En resum, hem obtingut una singularitat coordenada per a un valor de r separat del valor clàssic ($r = 2M$) per una distància de l'ordre de la longitud de Planck ($\sqrt{\hbar}$). La singularitat representa la gola d'un forat de cuc. El següent pas lògic és estendre la mètrica més enllà d'esta singularitat coordenada, com es fa en el cas clàssic. En l'article proposem una extensió de tipus Morris-Thorne, convenient per a una mètrica de forat de cuc, definida pel canvi $l(r) \equiv \int_{r_0}^{r} \sqrt{g_{rr}(r')}dr'$. La gola del forat de cuc està situada en $l = 0$. L'extensió de la mètrica a la regió $l < 0$ dona com a resultat un forat de cuc asimètric. A més a més, trobem una nova singularitat situada en $l_s \sim -0.278\hbar^{1/4}\sqrt{M}$. A l'article provem que esta singularitat és de curvatura i està situada sobre una hiper-superfície de tipus nul. En la Figura 0.1 mostrem un diagrama de Penrose qualitatiu d'esta solució. Així mateix demostrem que esta singularitat està situada a una distància geodèsica d'ordre $\mathcal{O}(\sqrt{\hbar})$ respecte de la gola, de manera que un observador travessant el forat de cuc trobaria quasi immediatament la singularitat. La forma d'esta solució (forat de cuc asimètric amb una singularitat de curvatura nul·la) coincideix qualitativament amb l'obtinguda mitjançant l'aproximació 2-dimensional [41], la qual cosa reforça la validesa d'esta aproximació.

Esta solució de forat de cuc és l'extensió maximal de la solució de les equacions semi-clàssiques d'Einstein de buit (quàntic) pur. Però u es pot plantejar empalmar esta mètrica amb l'interior d'una estrela estàtica i amb simetria esfèrica. La inclusió de matèria pot generar objectes estel·lars ultra-compactes [48, 44, 45, 46, 47]. Si empalmem estes solucions a la nostra mètrica per a l'exterior de l'estrela, el nostre resultat imposa un valor màxim per a la compacitat d'estos objectes, que ve donat pel mínim de la funció radial (la gola del forat de cuc). En concret obtenim que el màxim de compacitat (mesurat com $2M/r$) seria

$$\frac{2M}{r_0} \sim 1 - 0.01686\frac{\sqrt{\hbar}}{2M}\,. \tag{30}$$

Esta és una important restricció per als ECOs, que considerem com un dels principals resultats de l'article.

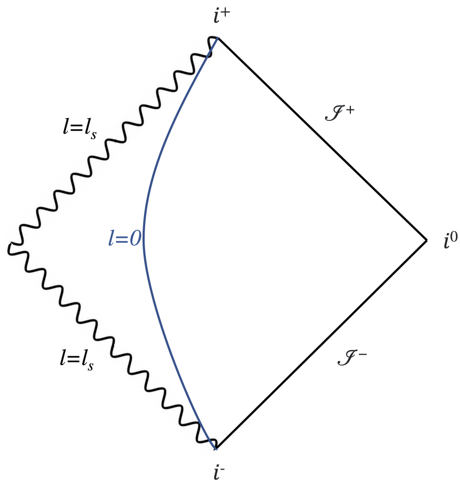

Figure 0.1: Diagrama de Penrose mostrant la gola del forat de cuc $(l = 0)$ i la singularitat de curvatura nul·la $(l = l_s)$.

Per últim, comprovem que les implicacions físiques de les correccions quàntiques lluny de $r = 2M$ són menyspreables i no serien detectables amb els interferòmetres actuals. En particular, a mode d'exemple, obtenim la correcció quàntica (a primer ordre en $\hbar$) de les freqüències dels modes del *light-ring* generat per pertorbacions escalars i electromagnètiques. Per a fer-ho utilitzem l'aproximació analítica WKB [49, 50]. Obtenim resultats de la forma $\omega^2 = \omega^2_{Sch} + \mathcal{O}(\hbar)$, on ω^2_{Sch} són les freqüències per al cas de Schwarzschild. Per exemple, en el cas de pertorbacions electromagnètiques obtenim

$$\omega^2 = \omega^2_{\mathrm{Sch}} + \frac{\hbar}{17010\pi M^2}\left(-13\,\mathrm{Re}\left[\omega^2_{\mathrm{Sch}}\right] + 11i\,\mathrm{Im}\left[\omega^2_{\mathrm{Sch}}\right]\right)\,. \tag{31}$$

Podem veure que les correccions quàntiques a estos observables són menyspreables. Açò és el que esperàvem, ja que el *light-ring* es troba entorn a $r = 3M$

que està suficientment allunyat de la regió de la gola, on s'espera que les correccions quàntiques tinguen més pes. Concloem per tant que, tot i que els efectes quàntics impliquen canvis dràstics en la geometria del forat negre prop de l'horitzó, no sembla que impliquen correccions significatives en l'exterior. És a dir, des del punt de vista d'un observador distant, esta solució semi-clàssica és indistingible d'un forat negre sense rotació.

Contents

Chapter 1

Introduction and motivation

There are two fundamental theories that underpin modern physics today: General Relativity (GR), which explains gravitation and the physics of the universe; and Quantum Field Theory, which is able to describe a wide range of quantum phenomena, ranging from condensed matter to the physics of particle interactions. Both theories have been tested experimentally and are widely accepted by the scientific community, although they are far from solving all the problems of current physics. One of the most fundamental problems remaining nowadays is precisely how to fit these two theories together in order to explain physical phenomena involving both quantum and gravitational effects. A unified theory may shed light on problems that physicists have been trying to solve for decades. The two most important in this regard are: discovering the true nature of black holes and the physics around their curvature singularity, as well as the physical processes that took place at the origins of the universe and led to the creation of matter.

We are far from finding a unified theory, but that does not mean that these problems are a complete mystery. Theories that go beyond the Standard Model [51], modified theories of gravity that review and provide corrections to classical General Relativity [52], quantum theories for gravity [53], etc, have their own proposal. However, all of them require additional hypoThesis that need to be confirmed by experiments. If there is a theory that stands out for its reliability, as well as for its practicality to study and to produce results of physical interest involving both gravity and quantum fields, this is Quantum Field Theory in curved spacetimes. This approach was originally started by Leonard Parker in 1966 [1, 2, 3, 4], where it was discovered that elementary particles can be created by a time-dependent gravitational field. These works launched a new and fruitful field in physics, which was subsequently analyzed and extended by many authors and summarized in standard monographs [5, 6, 7, 8]. This theory is a semiclassical approximation, in which quantum fields describing matter and/or radiation are coupled to an external, background gravitational field, which is approximated by a classical field in the weak-field regime. The semiclassical approximation is widely used and has historically paved the way for significative advances in our understanding of physical interactions, for example, in the early years of the quantum electrodynamics during the past century. The basic idea of semiclassical gravity is to apply this idea to the gravitational field. If gravity plays an important role in the dynamics of a quantum field (this happens, for example, in the vicinity of a black hole or in the early universe), but in such a way that quantum gravitational fluctuations are still negligible, then the gravitational field can be treated as a classical, external field. This avoids dealing with the unresolved problem of quantizing gravity in those situations where quantum fluctuations of the spacetime metric are expected to be negligible.

Quantum field theory in curved spacetimes is, in this sense, an effective theory that does not unify gravity and quantum physics, as it does not propose a method to quantize gravity, but it has nevertheless proven to be very useful in explaining phenomena that could not be tackled otherwise.

One of these is the well-known Hawking radiation [10, 11], which shows how, contrary to what has been assumed for decades, black holes are not simply sinks of matter that do not allow any kind of particle to pass beyond their boundaries. Black holes radiate matter, and this is an effect that can only be explained if the quantum fluctuations excited around the horizon are taken into account. Another important effect that the semiclassical theory is able to explain arises in cosmology. This is the spontaneous creation of particles that results from the expansion of the universe, an effect that was first discovered by L. Parker [1]. This phenomenon reveals the fundamental property that in a non-static spacetime there is no unique or preferred choice for a vacuum state in quantum field theory: observers at both early and late times differ in their "natural" notions of vacuum state. This results in a different measurement of the number of particles at different instants. This effect becomes particularly important during the first instants of time of the universe after the Big Bang and to explain the creation of matter [12, 13]. In fact, the phenomenon of spontaneous particle creation is not only present in curved spacetimes, but it can also occur in flat spacetimes. More precisely, an intense electric field can also generate particle-antiparticle pairs out of the quantum vacuum. This is known as the Schwinger effect [14, 15], a non-perturbative effect that can only be obtained through the semiclassical approach (via the one-loop effective action).

In conclusion, in all these scenarios (Electrodynamics, Cosmology, and Black holes), we see that quantum fields in the presence of external classical fields exhibit interesting properties that must be explored using semiclassical theories. This is the theoretical framework of this Thesis. The articles that make up this Thesis (shown in part III) are focused on the study, development, and application of the semiclassical theory in these physical scenarios (anomalies, renormalization, backreaction effects...).

Regarding Electrodynamics, we examined how certain well-known phenomena in the context of QFT in curved spacetimes can also be recovered in the context of an electromagnetic background. For instance, in the Article 1 of this Thesis we study whether the so-called *gravitational anomaly*, which is known to occur for a 2-dimensional Weyl fermion coupled to gravity, arises when the background is an electric field. We will see that in this second case, an anomaly also arises in the conservation of momentum. On the other hand, in the Article 2, we study whether the electromagnetic case maintains the *adiabatic invariance* of the number of particles, which consists of no particle creation in the limit of an infinitely slowly expanding universe. We will see that, under certain conditions, this invariance is broken in this second case. In fact, this phenomenon is intimately related to the well-known *axial anomaly*. Finally, in the Article 3, we extend the method of *adiabatic renormalization* (which is very useful in the cosmological context) to the case of a 4-dimensional Dirac field coupled to an electric background.

Regarding the field of Cosmology, in the Article 4, we extend to the cosmological context a recent and successful regularization method (*pragmatic mode-sum regularization*) that had only been applied in the context of black holes up to now. Finally, in the context of black holes, we study quantum corrections to the Schwarzschild metric, specifically in the Article 5. These corrections come from the effects of *backreaction* generated by the quantum fields on the gravitational background itself. We will see that the geometry of the newly generated spacetime presents significant differences compared to that of black holes. This result may contribute to the study of the formation of ultra-compact objects that mimic the physics of black holes.

1.1 Structure and conventions

The Thesis is organized as follows. In Part I we review the main concepts that make up the theoretical framework of this Thesis (placing them historically), which are necessary to facilitate the understanding of the articles. In Part II, we summarize the results and conclusions that have been obtained throughout the PhD. Finally, in Part III, we include the articles that support the Thesis.

Regarding conventions, throughout the Thesis we work in natural units, i.e., $G = c = \hbar = 1$, unless otherwise stated. For the signature of the metric we use $(+, -, -, -)$, except in the Article 5 where we use the opposite signature (and where we leave $\hbar$ explicitly). For curvature tensors, we follow the conventions of [5].

1.2 Methodology

For the development of the articles, it has been consulted updated literature from the different theoretical areas involved in the work. Regarding the calculations that were necessary for the development of the articles, it should be noted mainly the use of *Mathematica* for analytical calculations (including the *x-Act* package for tensorial calculations), as well as the use of *Matlab* for more involved numerical calculations.

The articles have been developed in collaboration with other members of the research group and external scientists, through meetings and task distribution. Therefore, the authorship of the articles is distributed equally (the names of the authors in the articles are listed alphabetically, as is customary in publications in this field). Lastly, conferences have also been attended to communicate the results obtained with other research groups, as well as to learn the work of other authors in related subjects.

Part I

Theoretical framework

Chapter 2

Review on Quantum Field Theory in curved spacetimes

All the articles that comprise this Thesis are, to a greater or lesser extent, based on Quantum Field Theory in curved spacetimes, as well as on semi-classical electrodynamics. The underlying idea in both theories is the same: to consider the gravitational/electromagnetic field as a classical background and to couple them to quantized fields. This approach allows the analysis of non-perturbative effects such as the spontaneous creation of particles. In this first section we will briefly explain the basic concepts of Quantum Field Theory in curved spacetimes, as well as a proof of the phenomenon of particle production in the cosmological context. [A more detailed derivation can be found in [5]]. In the next section we will introduce its electromagnetic analogue.

Let us consider a set of scalar fields $\phi_i(x)$ propagating in a n-dimensional, globally hyperbolic spacetime background (M, g_{ab}), where the manifold can be decomposed as $M \simeq \mathbb{R} \times \sigma$, for some $n-1$-dimensional spacelike hypersurface σ. Global hyperbolicity is required to guarantee that the time evolution of the fields is mathematically well-posed. In coordinates the metric can be expressed as $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$, where we will denote $x^0 = t$ as the time coordinate, and $(x^1, ...x^{n-1}) = \vec{x}$ as the spatial ones. The classical field theory is described by an action functional, which in curved spacetime also depends on the metric, so it must have the form $\mathcal{S}(\phi_i(x), \nabla\phi_i(x), g_{\mu\nu}(x))$. Such action can be constructed by following the *minimal coupling* prescription, which is consistent with the Einstein principle of equivalence. Starting from the usual action for the field in Minkowski spacetime, the prescription consists in replacing the flat Minkowski metric η_{ab} by the curved metric g_{ab}, the flat covariant derivative $\nabla_a[\eta]$ by the non-flat connection $\nabla_a[g]$, and the measure $d^n x\sqrt{-\eta}$ by the invariant volume element $d^n x\sqrt{-g}$, where $g = \det(g_{\mu\nu})$. As we will see later, one can add additional terms involving higher derivatives of the metric, but for now we will only consider the simplest case of minimal coupling.

The resulting action in curved spacetime can then be expressed in terms of a lagrangian density $\mathcal{L}$, as

$$\mathcal{S} = \int d^n x \mathcal{L}(\phi_i, \nabla_\mu\phi_i, g_{\mu\nu}) \,, \tag{2.1}$$

This functional must be invariant under general coordinate transformations. Requiring invariance of the action under variations of $g_{\mu\nu}$ induced by an infinitesimal coordinate transformation leads to the conservation law

$$\nabla_\mu T^{\mu\nu} = 0 \,, \tag{2.2}$$

where $T_{\mu\nu}$ is the symmetric stress-energy tensor, defined by

$$T^{\mu\nu} = -\frac{2}{|g|^{1/2}}\frac{\delta S}{\delta g_{\mu\nu}} \,. \tag{2.3}$$

On the other hand, requiring invariance of the action under variations of the fields ϕ_i yields the Euler-Lagrangian equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_i} = 0 \,. \tag{2.4}$$

Similarly to QFT in Minkowski spacetime, the fields ϕ_i are quantized by following the canonical procedure. More precisely, we impose the usual commutation relations

$$\begin{aligned} \left[\phi_i(t,\vec{x}),\pi_j\left(t,\vec{x}'\right)\right] &= \delta_{ij}\delta\left(\vec{x}-\vec{x}'\right) , \\ \left[\phi_i(t,\vec{x}),\phi_j\left(t,\vec{x}'\right)\right] &= \left[\pi_i(t,\vec{x}),\pi_j\left(t,\vec{x}'\right)\right] = 0\,, \end{aligned} \tag{2.5}$$

for bosonic fields, and

$$\begin{aligned} \left\{\phi_i(t,\vec{x}),\pi_j\left(t,\vec{x}'\right)\right\} &= \delta_{ij}\delta\left(\vec{x}-\vec{x}'\right) , \\ \left\{\phi_i(t,\vec{x}),\phi_j\left(t,\vec{x}'\right)\right\} &= \left\{\pi_i(t,\vec{x}),\pi_j\left(t,\vec{x}'\right)\right\} = 0\,, \end{aligned} \tag{2.6}$$

for fermionic fields. Here π_i are the canonical conjugated momentum defined by $\pi_i = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)}$, δ_{ij} is the Kronecker delta, and $\delta\left(\vec{x}-\vec{x}'\right)$ is the Dirac delta. These relations are covariant under transformations of the spatial coordinates, and they remain valid for any $t =$ constant hypersurface.

2.1 Scalar field

In this section we briefly explain how this theory can be applied to the simplest case, a scalar field (spin 0) coupled to a gravitational background. We will then restrict to cosmological spacetime backgrounds, consisting of expanding universes. This is a particularly useful arena for understanding the phenomenon of particle creation.

For Minkowski spacetime the Lagrangian density of a real scalar field with mass m is given by $\mathcal{L} = \frac{1}{2}(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)$. Therefore, following the minimal coupling prescription described in the previous section, the Lagrangian density in a curved spacetime becomes

$$\mathcal{L} = \frac{1}{2}|g|^{1/2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) . \tag{2.7}$$

(the factor $|g|^{1/2}$ comes from the change of the volume element in (2.1), and the covariant derivatives reduce to partial derivatives for scalar fields). Besides the minimal coupling prescription, there is also the freedom of adding a coupling with curvature in the following form

$$\mathcal{L} = \frac{1}{2}|g|^{1/2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right)\,, \tag{2.8}$$

where ξ is a dimensionless real number known as *coupling constant* between the field and the background, and R is the scalar curvature. This extra term is needed to ensure the renormalizability of the theory when interaction terms are included in the Lagrangian (see section 6.7 of [5]). Note that the case $\xi = 0$ corresponds to the minimal coupling prescription. On the other hand, when $\xi = 1/6$ and $m = 0$ the Lagrangian density is invariant under conformal transformations of the spacetime. For this reason the value $\xi = 1/6$ is known as *conformal coupling.* As we will explain in more detail in section 6.2, conformal symmetry implies that the trace of the stress-energy tensor (2.3) is 0 for solutions of the equation of motion, but in the quantum theory this is no longer true (and it is known as the Trace Anomaly).

The Euler-Lagrange equation of the Lagrangian density (2.8) produces the Klein-Gordon equation for curved spacetimes

$$\left(\Box + m^2 + \xi R\right)\phi = 0\,. \tag{2.9}$$

The operator $\Box$ in curved spacetimes acts as $\Box\phi = |g|^{-1/2}\partial_\mu\left(|g|^{1/2}\partial^\mu\phi\right)$. Since the field equations are linear, the space of solutions has the structure of a vector space. This vector space can be endowed with a symplectic structure. Consider two functions f_1 and f_2 that are solutions of the above equation. We define the Klein-Gordon inner product

$$(f_1, f_2) := i\int_\sigma d\sigma |g|^{1/2} n^\mu f_1^* \overleftrightarrow{\partial_\mu} f_2\,, \tag{2.10}$$

where σ is an arbitrary spacelike hypersurface, and n^μ is the future-pointing, unit normal vector. It can be proven that this product is independent of the choice of σ. In particular, if one considers a $\{t = \text{constant}\}$ hypersurface of dimension $n-1$, this product becomes

$$(f_1, f_2) = i\int_\sigma d^{n-1}x|g|^{1/2}n_\nu g^{\nu\mu} f_1^* \overleftrightarrow{\partial}_\mu f_2 \qquad (2.11)$$

and using the field equations it is not difficult to find that this quantity is conserved in time. Throughout this text the notation $(,)$ will refer to this product.

At this point, one can proceed to do the quantization of the scalar field by following the same procedure as in Minkowski spacetime. For convenience, let us consider finite spacelike hypersurfaces σ consisting of a cube of side length L (this length will be taken to infinity at the end of the analysis), and impose periodic boundary conditions on the field. Given a complete basis for the space of solutions of the field equation (2.9), $f_{\vec{k}}$, labelled by $n-1$ real numbers $\vec{k}$, the quantum field can be expanded in terms of the usual annihilation and creation operators ($A_{\vec{k}}$, $A^\dagger_{\vec{k}}$) as

$$\phi(t, \vec{x}) = \sum_{\vec{k}} \left(A_{\vec{k}} f_{\vec{k}}(t, \vec{x}) + A^\dagger_{\vec{k}} f^*_{\vec{k}}(t, \vec{x}) \right) . \qquad (2.12)$$

The mode functions $f_{\vec{k}}$ are orthonormal with respect to the symplectic inner product

$$(f_{\vec{k}}, f_{\vec{k}'}) = \delta_{\vec{k},\vec{k}'} \ , \qquad (f_{\vec{k}}, f^*_{\vec{k}'}) = 0 \, . \qquad (2.13)$$

Since this product is conserved in time, these conditions will be valid for all t. Using these relations and the properties of the δ distribution, one can prove that the usual commutation relations (2.5) are equivalent to

$$[A_{\vec{k}}, A^\dagger_{\vec{k}'}] = \delta_{k,k'} \quad , \quad [A_{\vec{k}}, A_{\vec{k}'}] = [A^\dagger_{\vec{k}}, A^\dagger_{\vec{k}'}] = 0 \, , \qquad (2.14)$$

which are the same relations as in QFT in Minkowski spacetime. As we will see in the next subsection, if the spacetime evolves from an asymptotically flat region at early times, one can assign $\{A_{\vec{k}}, A^{\dagger}_{\vec{k}}\}$ a notion of annihilation / creation operators of particles with momentum $\vec{k}$ with respect to observers that remain static at early times, i.e. with respect to the integral curves of the asymptotic Killing vector field $\partial/\partial t$. However, at later times when spacetime is dynamical (non-vanishing curvature) this statement is no longer true. This is the origin of the phenomenon of spontaneous particle creation by the curvature of the spacetime, that we will introduce in the following section.

2.2 Particle creation in an expanding universe

To analyze the phenomenon of particle creation it is convenient to consider a particular cosmological model: an expanding universe that is asymptotically flat at early and late times. This configuration is illustrative because the annihilation and creation operators (and therefore the concept of vacuum state, and in turn the notion of particle) can be given a clear physical meaning at both early and late times. Therefore, in this model one can obtain a rigorous calculation of the number of particles created by the spacetime expansion by comparing how the two notions of vacuum state relate to each other.

Let us then consider an isotropical and homogeneous spacetime in four dimensions, given by the well-known Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$ds^2 = dt^2 - a(t)^2 d\vec{x}^2 \,, \tag{2.15}$$

where $a(t)$ is a real function, known as *scale factor*. The asymptotically flatness condition requires that the scale factor behaves as

$$a(t) \to \begin{cases} a^{in} & \text{as } t \to -\infty \\ a^{out} & \text{as } t \to \infty \end{cases} , \tag{2.16}$$

where a^{in} and a^{out} are positive constants. Since the background metric is spatially homogeneous, the field will be propagated spatially as a free wave, so we can consider the following ansatz for the mode functions in (2.12)

$$f_{\vec{k}}(t,\vec{x}) = \frac{1}{\sqrt{2L^3 a(t)^3}} h_{\vec{k}}(t) e^{i\vec{k}\vec{x}} \, . \tag{2.17}$$

(The factor $1/\sqrt{2L^3 a(t)^3}$ is introduced for simplicity in the following calculations). Therefore, the Klein-Gordon equation (2.9) with these assumptions is reduced to the differential equation

$$\frac{d^2 h_{\vec{k}}}{dt^2} + \left(\omega_k^2 + \sigma\right) h_{\vec{k}} = 0 \, , \tag{2.18}$$

where $\omega_k = \sqrt{m^2 + \left(\frac{k}{a}\right)^2}$, and

$$\sigma = \left(6\xi - \frac{3}{4}\right) \frac{\dot{a}^2}{a^2} + \left(6\xi - \frac{3}{2}\right) \frac{\ddot{a}}{a} \, . \tag{2.19}$$

This last term is called the frequency scale of the background. Notice that when the spacetime tends to a flat region ($a \to$ const.) then $\sigma \to 0$ and the solution of this equation is just a free oscillating mode propagating with frequency ω_k. Then, at early and late times the modes $h_{\vec{k}}$ tend to be a linear combination of the Minkowskian positive ($e^{-i\omega_k t}$) and negative ($e^{i\omega_k t}$) frequency solutions.

On the other hand, the normalization relations (2.13) imply the following normalization condition

$$h_{\vec{k}} \dot{h}^*_{\vec{k}} - h^*_{\vec{k}} \dot{h}_{\vec{k}} = 2i \, . \tag{2.20}$$

Since the inner product is conserved in time, this equation will also be preserved. Therefore if we impose initial conditions which verify this condition, it will be ensured for all times. In fact, $h_{\vec{k}}\dot{h}^*_{\vec{k}} - h^*_{\vec{k}}\dot{h}_{\vec{k}}$ is a Wronskian of the equation (2.18), which is a conserved quantity. This equation fixes an integration constant of the differential equation (2.18), while the other one must be fixed by choosing the vacuum state $|0\rangle$ of the theory. For instance we can choose the vacuum state to be fixed at early times, i.e., no particles are present in the early universe. This is equivalent to fix that when $t \to -\infty$ the modes behave as positive frequency solutions with respect to the Killing vector $\partial/\partial t$, i.e.,

$$h_{\vec{k}}(t \to -\infty) \sim \frac{1}{\sqrt{\omega_k^{in}}} e^{-i\omega_k^{in} t}\,, \tag{2.21}$$

where $\omega_k^{in} = \sqrt{(\frac{k}{a^{in}})^2 + m^2}$. Therefore, A_k and $A_k^\dagger$ in (2.12) are the annihilation/creation operators of particles in the early universe, and it is verified $A_{\vec{k}}|0\rangle = 0$.

However, at late times the solution will be a combination of positive and negative frequency solutions, as a consequence of the time evolution of the differential equation, i.e.

$$h_{\vec{k}}(t \to \infty) \sim \frac{1}{\sqrt{\omega_k^{out}}} (\alpha_k e^{-i\omega_k^{out} t} + \beta_k e^{i\omega_k^{out} t})\,, \tag{2.22}$$

where $\omega_k^{out} = \sqrt{(\frac{k}{a^{out}})^2 + m^2}$ and α_k, β_k are dimensionless integration constants. As we will see later, this fact is the reason behind the particle production effect. The normalization equation (2.20) implies the relation

$$|\alpha_k|^2 - |\beta_k|^2 = 1\,. \tag{2.23}$$

Since in general $\beta_k \neq 0$, the operators $A_k, A_k^\dagger$ are no longer the standard annihilation and creation operators in the late universe. Introducing (2.22) into the field expansion (2.12) and regrouping terms one can find the expression for the annihilation/creation operators at late times in terms of the corresponding operators at early times

$$A^{out}_{\vec{k}} = \alpha_k A_{\vec{k}} + \beta^*_k A^\dagger_{-\vec{k}} \,. \tag{2.24}$$

These relations are known as Bogoliubov transformations, and have their origin in condensed matter physics [54, 55]. Using (2.23) it can be proved that these new operators verify the usual commutation relations (2.14).

We have all the elements now to prove the production of particles by the expanding universe. The particle number operator at late times for modes with momentum $\vec{k}$ is defined by $A^{\dagger out}_{k} A^{out}_{k}$. The vacuum expectation value of this operator can be easily computed using the commutation relations, and yields

$$\langle N^{out}_k \rangle = \langle 0 | A^{\dagger out}_k A^{out}_k | 0 \rangle = |\beta_k|^2 \,. \tag{2.25}$$

For a general changing scale factor $a(t)$, the coefficients $\beta_{\vec{k}}$ are different from zero. Therefore, particles can be created during the expansion because of the curvature of the spacetime. The momentum distribution of the created particles is given by the Bogoliubov coefficients $|\beta_k|^2$. Its expression depends on the particular form of $a(t)$, and for some simple cases it can be obtained in closed analytical form. Summing this distribution for all the possible momenta and dividing by the comoving volume of the universe at late times $(L\, a^{out})^3$, we obtain the average density of created particles

$$\langle n^{out} \rangle = \frac{1}{(L\, a^{out})^3} \sum_{\vec{k}} |\beta_k|^2 \,. \tag{2.26}$$

In the continuum limit $(L \to \infty)$ one obtains:

$$\langle n^{out} \rangle = \frac{1}{(2\pi^2 a^{out})^3} \int_0^\infty dk k^2 |\beta^{out}_k|^2 \,. \tag{2.27}$$

This is the total number of particles created by the dynamics of the spacetime between the two asymptotically flat limits at early and late times. It can be proved that $|\beta_k|^2$ decreases faster than any negative power of k when $k \to +\infty$, so this integral converges (see, for instance, [7]).

A very important property of the number of particles created is that it is an adiabatic invariant. This means that in the limit of an infinitely slow expanding universe, no particles would be created (see [17] for a historical review). In the Article 2 of this Thesis (shown in part III) we explain this phenomenon in detail, and we analyze whether this property holds when the background is not a gravitational field but an electromagnetic one. We prove that, in certain cases, this adiabatic invariance is broken in the presence of electromagnetic fields.

Chapter 3

Semiclassical Electrodynamics and the Schwinger Effect

As explained in Chapter 1, the semiclassical approach (i.e. coupling a quantum field to a classical background field) is very useful to study quantum effects in gravity because Einstein's equations are highly non-linear and the quantization of the full theory is far from obvious. But this is not the only case where this approach works. It is well-known that in Quantum Field Theory (in flat space), the semiclassical prescription gives interesting results. For instance, in Quantum Electrodynamics (QED) this method allows the analysis of non-perturbative effects that arise when the background is very strong (yet, classical) and are difficult to examine in the full theory. This is the case of the well-known Schwinger effect, which involves the spontaneous creation of particle-antiparticle pairs generated by the effect of a strong electric field. Note that this is analogous to the particle creation by the expansion of the universe that we explained above, but in this case, the strong background field is an electric field, instead of a gravitational one. The instability of the vacuum caused by an electric field was, however, discovered much earlier. It was predicted for the first time in 1936 by W. Heisenberg and H. Euler [56], inspired by the work of F. Sauter on the Klein paradox [18]. Some years later, it was formalized in Quantum Electrodynamics (QED) by J. Schwinger [14, 57, 58]. This phenomenon is of particular interest from an experimental point of view, as it may soon be possible to detect it in high intensity lasers [59] and beam-beam collisions [60]. Additionally, this effect holds significant importance in some scenarios in astrophysics [61, 62] and cosmology [63, 64, 65, 66, 67], as well as in non-equilibrium processes that are induced by strong fields [68, 69, 70]. On the other hand, recent works have resumed the study of semiclassical electrodynamics and the Schwinger effect to analyze technical aspects, such as the renormalization of physical observables associated with this effect [25, 26] (and also the Article 3 of this Thesis). Other works have also analyzed the ambiguities in defining the vacuum state at times when the electric field is acting, and have proposed criteria for selecting vacuum states that allow estimating the number of particles at these instants [71, 72, 73, 74, 75].

In this section we will introduce this phenomenon for the case of a constant electric background and will make some comments on its extension to the time-dependent case, which can be studied in a very analogous way to the particle creation in expanding universes. This will be useful for understanding the first three articles of the Thesis (shown in part III), which are based on semiclassical electrodynamics and are closely related to the Schwinger effect. We will show how this effect was derived for the first time, i.e., by using the Euler-Heisenberg effective Lagrangians [56]. These Lagrangians take into account the effect of vacuum polarization to one loop and describe the dynamics of a quantum field coupled to a strong and slowly varying electromagnetic field that is considered classical. Before the development of renormalization theory, these Lagrangians were employed to describe the nonlinear dynamics of electromagnetic fields in the vacuum, obtaining significant results, such as the Schwinger effect, as we will see briefly next (for a more detailed analysis see [76]).

Let us consider the case of charged bosons (spin 0) and Dirac fermions (spin 1/2). Note that, unlike the gravitational case, this effect can only produce charged particles, since the electromagnetic field couples only with charged fields. The effective Lagrangian densities for bosons and fermions (respectively), with mass m and charge q, coupled to a strong electromagnetic background field $F_{\mu\nu}$, can be written as [56, 15]

$$\mathcal{L}_b = -\frac{1}{16\pi^2}\int_0^\infty ds \frac{e^{-\frac{m^2 s}{\hbar}}}{s^3}\left(\frac{q^2 abs^2}{\sinh(qbs)\sin(qas)} + \frac{q^2 s^2}{6}(b^2-a^2) - 1\right) , \quad (3.1)$$

$$\mathcal{L}_f = -\frac{1}{8\pi^2}\int_0^\infty ds \frac{e^{-\frac{m^2 s}{\hbar}}}{s^3}\left(\frac{q^2 abs^2}{\tanh(qbs)\tan(qas)} - \frac{q^2 s^2}{3}(b^2-a^2) - 1\right) , \quad (3.2)$$

where

$$a = \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2} - \mathcal{F}} \quad , \quad b = \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2} + \mathcal{F}} , \quad (3.3)$$

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\vec{E}^2 - \vec{B}^2) \quad , \quad \mathcal{G} = \frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} = -\vec{E}\cdot\vec{B} , \quad (3.4)$$

and $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively. Note that if $\vec{E}$ and $\vec{B}$ are parallel then $a = |\vec{E}|$ and $b = |\vec{B}|$. For simplicity, we consider the case of a purely electric background ($\vec{B} = 0$), which is enough to produce the creation of particles. Then the Lagrangian densities become

$$\mathcal{L}_b = -\frac{1}{16\pi^2}\int_0^\infty ds \frac{e^{-\frac{m^2 s}{\hbar}}}{s^3}\left(\frac{qEs}{\sin(qEs)} - \frac{(qEs)^2}{6} - 1\right) , \quad (3.5)$$

$$\mathcal{L}_f = -\frac{1}{8\pi^2}\int_0^\infty ds \frac{e^{-\frac{m^2 s}{\hbar}}}{s^3}\left(\frac{qEs}{\tan(qEs)} + \frac{(qEs)^2}{3} - 1\right) , \quad (3.6)$$

where we denoted $E = |\vec{E}|$.

As a first approach to the problem, let us consider a constant electric field E. In this case, the Lagrangians above are just constants, and so is the effective action $S = \int d^4x\mathcal{L}$ ($\mathcal{L}$ can be either $\mathcal{L}_b$ or $\mathcal{L}_f$). This implies that the scattering matrix of this theory is just e^{iS}. Therefore, the probability that no particles are created, i.e., that the vacuum state $|0\rangle$ remains the vacuum, is given by

$$P(|0\rangle \to |0\rangle) = |\langle 0| e^{iS} |0\rangle|^2 = |e^{iS}|^2 . \tag{3.7}$$

Since the Lagrangians are constants, we have $S = VT\mathcal{L}$, where V and T are the volume and time scales of the experiment. Then $|e^{iS}|^2 = e^{-2VT\mathrm{Im}[\mathcal{L}]}$. As a result, assuming $2\mathrm{Im}[\mathcal{L}]$ is small, the quantity $2\mathrm{Im}[\mathcal{L}]$ can be regarded as the probability per unit time and volume that any number of pairs are created. By using contour integration over the poles of the integrals (3.5) and (3.6), one can calculate the imaginary part of the lagrangians, yielding

$$2\mathrm{Im}(\mathcal{L}_b) = \frac{q^2E^2}{8\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-\frac{n\pi m^2}{\hbar qE}} , \tag{3.8}$$

$$2\mathrm{Im}(\mathcal{L}_f) = \frac{q^2E^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{n\pi m^2}{\hbar qE}} . \tag{3.9}$$

These expressions give the rates for Schwinger pair production by a constant electric field.

One can see the non-perturbative nature of the Schwinger effect, as the argument of the exponential is proportional to the inverse of the charge, making it impossible to expand the expressions in power series of q, as typically done in perturbative QED. Consequently, pair production cannot be observed at any fixed order in perturbative QED.

According to equations (3.8) and (3.9), these rates are insignificant for low values of E, but become relevant when $E \gtrsim E_c = \frac{m^2}{\hbar q}$. This is known as the Schwinger limit, beyond which the electric field becomes nonlinear.

For instance, in the case of electrons, $E_c \approx 10^{18} V/m$, indicating that the Schwinger effect can only be produced by highly powerful electric fields, that are otherwise incredibly challenging to generate in laboratories. This is why the Schwinger effect has yet to be observed experimentally. However, with current laser technology, it may be possible to observe the Schwinger effect in laser experiments in the near future [59, 77, 78, 79].

We have analyzed this effect in the ideal scenario of constant electric fields, but it can be extended to the case in which the background is time-dependent. One possible approach to address this problem is through Bogoliubov transformations (see, for example, [15]). As mentioned before, particle production by electric fields is closely analogous to particle creation in expanding universes. The method described in section 2.2 can be straightforwardly extended to the case of an electromagnetic background. For example, consider a scalar field ϕ with mass m and charge q coupled to a time-dependent electric background (semiclassical scalar QED). The field equation is given by

$$(D_\mu D^\mu + m^2)\phi = 0\,, \tag{3.10}$$

where $D_\mu = (\partial_\mu + iqA_\mu)$ is a covariant derivative and A^μ is the 4-vector potential associated with the electric field. Expanding the scalar field in Fourier modes, one obtains harmonic oscillator-type differential equations with time-dependent frequencies, similarly to the gravitational case. Now the frequencies of the equations depend on the vector potential A^μ, which plays the role of the scale factor $a(t)$ in gravity. If we consider a configuration where the electric field is asymptotically vanishing at both early and late times, then the vacuum states at early and late times will in general differ, and as a consequence the creation and annihilation operators associated with early and late times can be related through Bogoliubov transformations. By calculating the expectation value of the number of particles at late times, one can find that particles have been spontaneously created by the dynamics of the electric background.

In the Article 2 of this Thesis (shown in part III) we perform this calculation in 2 dimensions, for both scalar and Dirac fields. [A 4-dimensional approach can be found, for instance, in [15].] In addition, we obtain the momentum spectrum of the number of particles created for a specific form of the electric field (a Sauter pulse), which allows us to solve the problem in closed analytical form. Furthermore, we analyze how this observable behaves in the limit in which the electric field varies adiabatically, and prove that the well-known adiabatic invariance of the particle number in expanding universes is not preserved in certain cases for an electromagnetic background. On the other hand, in the Article 3, we study the extension of the adiabatic renormalization method in the context of semiclassical electrodynamics. Finally, in the Article 1, we study the emergence of a momentum conservation anomaly associated with this effect in the case of Weyl fermions.

Chapter 4

Black Holes in the presence of Quantum Fields

As we have seen in previous sections, even without having a complete theory of quantum gravity at our disposal there are ways to deal with scenarios that combine gravity and quantum physics, that give rise to phenomena of great interest. In addition to cosmology, in this Thesis we have also addressed one of the most important frameworks that combine these two branches of physics: black holes. The prediction of black holes in General Relativity dates back to the early 20th century. However, it is worth noting that the first indication of objects of this type dates back to the late 18th century. J. Michel and P. S. Laplace independently proposed, based on Newtonian gravitation, that a very massive star could gravitationally attract light, turning it into an invisible object to our eyes [80, 81]. In Laplace's words, "*The gravitation attraction of a star with a diameter 250 times that of the Sun and comparable in density to the earth would be so great no light could escape from its surface. The largest bodies in the universe may thus be invisible by reason of their magnitude*". Much later, the development of Albert Einstein's theory of General Relativity gave rise to the modern concept of a black hole. Although the metric that describes the gravitational field of a non-rotating black hole was first obtained by K. Schwarzschild already in 1916 (what we now call the Schwarzschild metric [82], it was not until 1939 when the physical meaning of this vacuum solution of Einstein's equations was fully understood. This was the seminal paper by Oppenheimer and Snyder that describes the process of gravitational collapse of stars [83] (see [84] for a historical review).

The notion of black hole horizons is one of the most fascinating predictions of the theory of General Relativity. By definition, a black hole consists in a region of spacetime that concentrates a strong gravitational field, so strong that no particle can escape its influence, not even light. The boundary of this region is what we call the horizon. The formation of these objects is a consequence of the accumulation of matter in small regions of space, which inevitably collapses to a singular point when it exceeds a certain limit. For instance, for spherically symmetric spacetimes the Buchdahl theorem imposes a limit to the compacity of a star. Specifically, a star with a mass and radius such that $M/R < 4/9$ is unstable and will collapse inevitably in a black hole [85].

In the last decades the analysis of quantum fluctuations of fields around black holes has provided fundamental insights in our understanding of quantum gravity. In fact, the inclusion of quantum fields in the physics of black holes has called into question various basic properties that General Relativity predicted about these objects. On the one hand, the presence of quantum fields around black hole horizons leads to the emission of the well-known Hawking radiation, an effect that challenges the old statement that nothing can escape from black holes. This is perhaps the cornerstone of Quantum Field Theory in curved spacetimes. On the other hand, quantum effects also allow to bypass the assumptions of Buchdahl's theorem, thus opening up the possibility of forming exotic compact objects (ECOs) in astrophysics that may mimic the physics of black holes. This second question has been addressed in this Thesis, in particular in the Article 5 (shown in part III). In the following sections we will go through all these topics.

4.1 Hawking Radiation

The phenomenon of Hawking radiation, discovered by S. Hawking in 1974 [10, 11] refutes the idea that black holes are really "black". Instead, they emit thermal radiation composed of particles of any quantum field, which are excited by the gravitational collapse during the formation of the black hole. In Chapter 2 we have already seen how an intense and time varying gravitational background, like an expanding universe, can generate the spontaneous creation of particles that emerge from the quantum vacuum. The gravitational collapse of a star is physically a similar process and particles can be created in a similar fashion. Let us briefly see how this effect can be derived, as originally calculated by Hawking.

Consider for definiteness a spherically symmetric spacetime that describes a non-rotating, collapsing star into a black hole. This spacetime can be represented by the Penrose diagram in Fig. 4.1. In the region outside the star the metric is given by the Schwarzschild metric (Birkhoff's theorem [86]:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \quad (4.1)$$

where M is the mass of the black hole. Now we couple this background to a quantum field propagating along the spacetime. For simplicity, consider a massless, minimally coupled ($\xi = 0$) scalar field ϕ, which evolves according to the Klein-Gordon equation $\Box\phi = 0$. Following the exposition given

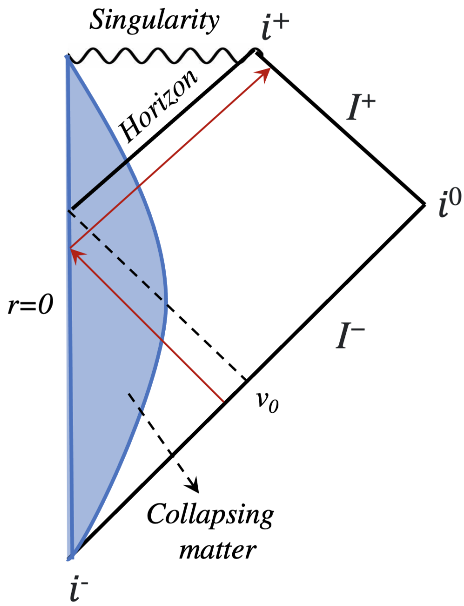

Figure 4.1: Penrose diagram of a spherically symmetric collapsing body into a black hole. I^- and I^+ are past and future null infinities respectively, i^- and i^+ are past and future time-like infinities respectively, and i^0 is space-like infinity. The red arrows represent a field mode that propagates from I^-, goes through the collapsing star and reaches I^+.

in Chapter 2, we can expand the quantum field in terms of the usual creation/annihilation operators as

$$\phi(x) = \sum_i \left(f_i(x) A_i + f_i^*(x) A_i^\dagger \right) , \tag{4.2}$$

where the set of functions $f_i(x)$ form an orthonormal family of solutions of the Klein-Gordon equation (the subindex i labels the quantum numbers of each mode). $f_i(x)$ are solutions of positive frequency at past null infinity I^-

with respect to the affine parameter v, which is the Eddington-Finkelstein coordinate defined by $v = t + r + 2M \log \left|\frac{r}{2M} - 1\right|$. This positive-frequency condition is equivalent to say that the field modes behave as $e^{-i\omega v}$, where ω is the frequency of the modes. On the other hand, A_i and $A_i^\dagger$ represent the annihilation and creation operators of particles at I^- (ingoing particles).

We can also express the field in terms of a different orthonormal family of solutions $g_i(x)$, that is,

$$\phi(x) = \sum_i \left(g_i(x) B_i + g_i^*(x) B_i^\dagger \right) , \tag{4.3}$$

such that B_i and $B_i^\dagger$ are the annihilation and creation operators of particles at future null infinity I^+. g_i form a family of positive frequency solutions at future null infinity I^+ with respect to the affine parameter u (outgoing particles), which is the outgoing Eddington-Finkelstein coordinate defined by $u = t - r - 2M \log \left|\frac{r}{2M} - 1\right|$. That is, they behave as $e^{-i\omega u}$. It is important to note that this is not completely accurate, since I^+ is not by itself a Cauchy surface, the event horizon must be taken into account. Therefore, to have a *complete* family of solutions we would need to add the contribution of the operators of particles that cross the event horizon. Nevertheless, this contribution will not play any crucial role in the calculation of particle emission to I^+, so we will ignore it here.

Having defined the two families of solutions, we can write one as a linear combination of the other. [We are ignoring incoming modes at the horizon at late times.] In other words, the early and late time operators are related by a Bogoliubov transformation of the form

$$B_i = \sum_j \left(\alpha_{ij}^* A_j - \beta_{ij}^* A_j^\dagger \right) . \tag{4.4}$$

The vacuum state at I^- is defined by $A_i|0\rangle = 0$. However, due to the dynamics (curvature) of the spacetime, this state will not be observed as a vacuum state to an observer at I^+. In fact, if we calculate the expectation value of the number of particles (for a particular i) at I^+, we will obtain

$$\langle N_i \rangle = \langle 0 | b_i^\dagger b_i | 0 \rangle = \sum_j |\beta_{ij}|^2 \ , \tag{4.5}$$

which is generally different from zero. The calculation of the coefficients β_{ij} depends on the details of the collapse, but its asymptotic expression for late times turns out to be independent of this. As we will see shortly, it only depends on the mass of the black hole.

Consider a positive frequency mode $f_\omega \sim e^{-i\omega v}$ of the field propagating from I^-. If v is below the threshold v_0 indicated in Fig. 4.1, this mode will pass through the collapsing star and reach I^+ as a combination of positive and negative frequency solutions, $f_\omega \sim \alpha_\omega e^{-i\omega u} + \beta_\omega e^{i\omega u}$. This distortion of the modes implies the spontaneous creation of particles, as explained in Section 2.2. But not all modes that leave I^- generate this effect. Those that start with $v > v_0$ will inevitably fall into the horizon. Therefore, we are only interested in the modes that start with $v < v_0$. In addition, there is a cumulative effect that causes more particles to be generated as the initial value of v approaches v_0. Therefore, the coefficient β can be estimated by considering only the modes that start from a value of v close to (and less than) v_0. A detailed computation shows that the expectation value of the number of particles created with frequency ω (see [11] or [40] for more details in the calculation, including grey-body factors) is

$$\langle N_\omega \rangle \approx \frac{1}{e^{8\pi M\omega} - 1} \, . \tag{4.6}$$

Note that this expression is equivalent to the Planck distribution for bosons that describes the thermal radiation of a black body. This is given by $1/(e^{\hbar\omega/(k_B T)} - 1)$, where k_B is the Boltzmann constant. Therefore, comparing both expressions, one can identify a temperature for a Schwarzschild black hole

$$T_H = \frac{\hbar}{8\pi k_B M} , \tag{4.7}$$

which is known as the Hawking temperature. This is one of the most prominent results in QFT in curved spacetimes. Although the analysis given here was restricted to a scalar field, this effect holds for any quantum field.

4.2 Exotic Compact Objects (ECOs)

In the last years some researchers have given strong indications supporting the idea that the dark, compact, massive objects that are observed in astrophysics may not be necessarily black holes, in the sense that they may not possess an actual event horizon. The advent of gravitational-wave astronomy has triggered a lot of interest in this direction. Different phenomenological models propose the existence of ultra-compact dark massive objects, named as Exotic Compact Objects (ECOs), that can mimic the physics of black holes in observations. However, they all require physics beyond the standard model. The possibility offered by gravitational wave interferometers to shed light on this matter has sparked special interest in recent years in studying possible types of ECOs, as well as in ways to distinguish them from classical black holes through several mechanisms [39, 87]. An interesting question is whether quantum fluctuations of fields may be capable of preventing the formation of black holes in situations where, from a classical point of view, collapse would be inevitable, leaving as a result the formation of an ECO.

The problem that arises when trying to construct such objects is that General Relativity establishes a limit on the compactness of self-gravitating objects, meaning that if this limit is exceeded, the object inevitably collapses into a black hole. This limit is given by the well-known Buchdahl theorem [85]: *Let there be a static, spherically symmetric star composed of a perfect, isotropic fluid with total mass M and radius R. Assuming that its radial* *pressure is positive and its energy density is positive and decreasing with radius, then the object can only be stable if* $\frac{R}{2M} \geq \frac{9}{8}$. This theorem can be proved just using the Tolman-Oppenheimer-Volkoff (TOV) equations, which are the Einstein's equations for a static and spherically symmetric configuration. For a long time, this theorem has maintained the idea that stable stars with similar compactness to a black hole cannot exist, but in recent years this idea has been called into question. Various groups have proposed scenarios in which the assumptions of the theorem are not met, opening up the possibility of the existence of ECOs. In [39] one can find a compendium of the types of ECOs that can be constructed depending on the assumptions of the theorem that are relaxed.

The presence of quantum fields can imply the violation of some of the assumptions of the theorem. To give an example, quantum fields can generate anisotropies in the stress-energy tensor, and/or negative pressures and densities. Thus, a negative pressure may counteract the gravitational attraction and generate stable configurations of high compactness. The idea that quantum effects may play a crucial role in the formation of astrophysical objects dates back to Chandrasekhar in 1931 [88], who showed that the quantum degeneracy pressure of fermions due to the Pauli exclusion principle can counterbalance the gravitational force and prevent collapse into a black hole, forming as a result a White Dwarf. Three years later, W. Baade and F. Zwicky proposed the existence of what is known as neutron stars [89], which are based on the Chandrasekhar's idea. Unfortunately such stars do not reach sufficiently high values of compactness so as to be able to mimic black holes in gravitational-wave observations, and in fact they can be identified with other techniques in astrophysics. Other types of fermion star configurations based on the same idea were subsequently proposed in [90, 87]. Most of these models consider a polytropic equation of state for the fluid instead of a pure perfect fluid, which violates the assumptions of the Buchdahl theorem. However they also have not found stable solutions with compactness similar to that of black holes.

In parallel, various groups have worked on extending this idea to the case of bosonic fields, leading to what is known as boson stars (see for instance [90, 91, 92, 93, 94]). Depending on how the interaction between the bosons is defined, the maximum mass and compactness of these stars varies. Additionally, the stress-energy tensor of these objects presents anisotropies, which bypasses the assumptions of Buchdahl's theorem. However, the compactness of this type of star is around $R/(2M) \approx 1.4$, still not surpassing the Buchdahl's limit [95].

The type of ultra-compact stars that allows for the highest compactness, even violating Buchdahl's theorem, are anisotropic stars, which are configurations of matter that exhibit high anisotropies in their pressures [96]. There are many types of anisotropic stars, depending on the origin of these anisotropies (see for instance [97, 98, 99, 100, 101, 102]). Some of these models reach compactness very close to $r = 2M$. Many other ECO models have been proposed that accept compactness similar to that of black holes, most of which are based on quantum effects. This is the case for Gravastars (based on one-loop QFT in curved spacetimes [103]), Fuzzballs (based on String Theory [104, 105, 106, 107, 108]), or Firewalls (black holes surrounded by some hard structure made of quantum matter that behaves as a compact horizonless object [109, 110, 111]).

Wormholes

Ultra-compact stars are not the only type of horizonless objects that can mimic the behavior of black holes. Wormholes are an interesting candidate in this regard. They can be defined as astrophysical objects that connect two regions of spacetime, so that matter could pass through the wormhole and go from one zone to another. This idea was firstly introduced by A. Einstein and N. Rosen [112], who constructed a spacetime formed by coupling two Schwarzschild exteriors, which is known as Einstein-Rosen bridge. Several decades later, wormholes regained interest as possible exotic objects that mimic the physics of black holes [113, 114]. There are different ways to construct wormholes, but exotic matter is usually required. For instance, the Einstein-Rosen bridge can be obtained as the solution of Einstein's equations for a thin shell of matter located at the throat of the wormhole [113].

Another, more complex way of constructing a wormhole is by means of some type of matter distribution that generates a decoupling between the components of the metric, so that, the tt-component does not vanish in the region where the rr-component diverges (as it happens in black holes). This implies that this region is not a horizon, but the throat of a wormhole. The simplest example would be a spacetime given by [115]

$$ds^2 = \left(1 - \frac{2M}{r} + \lambda^2\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \,, \tag{4.8}$$

where λ is a constant that is usually taken to be very small. These types of objects can be extremely compact, but it is not yet clear what process could lead to their formation, nor is it clear whether the required matter configuration can be stable. In the Article 5 of this Thesis (shown in part III) we obtain a wormhole of this kind by considering the quantum vacuum as the matter source. We will explain in the following subsection how to address this type of problems.

It is important to remark that exotic matter is not the only way to construct wormholes. There are several works in different Modified Gravity theories that have recently proposed vacuum solutions which give rise to wormhole-like spacetimes [116, 117, 118, 119, 120].

Possibility of ECOs via Semiclassical Gravity

We have seen several proposals, in different theoretical frameworks, to construct ECOs or wormholes that mimic black hole physics. QFT in curved spacetimes is no exception. Several works framed in this theory have proposed the possibility that semiclassical effects can prevent the formation of black holes. Likewise, these effects could facilitate the formation of ultra-compact stars known as dark stars [121, 48, 122], as well as wormhole-type spacetimes [41, 42]. This is the idea that we explore in this Thesis.

To study the quantum effects in the context of semiclassical gravity, it is necessary to analyze the expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$. This calculation is considerably complex due to the renormalization process (we will see it in detail in the following section). The methods applied to calculate this observable usually assume a fixed background metric. This is a good approximation in general, but it has its limitations and does not give a complete picture of the problem. To thoroughly study the quantum effects in black holes and obtain the quantum corrections to the metric, the complete semiclassical Einstein equations should be solved

$$G_{\mu\nu} = 8\pi (T_{\mu\nu}^{\text{classical}} + \langle T_{\mu\nu} \rangle) \,. \tag{4.9}$$

That is, it is necessary to include the backreaction effects generated by quantum fields on the metric itself. To solve these equations, the stress-energy tensor is required as a function of a general metric, which unfortunately we do not have. The main problem in obtaining a general expression lies in the complexity of the renormalization process. As we will see in the next section, there are methods to obtain the renormalized stress-energy tensor for given a metric [123, 33], but they are highly complicated to implement in terms of a general metric, since they require very involved numerical calculations.

However, there is a particular case in which the semiclassical equations can be solved exactly and analytically. This consists in freezing the angular degrees of freedom and working in an effective two-dimensional spacetime (the $t-r$ plane). Several groups have studied the semiclassical Einstein's equations with backreaction in this 2-dimensional context, and have explored the possibility of constructing ECOs or wormholes from semiclassical effects [40, 41, 42, 43, 44, 45, 46, 47]. In the Article 5 of this Thesis (shown in part III), we propose a novel way to solve the equations directly in the *4-dimensional* spacetime. The main idea is to treat the components of the quantum stress-energy tensor as variables, to be determined by the differential equations. By considering one mild assumption on the quantum state, and using the trace anomaly (which we will introduce in detail in the section 6), we manage to solve the full problem in closed form. In particular, we obtain the quantum corrections to the Schwarzschild metric generated by the quantum vacuum, and we study which role these corrections play in the construction of ECOs or wormholes.

Chapter 5

Renormalization in Curved Spacetimes

To explore quantum effects that arise in the extreme universe, it is helpful to calculate relevant physical observables, such as the expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$. However, the direct evaluation of the expectation value $\langle T_{\mu\nu} \rangle$ by expanding the quantum field in modes leads to integrals that present ultraviolet (UV) divergences. These divergences are not physical and, as usual in QFT, renormalization is required to obtain physical results.

The usual renormalization techniques in QFT employed in Minkowski space do not work in curved spacetimes, because the presence of curvature reveals new UV divergences that are absent in flat space. As a result, an important field of research started in the 70's to develop local, covariant renormalization techniques that manage to get physically sensible results, and that reduce to the usual expressions in the flat limit. The widely accepted method to renormalize expectation values of quadratic operators is based on the covariant point-splitting regularization technique [124, 125, 126, 127], which we explain in detail in the following section. The applicability of this method to specific cases is not straightforward, so different variants have been developed (depending on the type of spacetime background). In this chapter, we will introduce some of them, in particular the *adiabatic renormalization* method [5, 6], useful in the context of cosmology and electrodynamics, as well as methods for the case of black holes and stellar configurations [128, 123, 33].

The different problems addressed in this Thesis have required one or another renormalization method. Moreover, in some articles we have focused on studying and extending these methods. This is the case in Article 3 of this Thesis (shown in part III), where we extend the adiabatic renormalization method to the case of spin 1/2 fields with an electric background (which had only been studied in two dimensions). Furthermore, in Article 4 we analyze a recent renormalization method of increasing interest in the context of black holes and extend it to the case of an expanding universe.

5.1 Point-Splitting regularization

In [125] B.S. DeWitt outlined a method, known as geodesic point separation or point-splitting regularization, to regularize the divergences in the vacuum expectation values of the stress-energy tensor in a manifestly covariant way. The method is supplemented with the DeWitt-Schwinger proper time algorithm [14, 124]. The overall procedure resulted in the development of a successful regularization and renormalization framework to deal with divergences in quantum field theory in curved spacetimes [126, 127].

To give an overview of the point-splitting method, consider the vacuum expectation value of a quadratic observable, for example $\langle \phi(x)^2 \rangle$, where ϕ is a scalar field. As commented above, this quantity is ill-defined. We replace then one of the $\phi(x)$ by $\phi(x')$, with x' a point in a normal neighborhood of

x, and define the *two-point function* $G^{(1)}(x, x') = \langle\{\phi(x), \phi(x')\}\rangle$, which is also known as the Hadamard elementary function. This is a well-defined bi-distribution in the spacetime. The UV divergence arises in the limit when the two points merge $x' \to x$, in which $G^{(1)}(x, x)$ fails to be well-defined even in a distributional sense. Thus, this splitting of points allows us to regularize the UV divergences. The potentially divergent part that has to be subtracted is known as the DeWitt-Schwinger counter-term, which we will denote as $G^{(1)}_{DS}(x, x')$. This term completely captures the singular structure of the two-point function, maintaining the covariance of the full observable. To obtain the renormalized observable we subtract this bi-distribution and take the limit $x' \to x$,

$$\left\langle\phi^2(x)\right\rangle_{\text{ren}} = \lim_{x'\to x}\left[G^{(1)}\left(x, x'\right) - G^{(1)}_{DS}\left(x, x'\right)\right] . \tag{5.1}$$

Instead of $G^{(1)}$, it is generally more useful to work with the Feynman Green's function, defined by

$$G\left(x, x'\right) = \bar{G}\left(x, x'\right) - \frac{1}{2} i G^{(1)}\left(x, x'\right) , \tag{5.2}$$

where $\bar{G}(x, x')$ is the principal-value function (one-half the sum of the advanced and retarded Green's functions). Either $G(x, x')$ or $G^{(1)}(x, x')$ are calculated by expanding the quantum field in field modes, which have to be obtained by solving the corresponding equations of motion. The only remaining question is then getting an expression for $G^{(1)}_{DS}(x, x')$.

The method for calculating $G^{(1)}_{DS}(x, x')$ must be completely covariant. DeWitt proposed to extend the Schwinger's proper-time technique to curved spaces. The Feynman Green's function for a scalar field (with mass m and coupling constant ξ) in a curved spacetime $g_{\mu\nu}$ satisfies the equation

$$\left(\Box + m^2 + \xi R\right) G\left(x, x'\right) = g^{-1/2}(x)\delta\left(x - x'\right) , \tag{5.3}$$

where R is the scalar curvature and $g = |\det(g_{\mu\nu})|$. The solution to this equation admits the following asymptotic expansion (see [126] for more details of the calculation)

$$G(x,x') \sim \frac{\Delta^{1/2}(x,x')}{(4\pi)^2} \int_0^\infty \frac{ds}{(is)^2} e^{-i(m^2 s + \frac{\sigma}{2s})} \sum_{n=0}^{\infty} a_n(x,x')(is)^n \,, \tag{5.4}$$

where σ is the geodesic distance squared associated with the geodesic connecting x and x' (which is unique if x' is in a normal neighborhood of x [129]), and Δ is the Van Vleck-Morette determinant defined by $\Delta(x,x') = -|g(x)|^{-1/2}\det\left[-\partial_\mu\partial_{\nu'}\sigma(x,x')\right]|g(x')|^{-1/2}$. The functions $a_n(x,x')$ are obtained through the recurrence relations

$$(\partial^\mu\sigma)(\partial_\mu a_{n+1}) + (n+1)a_{n+1} = \Delta^{-1/2}\Box\left(\Delta^{1/2}a_n\right) - \xi R a_n \,, \tag{5.5}$$

starting with $a_0 = 1$. The integration can be solved in terms of second-order Hankel functions, and expanding these in a power series of σ yields the following expression for $G^{(1)}$

$$\begin{aligned} G^{(1)}\left(x,x'\right) \sim \frac{\Delta^{1/2}}{8\pi^2}\Bigg[a_0\left(-\frac{1}{\sigma} + m^2\left(\gamma + \frac{1}{2}\log\left(\frac{m^2|\sigma|}{2}\right)\right) - \frac{m^2}{2}\right) \\ -a_1\left(\gamma + \frac{1}{2}\log\left(\frac{m^2|\sigma|}{2}\right)\right)\Bigg] + \cdots \,, \end{aligned} \tag{5.6}$$

where γ is the Euler constant. The terms shown in (5.6) capture the full singular structure of the two-point function, since these divergences arise when taking the limit $x' \to x$, i.e., $\sigma \to 0$.

To proceed further, it is convenient to expand the bi-scalars (dependent on x and x') as functions of x and the tangent vector $\partial^\mu\sigma$. By definition σ can be expressed as $\sigma = \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma)$. Likewise, $\Delta\left(x,x'\right)$ is expanded as [126]

$$\Delta^{1/2} = 1 + \frac{1}{12}R_{\mu\nu}\partial^\mu\sigma\partial^\nu\sigma + \cdots \,. \tag{5.7}$$

With these expressions and using the recurrent relation (5.5), the expansions of a_n can be obtained. In particular we have that the leading order of a_1 (the only one necessary to renormalize the two-point function) is

$$a_1(x,x') = \left(\frac{1}{6} - \xi\right) R + \cdots \,. \tag{5.8}$$

Using these results, the covariant expansion of $G^{(1)}$ up to order $\mathcal{O}(\sigma^0)$ can finally be obtained. This is what is identified as the DeWitt-Schwinger counter-term for the two point function, $G^{(1)}_{DS}$, and it is given by

$$G^{(1)}_{DS}(x,x') = \frac{1}{8\pi^2}\left[-\frac{1}{\sigma} + \left(m^2 + \left(\xi - \frac{1}{6}\right) R\right)\left(\gamma + \frac{1}{2}\log\left(\frac{m^2|\sigma|}{2}\right)\right) - \frac{m^2}{2} + \frac{1}{12} R_{\mu\nu}\frac{\partial^\mu\sigma\partial^\nu\sigma}{\sigma}\right]. \tag{5.9}$$

This expression capture all the divergences of the two-point function. To obtain the renormalized expression of $\langle\phi(x)^2\rangle$ in a given spacetime, the procedure is to select a point-splitting direction, calculate $\sigma(x,x')$, evaluate the previous expression, and then compute the subtraction and limit described in (5.1).

The procedure to follow for any other observable is the same. For instance, $\langle T_{\mu\nu}\rangle$ can be expressed in terms of $G^{(1)}$ and its second order derivatives, as [5]

$$\begin{aligned}\langle T_{\mu\nu}\rangle = \lim_{x'\to x}\Bigg[&\left(\frac{1}{2} - \xi\right)\left(\nabla_\mu\nabla_{\nu'}G^{(1)} + \nabla_{\mu'}\nabla_\nu G^{(1)}\right) + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu}\nabla_\alpha\nabla^{\alpha'}G^{(1)} \\ &-\xi\left(\nabla_\mu\nabla_\nu G^{(1)} + \nabla_{\mu'}\nabla_{\nu'}G^{(1)}\right) + \xi g_{\mu\nu}\left(\nabla_\alpha\nabla^\alpha G^{(1)} + \nabla_{\alpha'}\nabla^{\alpha'}G^{(1)}\right) \\ &+\left(-\xi R_{\mu\nu} + \frac{1}{2}(\xi R + m^2) g_{\mu\nu}\right) G^{(1)}\Bigg],\end{aligned} \tag{5.10}$$

where the primes refer to derivatives respect to x'. Applying the procedure explained above to each component of $\langle T_{\mu\nu}\rangle$, one can obtain the renormalized expression for the stress tensor (see [126] for more details).

The point-splitting renormalization method is widely accepted today, but its practical implementation faces some challenges in many cases due to the complexity of its expressions. The main disadvantage is that, since the solution for the field modes is only available numerically for most spacetimes, it is not possible to take the limit $x' \to x$ directly. As a result, other methods equivalent to point-splitting renormalization were developed, valid under special circumstances, but otherwise easy to implement. In the following subsections, we briefly summarize the most accepted renormalization methods in the fields of cosmology and black holes.

5.2 Adiabatic renormalization

In the field of cosmology, the method of *adiabatic renormalization* is well-known and has produced fruitful results. It was introduced by L. Parker and S. Fulling in 1974 [20, 21] to renormalize the stress-energy tensor of scalar fields in an expanding universe. It was further analyzed in [130, 131, 132, 133], and extended to the case of spin 1/2 fields [134, 135, 136, 137, 138, 139]. Moreover, it was applied to study inflationary cosmology [140, 141], to analyze preferred vacuum states in cosmology [142, 143, 144, 145] and to obtain running coupling constants [31, 146].

This renormalization method is equivalent to the DeWitt point-splitting procedure [27, 28] (and also to the Hadamard's renormalization method [147]), but much more manageable, since it writes the subtraction terms as integrals over momenta so that the subtraction can be performed under the integral of modes, obtaining finite integrals that can be calculated numerically. The method works in homogeneous spacetime backgrounds, as for instance in FLRW metrics. But this is not the only case where the adiabatic method has shown its usefulness. Recent works have shown that it can also be applied to a homogeneous, time dependent electromagnetic background [25, 26, 148, 149]. Despite this, there is still considerable work to do in this field. In particular, for Dirac fields coupled to external, classical electric fields the adiabatic method was only known in 2 dimensions due to the difficulty of finding a suitable ansatz in 4 dimensions. In the Article 3 of this Thesis (shown in part III) we solved this problem.

The adiabatic renormalization method is based on the adiabatic expansion of the field modes [5]. This is an asymptotic expansion of the solutions of the field equations in which the n-th adiabatic order in the expansion involves n time derivatives of the metric. To be more precise, let us consider a scalar field propagating on a FLRW metric $ds^2 = dt^2 - a^2(t)d\vec{x}^2$. By using an appropriate ansatz for the field modes in the Klein-Gordon equation (based on the WKB approximation), one can solve the differential equation for the modes order by order in the number of derivatives of $a(t)$. Thus, the zero adiabatic order will only involve the scale factor $a(t)$, the first order includes the first derivative $\dot{a}(t)$, the second order includes $\ddot{a}(t)$ and $\dot{a}(t)^2$, and so on. Since higher order terms in the adiabatic expansion involve more and more derivatives, the leading order terms recover the usual modes in Minkowski spacetime (in which the scale factor is strictly constant). The idea is that higher order adiabatic terms capture, or sense, the dependence of the field modes on the curvature of the spacetime.

Given the vacuum expectation value of a quadratic operator expanded in field modes, we can use the adiabatic expansion of the modes to produce an adiabatic expansion of this observable. It turns out that the leading order terms in the expansion contain all the UV divergences. We can then do the renormalization by subtracting (inside the formal integrals or sums of modes) a truncated expansion to a given adiabatic order. The specific order of truncation depends on the observable to work.

For example, consider the stress-energy tensor of a given field in a FLRW metric. Because the background is homogeneous the field modes admit a

Fourier expansion in momenta $\vec{k}$. The stress-energy tensor can be formally expressed in the form

$$\langle T_{\mu\nu}(x)\rangle = \int d^3k \; T_{\mu\nu}(\vec{k}, x)\,, \tag{5.11}$$

and the renormalized expression takes the form

$$\langle T_{\mu\nu}(x)\rangle_{ren} = \int d^3k \left[T_{\mu\nu}(\vec{k}, x) - T^N_{\mu\nu}(\vec{k}, x)\right]\,, \tag{5.12}$$

where $T^N_{\mu\nu}$ is the adiabatic expansion of $T_{\mu\nu}(\vec{k}, x)$ truncated at order N, which is chosen to be the minimum order required to cancel the divergences of the integral. It is crucial to note that even if the last adiabatic order that we subtract contains both divergent and convergent terms, we must subtract all of them. This condition is a requirement to ensure the covariance of the method. In particular, the subtractions constructed this way preserve the covariant conservation law, $\nabla^\mu T^N_{\mu\nu} = 0$, thereby ensuring the conservation of the renormalized stress-energy tensor.

There exists also a specific rule based on dimensional grounds to identify the adiabatic orders that need to be subtracted. The highest adiabatic order that can potentially contain divergences corresponds to the scaling dimension of the observable in question. For instance, the dimensions of the stress-energy tensor in a 4-dimensional spacetime are k^4, and therefore, the term $T_{\mu\nu}(\vec{k}, x)$ in (5.11) has dimensions of k^1. Consequently, in the general case, the 0^{th} adiabatic order of $T_{\mu\nu}(\vec{k}, x)$ behaves (at large k) as k^1, the first one as k^0, the second one as k^{-1}, and so on. To ensure convergence of the integral, it is necessary to subtract up to the 4^{th} adiabatic order in such a way that all the possible terms $k^{n\leq 3}$ are cancelled. This coincidence between the scaling dimension of the observable and the adiabatic orders that must be substracted can be used as a general rule for any observable.

It is important to apply this rule consistently in all cases, even when some of the adiabatic orders are convergent. Sometimes, specific parameter selections of the theory may result in the convergence of some terms, which would otherwise diverge. For instance, in the scalar theory, choosing the minimal coupling $\xi = 0$ causes the fourth adiabatic order of the stress-energy tensor to converge, whereas for a generic value of ξ, it diverges. However, the renormalization method cannot depend on the value of the parameters, and hence, the rule must be applied uniformly for all values of ξ.

As mentioned above, the adiabatic renormalization method is not restricted to cosmological scenarios and can be applied to cases involving an electric homogeneous background as well. In such cases, it is crucial to take into account an important consideration. Recent studies have demonstrated that when applying the adiabatic regularization method simultaneously to gravitational and electric fields, it is necessary to consider the potential vector as first adiabatic order, in order to ensure the local conservation of energy and the consistency with the trace and chiral anomalies [25, 26]. For instance, consider the case of a FLRW spacetime and a potential 4-vector of the form $A_\mu = (0, 0, 0, -A(t))$. While the scale factor $a(t)$ would be considered as zeroth adiabatic order, $A(t)$ must be treated as first order.

Scalar field in an expanding universe

It is illustrative to show this method applied to a specific case. Let us consider a real scalar field in a FLRW background. As shown in section 2.2, the equation for the field modes is given by

$$\frac{d^2 h_{\vec{k}}}{dt^2} + \left[\omega^2 + \left(6\xi - \frac{3}{4}\right)\frac{\dot{a}^2}{a^2} + \left(6\xi - \frac{3}{2}\right)\frac{\ddot{a}}{a}\right] h_{\vec{k}} = 0 \,, \tag{5.13}$$

where $\omega = \sqrt{m^2 + \left(\frac{k}{a}\right)^2}$, with $k = |\vec{k}|$. The modes must also satisfy the normalization condition

$$h_{\vec{k}} \dot{h}^*_{\vec{k}} - h^*_{\vec{k}} \dot{h}_{\vec{k}} = 2i \,. \tag{5.14}$$

To address this problem it is convenient to consider a Wentzel-Kramers-Brillouin (WKB) ansatz for the modes of the field

$$h_k = \frac{1}{\sqrt{W_k(t)}} e^{-i \int^t W_k(t')dt'} \,, \tag{5.15}$$

where $W_k(t)$ is a real function that can be expanded in adiabatic orders as $W_k(t) = \omega_k^{(0)} + \omega_k^{(1)} + \omega_k^{(2)} + ...$ Inserting this ansatz into the differential equation for the field, one obtains an algebraic equation for each order. Solving the equation order by order, we find

$$\begin{aligned}
\omega_k^{(0)} &= \omega \,, & (5.16)\\
\omega_k^{(1)} &= 0 \,, & (5.17)\\
\omega_k^{(2)} &= -\frac{3\dot{a}^2}{8a^2\omega} - \frac{3\ddot{a}}{4a\omega} - \frac{3k^2\dot{a}^2}{4a^4\omega^3} + \frac{k^2\ddot{a}}{4a^3\omega^3} + \frac{5k^4\dot{a}^2}{8a^6\omega^5} + \frac{3\xi\dot{a}^2}{a^2\omega} + \frac{3\xi\ddot{a}}{a\omega} \,, & (5.18)\\
&\cdots
\end{aligned}$$

It is worth noting that the solution at the zeroth adiabatic order, corresponding to a slow expansion ($\dot{a}(t) \sim 0$, $\ddot{a}(t) \sim 0$, ...), coincides with the solution in flat (Minkowski) spacetime, confirming that the ansatz in (5.15) is appropriate. It is also convenient to expand adiabatically the term W_k^{-1}

$$\begin{aligned}
\left(W_k^{-1}\right)^{(0)} &= \omega_k^{-1} \,, & (5.19)\\
\left(W_k^{-1}\right)^{(1)} &= 0 \,, & (5.20)\\
\left(W_k^{-1}\right)^{(2)} &= \frac{m^2\dot{a}^2}{2a^2\omega^5} + \frac{m^2\ddot{a}}{4a\omega^5} - \frac{5m^4\dot{a}^2}{8a^2\omega^7} - (\xi - \frac{1}{6})\left(\frac{3\dot{a}^2}{a^2\omega^3} + \frac{3\ddot{a}}{a\omega^3}\right) & (5.21)\\
&\cdots
\end{aligned}$$

We can use this expansion to renormalize, for example, the vacuum polarization, which is given by

$$\langle 0| \phi(x)^2 |0\rangle = \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 |h_{\vec{k}}|^2 = \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 \frac{1}{W_k} \,. \tag{5.22}$$

Since this observable has dimensions of k^2, one must subtract up to the second adiabatic order. The expression for the renormalized vacuum polarization is then

$$\left\langle 0 \left| \phi(x)^2 \right| 0 \right\rangle_{ren} = \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 \left[|h_k|^2 - \frac{1}{\omega} - \left(W_k^{-1} \right)^{(2)} \right] , \qquad (5.23)$$

where $W_k(t)^{-1}$ is given by (5.21). By integrating the finite terms and using the expression of the scalar curvature for a FLRW metric, $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$, we can rewrite (5.23) as

$$\left\langle 0 \left| \phi(x)^2 \right| 0 \right\rangle_{ren} = \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 \left[|h_k|^2 - \frac{1}{\omega} - \frac{\left(\frac{1}{6} - \xi\right) R}{2\omega^3} \right] - \frac{R}{288\pi^2} . \qquad (5.24)$$

This integral is convergent, and can be computed numerically for a given function $a(t)$.

In a similar manner, one can also apply the adiabatic renormalization method to obtain the vacuum expectation value of the stress-energy tensor. The procedure is the same, the only difference is that the subtractions must include up to fourth adiabatic order, since the stress-energy tensor has dimensions of k^4.

5.3 Renormalization in Black Holes

For black holes, Candelas was the first to implement the point-splitting method in a Schwarzschild metric [35]. Let us see how it would be applied, for example, to renormalize $\langle \phi^2 \rangle$ for a massless scalar field. The spherical symmetry and static nature of the spacetime allows us to express the formal two-point function as

$$\left\langle \phi(x)^2 \right\rangle = \int_0^\infty d\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left| Y_{lm}(\theta, \varphi) \right|^2 \left| \bar{\psi}_{\omega l}(r) \right|^2 , \qquad (5.25)$$

where $Y_{lm}(\theta, \varphi)$ are the spherical harmonics and $\bar{\psi}_{\omega l}$ is the field mode of frequency ω and angular momentum l. This integral is divergent and must be renormalized. Following the point-splitting method, the quadratic operator must be evaluated at two different points. Since it is a static spacetime, it is convenient (for simplicity of the calculation) to choose two points located in the same place and time-like separated, i.e., $x = (t, r, \theta, \phi)$ and $x' = (t + \epsilon, r, \theta, \phi)$. For this case, the two-point function is simply

$$\langle \{\phi(x), \phi(x')\} \rangle = \int_0^\infty d\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \cos(\omega\epsilon) \left|Y_{lm}(\theta, \varphi)\right|^2 \left|\bar{\psi}_{\omega l}(r)\right|^2 . \quad (5.26)$$

On the other hand, the DeWitt counter-term (5.9) reduces to $G^{(1)}_{DS} = \frac{1}{8\pi^2\sigma}$ for a massless field and Schwarszchild spacetime (where $R_{ab} = 0$), and the geodesic distance squared σ between these points is given by

$$\frac{1}{\sigma} = \frac{2}{(1 - 2M/r)\epsilon^2} + \frac{M^2}{6r^4(1 - 2M/r)} + O(\epsilon) . \quad (5.27)$$

As we can see the divergence is captured by the term $1/\epsilon^2$, but in order to obtain a manifestly finite quantity after subtracting this contribution in (5.26), it is convenient to find first an integral representation. To do this, Candelas proposed the Laplace transform

$$\int_0^\infty d\omega\omega \cos(\omega\epsilon) = -\frac{1}{\epsilon^2} . \quad (5.28)$$

Therefore, from (5.1), the renormalized expression of the two-point function can be written as

$$\langle \phi(x)^2 \rangle_{ren} = \lim_{\epsilon\to 0} \int_0^\infty d\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \cos(\omega\epsilon) \left[\left|Y_{lm}(\theta, \varphi)\right|^2 \left|\bar{\psi}_{\omega l}(r)\right|^2 - \frac{\omega}{4\pi^2(1 - 2M/r)} \right] - \frac{M^2}{48\pi^2 r^4(1 - 2M/r)} . \quad (5.29)$$

Similarly, Candelas obtained a renormalized expression for the stress-energy tensor.

These integrals require numerical calculations because the field modes $\bar{\psi}_{\omega l}(r)$ cannot be solved analytically. Still, the numerical computation of these integrals for any $r > 2M$ is not easy at all. [In particular, Candelas did not evaluate the integrals for all r, but focused only on the limits $r \to 2M$ and $r \to \infty$, for which he could infer analytical expressions using some tricks.] As a consequence, Candelas and Howard proposed an alternative strategy to obtain an analytical estimation of the renormalized two-point function (valid only for the Hartle-Hawking vacuum) [150, 128]. The method requires some knowledge from the field modes, in particular, they proposed a WKB expansion. By expanding the modes for large ω and l, one can construct an analytical approximation for $\langle\{\phi(x), \phi(x')\}\rangle$, which we will denote as $\langle\{\phi(x), \phi(x')\}\rangle_{WKB}$, that captures the full singular structure of the two-point function. Next, we add and subtract this expression from $\langle\phi^2(x)\rangle_{ren}$, obtaining

$$\begin{aligned} \left\langle\phi^2(x)\right\rangle_{ren} = & \lim_{\epsilon\to 0}\left[\langle\{\phi(x), \phi(x')\}\rangle - \langle\{\phi(x), \phi(x')\}\rangle_{WKB}\right] \\ & + \lim_{\epsilon\to 0}\left[\langle\{\phi(x), \phi(x')\}\rangle_{WKB} - G^{(1)}_{DS}(x, x')\right] . \end{aligned} \quad (5.30)$$

Thus, the expression is divided into a part that must be analyzed numerically (the first limit) and a completely analytical part (the second limit). They also showed that the analytical part gives much larger values, so it can be used as a first approximation for $\langle\phi^2\rangle_{ren}$. To be more precise, for the Hartle-Hawking vacuum they obtained

$$\langle\phi^2(x)\rangle_{ren} = \frac{1}{12(8\pi M)^2}\frac{1-(2M/r)^4}{1-2M/r} + \text{numerical part}\,, \quad (5.31)$$

which coincides with a previous result obtained with a different approach [151]. Howard extended the method to the case of the stress-energy tensor

[152], and later Anderson et. al. extended it to the case of a general static and spherically symmetric spacetime [153].

Despite these advances, the calculation of the numerical contribution above continues to be difficult. In view of this, Candelas and Howard proposed to go higher order in the WKB expansion of the field modes. This analysis is however a daunting task and requires several approximations. In particular, the main problem was the presence of a turning point. To overcome this difficulty, they proposed the idea of analytically continue the background metric to the Euclidean space through a Wick rotation ($t \to -i\tau$). In this Euclidean space there are no turning points and the WKB expansion works well.

The extension to the Euclidean space can be applied to any static metric, and in particular Anderson et al. applied it to solve the numerical part in a general static and spherically symmetric background [123]. However, time-dependent spacetimes generally do not admit an extension to the Euclidean space, so this method cannot be extended, for example, to analyze the evaporation process of black holes. Additionally, the WKB expansion is considerably complicated if the modes depend on two variables (t, r), which greatly hinders the resolution of the numerical part. In this context, a new and more general renormalization method has been recently developed by A. Levi and A. Ori [32, 33, 34]. It was called *pragmatic mode-sum method of regularization*. This method addresses the numerical problem from scratch, and it does not require any approximation nor analytical continuation to the Euclidean space. It only demands that the spacetime presents a symmetry so as to take the splitting of points in the corresponding spacetime direction. Roughly speaking, the method takes up the path originally proposed by Candelas [35], and solves the integral (5.29) using accurate numerical techniques. Therefore, for the Schwarzschild black hole this method can be understood as a completion of what Candelas initially started. Subsequently, this numerical method has been applied to obtain renormalized observables in different types of black hole frameworks, thus showing the power of the new technique [154, 155, 156, 157].

In the Article 4 of this Thesis (shown in part III) we provide a brief summary of this method, and extend it to a scenario where it had not yet been applied: cosmology. We show how the translational symmetry of the FLRW metric allows the implementation of these ideas, and we will prove that the final expressions for the renormalized observables are equivalent to those provided by the well-known adiabatic renormalization method explained in the previous section.

Chapter 6

Anomalies in QFT and Gravitation

In this section we introduce a concept that will be present in all the articles that comprise this Thesis: quantum anomalies. As is well known symmetries play a fundamental role in Physics. In particular, Noether's theorem establishes that they are in one-to-one correspondence with conservation laws. However, since the late 60's it is known that some symmetries of classical fields are broken when the fields are quantized. When this happens, we say that there is an anomaly. A quick way to understand this is by using the path integral framework. In classical field theory, one says that a certain transformation of a field ϕ is a symmetry if the transformation leaves the action $S[\phi]$ invariant. But in QFT what must remain invariant is the full quantum effective action, given by the path integral $\int D[\phi]e^{iS[\phi]}$. The measure $D[\phi]$ may not be necessarily invariant under the above transformation, and therefore the symmetry can be broken in the quantum theory. For a detailed study on anomalies in QFT see for instance [158] or [76].

There are several types of anomalies. On the one hand, there are anomalies that break global symmetries (global anomalies). This is the case of the well-known axial anomaly in QED, that we will explain in the next section. The Standard Model contains anomalies of this type, which leads for instance to the anomalous non-conservation of baryon number (which is important in studies of the asymmetry between matter and antimatter observed in the universe [159]). On the other hand, there are anomalies that break gauge symmetries (gauge anomalies). Many important theories in physics are gauge theories, that is, their Lagrangians are invariant under certain local transformations. The presence of a gauge anomaly in one of these theories indicates that the theory is inconsistent. For example, if we consider the theory of a single charged and massless fermion (Weyl fermion or chiral fermion) we obtain a gauge anomaly, since the gauge current is not conserved when the field is quantized. To make the theory consistent, a charged Weyl fermion of opposite chirality must be added so that the gauge symmetry is preserved [76].

Another type of anomalies are known as gravitational anomalies, which were discovered in 1984 by L. Álvarez-Gaumé and E. Witten [16]. As the name suggests, they appear in the context of gravity, and consist of a violation of the principle of general covariance. These anomalies can also be understood as gauge anomalies, since General Relativity can be understood as a gauge theory, where transformations from one coordinate system to another (diffeomorphisms) would be the gauge transformations. The principle of general covariance requires that physical laws must be invariant under these types of transformations. When quantum fields (in particular chiral fields) are coupled to the gravitational field, these anomalies arise, implying the non-conservation of the stress-energy tensor ($\langle \nabla_\mu T^{\mu\nu} \rangle \neq 0$). Fortunately, this does not occur in the 4-dimensional case, but rather it occurs in spacetimes of dimension $2, 6, 10...$ In the Article 1 of this Thesis (shown in part III) we point out the existence of an anomaly of this type in flat spacetimes. In particular, in the case of a Weyl fermion coupled to a time-varying homogeneous electric field in two dimensions.

Finally, there is another type of anomalies, the ones that break the scale invariance, which are known as trace anomalies (or conformal anomalies, or Weyl anomalies) [160]. [See [29] for a historical review.] In Minkowski spacetime the anomalous trace of the stress-energy tensor in massless theories is related to the well-known beta functions and the renormalization group flow. For instance, there is an anomaly of this kind in massless quantum electrodynamics, which has scale invariance in the classical theory, but it is broken when the theory is quantized. Trace anomalies also appear in the context of free field theories in curved spacetimes. In certain cases, such as the electromagnetic field or the massless Dirac field coupled to gravitational backgrounds, the action is invariant under conformal transformations of the metric. However, this symmetry is lost when quantizing the matter field, generating a trace anomaly (see, for instance, [5] for more details). At the end of this chapter we will further explore this topic.

Next, we will explain briefly the two anomalies that are most relevant to understand the articles of this Thesis: the axial anomaly in QED and the trace anomaly in QFT in curved spacetimes.

6.1 Axial anomaly in QED

Studying the decay $\pi^0 \to \gamma\gamma$, in 1949 J. Steinberger [161], and independently H. Fukuda and Y. Miyamoto [162], found an inconsistency between the theoretical prediction for the decay rate and the experimental results. The confusion persisted for some years and this question was known as the pion decay puzzle. It was not until 1969 when J. Bell and R. Jackiv [163], and independently S. L. Adler [164], solved the problem of the pion decay. By noting that the axial symmetry in QED fails in the quantum theory, they obtained theoretical predictions that matched perfectly with the experiments. This was the first discovery of an anomaly in quantum field theory.

The QED action for a Dirac field ψ of mass m and charge q, and an electromagnetic field given by the potential 4-vector A^μ, reads

$$\mathcal{S} = \int dx^4 \left[\bar{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] , \tag{6.1}$$

where $D_\mu \equiv \partial_\mu - iqA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and γ^μ are the Dirac matrices. One can easily see that, in the massless case ($m = 0$), this action is invariant under the transformation $\psi \to e^{-i\epsilon\gamma^5}\psi$ (chiral transformation), where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. The Noether's current associated with this transformation (axial current) is given by $j^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$. It can be verified that, for solutions of the Dirac equation,

$$\partial_\mu j^\mu_A = 2im\bar{\psi}\gamma^5\psi \, . \tag{6.2}$$

Therefore, in the case $m = 0$ one obtains $\partial_\mu j^\mu_A = 0$, that is, the axial current is conserved in the massless case.

But this is no longer true when quantizing the Dirac field. The formal vacuum expectation value of the divergence of the axial current would be

$$\langle \partial_\mu j^\mu_A \rangle = 2im\langle \bar{\psi}\gamma^5\psi \rangle \, . \tag{6.3}$$

When trying to compute the right-hand side one finds that it diverges. To obtain the physical result, a renormalization method must be applied. There are different ways to perform this calculation with different renormalization methods. For example, in [76] a derivation of the axial anomaly can be found using the proper-time Schwinger method of regularization. Similarly, the adiabatic regularization method explained in 5.2, which can be applied in the case of electromagnetic backgrounds, is also a useful tool for obtaining this anomaly. In [26], the anomaly is recovered using the adiabatic method in two dimensions. In the Article 3 of this Thesis (shown in part III), the adiabatic method is extended to 4 dimensions and could be easily applied to obtain the axial anomaly.

After renormalization $\langle\bar{\psi}\gamma^5\psi\rangle$ contains residual poles in the mass going like $1/m$. As a consequenece, the massless limit in (6.3) produces $\langle\partial_\mu j^\mu_A\rangle_{ren} \neq 0$. In particular, one obtains

$$\left\langle\partial_\mu j^\mu_A\right\rangle_{ren} = -\frac{q^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,, \tag{6.4}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol. This is the well-known expression of the axial anomaly in QED. In the Article 2 of this Thesis (shown in part III) we explain the relationship between this anomaly and the phenomenon of spontaneous particle creation by an electric field (Schwinger effect), as well as its relation with the breaking of adiabatic invariance that occurs in this context.

6.2 Trace Anomaly in QFT in curved spacetimes

As we have previously mentioned, trace anomalies (or conformal anomalies) occur when the scale symmetry (or conformal invariance) of a theory is broken upon quantization. This type of anomaly was discovered by D. Capper and M. J. Duff in 1974 [160]. This discovery was of great relevance in the field of Gravitation, and in the following years it found multiple applications, as for example in cosmology [165, 166, 167, 168], supersymmetry [169, 170], or string theory [171, 172].

Let us see what this anomaly consists of in the context of QFT in curved spacetimes. A field theory is said to be conformally invariant if it remains invariant under a conformal transformation, i.e., under a rescaling of the metric $g_{\mu\nu}$ of the form

$$\tilde{g}_{\mu\nu}(x) \quad = \Omega^2(x)g_{\mu\nu}(x), \tag{6.5}$$

where $\Omega(x)$ is an arbitrary function. For example, both the electromagnetic and massless Dirac fields are conformally invariant. Likewise, the scalar field can also be conformally invariant, but only under certain conditions, as we will see shortly. It is possible to prove that, as a consequence of this symmetry, particle creation does not occur for conformally invariant fields propagating in conformally flat spacetimes (as for instance in FLRW metrics). In particular, the expansion of the universe does not produce particles associated to conformal fields, like massless neutrinos or photons [5].

What is the conserved current associated to this symmetry? Consider the action S of a scalar field ϕ in a background $g_{\mu\nu}$. If it is invariant under conformal transformations, then

$$0 = \delta S = \int d^n x \left\{ \frac{\delta S}{\delta \phi} \delta_0 \phi + \frac{\delta S}{\delta g_{\mu\nu}} \delta_0 g_{\mu\nu} \right\} , \tag{6.6}$$

where $\delta_0 \phi$ and $\delta_0 g_{\mu\nu}$ are the infinitesimal variations associated to the conformal transformation. The first term vanishes if the field satisfies the Euler-Lagrange equation, which is equivalent to $\delta S / \delta \phi = 0$. On the other hand, $\delta_0 g_{\mu\nu}$ can be obtained by considering an infinitesimal variation of the metric, i.e., $\Omega^2(x) = 1 + \delta\Omega^2(x)$, where $|\delta\Omega^2(x)| << 1$. Therefore, $\delta_0 g_{\mu\nu} = g_{\mu\nu} \delta\Omega^2$. We finally obtain

$$\int d^n x \frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu} \delta\Omega^2 = 0 \, . \tag{6.7}$$

For this to hold for any $\delta\Omega^2$, we must impose

$$\frac{\delta S}{\delta g_{\mu\nu}(x)} g_{\mu\nu}(x) = 0 \, . \tag{6.8}$$

On the other hand, recall that the stress-energy tensor of the theory is given by (2.3), i.e.

$$T^{\mu\nu} \equiv -2|g|^{-1/2} \frac{\delta S}{\delta g_{\mu\nu}(x)} \, . \tag{6.9}$$

Combining the last two equations, it is straightforward to get

$$T^{\mu}{}_{\mu} = 0\,. \tag{6.10}$$

In summary, in the classical theory the trace of the stress-energy tensor of a conformally invariant theory vanishes for solutions of the field equations. But as we will see later, upon quantization this is no longer true.

As we advanced before, there are some conditions upon which a scalar field theory can be conformally invariant. Let us consider a scalar field with mass m and coupling constant ξ. Its Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}|g|^{1/2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right)\,. \tag{6.11}$$

Under a conformal transformation as in (6.5), we have $\tilde{g}^{\mu\nu} = \Omega^{-2}g^{\mu\nu}$ and $\tilde{g}^{1/2} = \Omega^4 g^{1/2}$. Therefore, from the first term one can see that for the Lagrangian to be invariant, the field must transform as $\tilde{\phi} = \Omega^{-1}\phi$. On the other hand, under a conformal transformation, the scalar curvature transforms as

$$\tilde{R} = \Omega^{-2}(R - 6\Omega^{-1}\Box\Omega)\,, \tag{6.12}$$

where $\Box\Omega = g^{-1/2}\partial_\mu(g^{1/2}\partial^\mu\Omega)$. If we now consider the case $m = 0$ and $\xi = 1/6$ (which, recall from section 2.1, is known as conformal coupling), we deduce that the Lagrangian transforms as (see [5] for more details in the calculation)

$$\tilde{\mathcal{L}} = \mathcal{L} - \partial_\mu\left(\frac{1}{2}|g|^{1/2}\Omega^{-1}(\partial^\mu\Omega)\phi^2\right)\,. \tag{6.13}$$

The Lagrangian is therefore conserved except for a total derivative that cancels in the integration of the action S. Only for the case $m = 0$ and. $\xi = 1/6$ can the Lagrangian be expressed in this way. Therefore, massless, conformally coupled ($\xi = 1/6$) scalar fields are conformally invariant. In fact, for solutions of the Klein-Gordon equation the trace of the stress-energy tensor of a scalar field is

$$T^{\mu}{}_{\mu} = (6\xi - 1)\left((\partial_\mu\phi)(\partial^\mu\phi) - \xi R\right) + 2(1-3\xi)m^2\phi^2\,, \qquad (6.14)$$

which vanishes in the case $m = 0$ and $\xi = 1/6$, as expected from the conformal symmetry.

But what happens if we quantize the field? As we mentioned several times earlier in this Thesis, quantization breaks conformal symmetry, giving rise to a non-zero trace of the stress-energy tensor, known as the trace anomaly. There are different ways to obtain the expression for this anomaly, for example, using the DeWitt-Schwinger renormalization method [126] or via the one-loop effective action [173, 174]. For a FLRW spacetime there is a much simpler way to derive it using the adiabatic renormalization method introduced in section 5.2, which is particularly illustrative. In fact, this is the method we have used in some articles of this Thesis to calculate anomalies. Therefore, it is convenient to detail this calculation here as an introduction, as it will be useful for understanding analogous calculations in the articles of the Thesis.

Let us take $\xi = 1/6$. The vacuum expectation value of the trace of the stress-energy tensor (6.14) reads

$$\langle T^{\mu}{}_{\mu}\rangle = m^2\left\langle \phi^2\right\rangle\,. \qquad (6.15)$$

This expression is divergent and must be renormalized. In section 5.2, we already renormalized the vacuum polarization using the adiabatic method, but the procedure now is not as simple as imposing $\langle T^{\mu}{}_{\mu}\rangle_{ren} = m^2\left\langle\phi^2\right\rangle_{ren}$. Recall that to renormalize the stress-energy tensor consistently in such a way that covariance is respected, one must subtract an adiabatic expansion truncated up to fourth order (in contrast to the vacuum polarization, where it is enough to truncate at second adiabatic order). Therefore, following the adiabatic procedure, we must calculate the adiabatic expansion of $\langle\phi^2\rangle$ up to fourth order, and then we arrive to

$$\langle T^{\mu}{}_{\mu}\rangle_{ren} = m^2 \left\langle \phi^2 \right\rangle_{ren} - \frac{m^2}{4\pi^2 a^3} \int_0^\infty dk k^2 \left(W_k^{-1}\right)^{(4)} , \tag{6.16}$$

where $\left\langle \phi^2 \right\rangle_{ren}$ is obtained in section 5.2 (Eq. (5.24)) and $\left(W_k^{-1}\right)^{(4)}$ is the fourth order in the adiabatic expansion of W_k^{-1} (defined in (5.15)).

The conformal symmetry occurs classically in the massless case, so let us see what happens when taking the limit $m \to 0$. The expression for $\langle \phi^2 \rangle_{ren}$ does not contain any subtraction term of the form $1/m^n$, and there is no reason to expect that the contribution from the modes becomes divergent in the massless limit, so we can assert that in the limit $m \to 0$ we have $m^2 \langle \phi^2 \rangle_{ren} \to 0$. But the same is not true for the fourth-order adiabatic subtraction. Following the adiabatic expansion procedure one can find its expression, which is

$$\begin{aligned}\left(W_k^{-1}\right)^{(4)} &= -\frac{m^2 \dot{a}^4}{2a^4\omega^7} - \frac{7m^2 \ddot{a}^2}{16a^2\omega^7} - \frac{m^2 \ddddot{a}}{16a\omega^7} - \frac{33m^2\dot{a}^2\ddot{a}}{16a^3\omega^7} - \frac{11m^2 \dot{a}\dddot{a}}{16a^2\omega^7} + \frac{49m^4\dot{a}^4}{8a^4\omega^9} \\ &+ \frac{21m^4\ddot{a}^2}{32a^2\omega^9} + \frac{35m^4\dot{a}^2\ddot{a}}{4a^2\omega^9} + \frac{7m^4\dot{a}\dddot{a}}{8a^2\omega^9} - \frac{231m^6\dot{a}^4}{16a^4\omega^{11}} - \frac{231m^6\dot{a}^2\ddot{a}}{32a^3\omega^{11}} + \frac{1155m^8\dot{a}^4}{128a^4\omega^{13}} . \end{aligned} \tag{6.17}$$

By integrating this expression in (6.16) and making a change of variable of the form $k \to k/(ma)$, one can infer that $\int_0^\infty dk k^2 \left(W_k^{-1}\right)^{(4)}$ is proportional to m^{-2}. Therefore, the massless limit $m \to 0$ of (6.16) does not vanish. Moreover, this integral is convergent and can be evaluated analytically, resulting in

$$\langle T^{\mu}{}_{\mu}\rangle_{ren} = \frac{1}{480\pi^2}\left(\frac{\ddot{a}^2}{a^2} + \frac{\ddddot{a}}{a} - 3\frac{\dot{a}^2\ddot{a}}{a^3} + 3\frac{\dot{a}\dddot{a}}{a^2}\right) . \tag{6.18}$$

Overall, we have shown that upon quantization of the scalar field, the trace is no longer zero, even in the case $m = 0$ and $\xi = 1/6$. Quantization breaks conformal symmetry.

As a last remark, it is convenient to express (6.18) in terms of covariant geometric scalars. In a FLRW universe, we have the following identities

$$R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} = 12\left(\frac{\dot{a}^4}{a^4}+\frac{\ddot{a}^2}{a^2}\right), \tag{6.19}$$

$$R^{\mu\nu}R_{\mu\nu} = 12\left(\frac{\dot{a}^4}{a^4}+\frac{\ddot{a}^2}{a^2}+\frac{\dot{a}^2\ddot{a}}{a^3}\right), \tag{6.20}$$

$$\Box R = 6\left(\frac{\ddot{a}^2}{a^2}+\frac{\ddddot{a}}{a}-\frac{5\dot{a}^2\ddot{a}}{a^3}+\frac{3\dot{a}\,\dddot{a}}{a^2}\right). \tag{6.21}$$

With a bit of algebra, we can finally express (6.18) as

$$\langle T^{\mu}{}_{\mu}\rangle_{ren} = \frac{1}{2880\pi^2}\left(-R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}+R^{\mu\nu}R_{\mu\nu}+\Box R\right), \tag{6.22}$$

in agreement with the general expression for the trace anomaly for a scalar field in a general curved spacetime [5].

The conformal anomaly also appears in other quantum fields, as mentioned earlier. The result is similar, except for changes in the numerical coefficients. The general expression for the trace anomaly for any quantum field has the form

$$\langle T^{\mu}{}_{\mu}\rangle = \frac{1}{2880\pi^2}(aC^{\mu\nu\sigma\rho}C_{\mu\nu\sigma\rho}+bR^{\mu\nu}R_{\mu\nu}+cR^2+d\Box R), \tag{6.23}$$

where $C^{\mu\nu\sigma\rho}$ is the Weyl tensor, which can be related to the other scalars via

$$C_{\mu\nu\sigma\rho}C^{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}-2R_{\mu\nu}R^{\mu\nu}+\frac{1}{3}R^2. \tag{6.24}$$

And a, b, c and d are constants that depend on the specific field. For instance, for conformal scalar fields they are $a = b = -1$ and $c = -1/3$; for massless Dirac fermions $a = -7/4, b = -11/2$ and $c = 11/6$; and for photons $a = 13, b = -62$ and $c = 62/3$ [6]. The coefficient d can take different values depending on the chosen renormalization method. This is because it is subject to an ambiguity related to the choice of the renormalization scheme, and then it can be chosen arbitrarily by adding a local counter-term in the Lagrangian [175]. In the Article 5 of this Thesis (shown in part III) we use this anomaly as an effective equation of state that allows us to solve the semiclassical TOV equations.

It is worth noting that there exists an analogous trace anomaly for quantum fields coupled to electromagnetic backgrounds [29]. For instance, in the case of a massless Dirac field, this is given by

$$\langle T^{\mu}{}_{\mu}\rangle_{ren} = \frac{q^2}{24\pi^2} F_{\mu\nu}F^{\mu\nu} \,. \tag{6.25}$$

In the Article 3 of this Thesis (see part III) we perform a test of the proposed method by computing this anomaly and verifying that we obtain (6.25).

Part II

Results and Conclusions

Chapter 7

Results and Conclusions

7.1 Article 1: Translational anomaly in electric backgrounds

Quantum anomalies are the failure of classical symmetries to survive quantization, as explained in detail in chapter 6. In addition to the well-known chiral anomalies, which entail the non-conservation of the axial current, gravitational anomalies imply the non-conservation of the expected value of the stress-energy tensor, i.e. $\langle \nabla_\mu T^{\mu\nu} \rangle \neq 0$. Gravitational anomalies are somewhat similar to gauge anomalies, signaling the inconsistency of the theory. They arise in theories with Weyl (or chiral) fermions coupled to gravity for spacetimes of dimension $2, 6, 10, ...$ In particular, a chiral field in two spacetime dimensions displays the following gravitational anomaly

$$\langle \nabla_\mu T^\mu_\nu \rangle = \frac{1}{96\pi\sqrt{-g}} \epsilon^{\alpha\beta} \partial_\beta \partial_\rho \Gamma^\rho_{\nu\alpha} \, . \tag{7.1}$$

In the Article 1 of the Thesis (shown in section III), we show that a gravitational-type anomaly can also appear in flat space, provided the Weyl fermion is coupled to an electric background. To be more concrete, we consider a Weyl field in two dimensions coupled to a homogeneous and time-dependent electric field $E(t)$. From a classical point of view, this system is invariant under translations in the spatial direction, which implies the conservation of momentum, i.e. $\partial_\mu T^{\mu 1} = 0$. However, upon quantization of the Weyl field, this symmetry is broken. To obtain this anomaly, we need to find the renormalized expression of the stress-energy tensor. To do this, we have applied the *adiabatic renormalization* method [5], which we explain in detail in section 5.2. With the help of this method, we arrive at the following result

$$\partial_\mu \left\langle T^{\mu 1}_{R,L} \right\rangle = \mp \frac{q^2 A \dot{A}}{2\pi} \,, \tag{7.2}$$

where R and L indicate the chirality (right-handed or left-handed) of the fermion under consideration, and $A(t)$ is the potential vector, defined by $E(t) = -\dot{A}(t)$. This result was unknown in the literature. As this anomaly breaks translational symmetry, we have named it *translational anomaly*. The appearance of this type of anomaly is an indication that the theory is incomplete. In fact, this physical system also exhibits a gauge anomaly (the electric current is not conserved). We show that both anomalies cancel out when a Weyl fermion of the opposite chirality is added. For a massless Dirac fermion ($\Psi = \Psi_R + \Psi_L$), the stress-energy tensor is the sum of the two chiral components, so we obtain:

$$\partial_\mu \left(\left\langle T^{\mu 1}_R \right\rangle_{\text{ren}} + \left\langle T^{\mu 1}_L \right\rangle_{\text{ren}} \right) = 0 \,. \tag{7.3}$$

On top of this, in the mentioned article we also show the relationship of this anomaly with the phenomenon of spontaneous particle creation by intense electric fields (which we explain in Chapter 3). For a Weyl field the created particles all move in the same direction, generating a total amount of momentum that coincides with the result of the anomaly. Instead, if we consider a Dirac field, we see that what is created are particle-antiparticle pairs. Particles and antiparticles travel in opposite directions, maintaining the conservation of the total momentum.

Finally, we also solve the full semiclassical Maxwell equation of the system to explore the backreaction effects generated by the created particles on the electric field. We verify that the amount of linear moment created by each chiral sector oscillates with the same frequency as $E(t)$. Likewise, we see that the sum of the oscillations of the two chiral sectors cancels perfectly.

7.2 Article 2: Breaking of adiabatic invariance in electromagnetic backgrounds

The phenomenon of particle creation in an expanding universe (which we explain in Section 2.2) has a fundamental property. In the limit of an infinitely slow expansion of the universe (adiabatic limit), no particle creation occurs. More precisely, the density of created particles tends to zero at each instant of time in the limit in which the Hubble rate approaches zero, even if the net change in the scale factor is large. This is why it is said that the particle number is an adiabatic invariant. This property was of major relevance in the pioneer papers on cosmological particle creation (for an historical review see [17]). In the Article 2 of this Thesis (shown in Part III), we study whether this property holds for the case of an electromagnetic background. We first analyze the 2-dimensional case for its simplicity and then extend it to the 4-dimensional case. We also study both the case of a charged scalar field coupled to the electromagnetic field (scalar QED) and the case of a Dirac field (QED).

Let us consider a homogeneous, time-dependent electric field acting in the spatial direction, $E(t)$. Its associated 2-vector potential is $A_\mu = (0, -A(t))$, where $E(t) = \dot{A}(t)$. The vector potential plays a similar role as the scale factor $a(t)$ in cosmology, so it is convenient to consider a similar adiabatic expansion for $A(t)$ which may allow us to define the number of particles in $t \to \pm\infty$. In order to study the problem analytically, we have considered a specific form for the electric field that is well known: a Sauter electric pulse [18]. In this case, the potential vector reads

$$A(t) = \frac{1}{2} A_0 (\tanh(\rho t) + 1)\,, \tag{7.4}$$

where A_0 and ρ are real-valued constants. It is not difficult to see that the potential tends to 0 in the limit $t \to -\infty$, and to A_0 as $t \to \infty$. The parameter ρ sets the rate at which the potential grows, so it can be considered as the adiabaticity parameter. The adiabatic limit (extremely slow growth) is given by $\rho \to 0$. The goal in the article is to examine whether in this limit the number of particles tends to 0 or not.

In the article we obtain that for bosons (b) and for fermions (f) the expected number of particles created by the external electric field reads

$$\langle N_{b/f} \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \frac{\cosh\left(2\pi \frac{\omega_-}{\rho}\right) \pm \cosh\left(2\pi \frac{\kappa_{b/f}}{\rho}\right)}{2 \sinh\left(\pi \frac{\omega_{\text{in}}}{\rho}\right) \sinh\left(\pi \frac{\omega_{\text{out}}}{\rho}\right)}, \tag{7.5}$$

where $\omega_{\text{in}} = \sqrt{k^2 + m^2}$, $\omega_{\text{out}} = \sqrt{(k - qA_0)^2 + m^2}$, $\omega_\pm = \frac{1}{2}(\omega_{\text{out}} \pm \omega_{\text{in}})$, $\kappa_b = \frac{1}{2}\sqrt{(qA_0)^2 - \rho^2}$, and $\kappa_f = qA_0/2$.

By studying the limit $\rho \to 0$ in these expressions we draw the following conclusions. For the massive case ($m \neq 0$), $\langle N_{b/f} \rangle \to 0$ in the adiabatic limit. That is, for an infinitely slow growth of $A(t)$ no massive bosons or fermions are created, thus maintaining the adiabatic invariance of the number of particles. However, the situation is different in the massless case ($m = 0$). Namely, we obtain that $\langle N_{b/f} \rangle \neq 0$ as $\rho \to 0$, or to be more precise,

$$\langle N_{b/f} \rangle = \frac{|qA_0|}{\pi}. \tag{7.6}$$

Therefore, we conclude that for a potential vector growing infinitely slowly, massless particles are indeed created, and adiabatic invariance is broken. We have also found that the moment spectrum of these massless particles created lies in the interval $k \in [-|qA_0|, |qA_0|]$. It should be noted that there is a clear difference between bosons and fermions that can be extracted from equation (7.5). Massless bosons tend to accumulate at $k = 0$ and $k = \pm qA_0$, while massless fermions are created in the same proportion for all k. This can be interpreted in terms of Pauli's exclusion principle, which does not allow fermions to accumulate in the same state. Moreover, and unlike the scalar case, the number of created massless fermions (as well as their spectrum in momenta) does not depend on the parameter ρ, i.e., it does not depend on the history of $A(t)$ but only on its initial and final values.

This is a remarkable outcome. In order to give consistency to this result, we have also calculated the expected value of the electric current and energy density of the quantum field using the adiabatic renormalization method (explained in section 5.2). Just like the number of particles, we observe that these quantities tend to 0 in the adiabatic limit, except in the case of massless particles. This residual energy and current correspond to the energy generated by the massless particles created. Furthermore, the simplicity of the theory of massless fermions allows getting an analytical expression of the renormalized electric current as a function of time, which we find as $\langle j^x \rangle_{\text{ren}} = -\frac{q^2 A(t)}{\pi}$. The semiclassical Maxwell equation then reads $\ddot{A} + \frac{q^2}{\pi} A = 0$. This harmonic oscillator equation takes into account the backreaction effects of the created particles on the electric field. The electric field that solves this semiclassical equation oscillates with a frequency of $|q|/\sqrt{\pi}$, as does the number of particles. It can be easily seen that the energy associated with the electric field and the energy of the created particles cancel out for all t, maintaining energy conservation. As a final remark, the value obtained for the frequency is consistent with the well-known fact that radiative corrections to the Schwinger model induce a photon mass of value $m_\gamma^2 = q^2/\pi$ [19].

We have also performed the above calculation in the 4-dimensional case, taking the electric field $\vec{E}(t)$ in the z direction for convenience. In this case we find $\langle N_{b/f} \rangle \to 0$ in the adiabatic limit, independently of m. Thus, the adiabatic invariance for an electric background does hold in 4 dimensions. However, the situation changes if we add a magnetic field. Let us consider, for simplicity, a constant magnetic field $\vec{B}$ acting in the direction parallel to $\vec{E}(t)$. The presence of the magnetic field generates a discretization of the momentum in the direction perpendicular to the fields, known as Landau levels, which drastically changes the picture. We find that, while for bosons of any mass the adiabatic invariance is respected, for massless fermions in presence of both electric and magnetic fields the adiabatic invariance is lost. This result holds for other directions of $\vec{B}$, except when it is perpendicular to the electric field. In that case, adiabatic invariance is preserved.

Overall, we have shown that the adiabatic invariance of the particle number is maintained for electromagnetic backgrounds except for some specific cases. These cases are: massless bosons and fermions in 2 dimensions, and massless fermions in 4 dimensions in the presence of electric and magnetic fields not perpendicular to each other. This indicates that there is a relationship between the phenomenon of breaking adiabatic invariance and the well-known axial anomaly [14], since it is present precisely in the mentioned cases. This is the anomaly associated to the classical axial symmetry of massless Dirac fields, resulting from the quantization of the theory (in section 6.1 we explain this anomaly in more detail). In 2 dimensions, the axial anomaly is given by the expression

$$\langle \partial_\mu j_5^\mu \rangle_{\text{ren}} = -\frac{q}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}, \tag{7.7}$$

which, for homogeneous fields, is equivalent to the statement that the chiral charge density j_A^0 is not conserved. In 2 dimensions this charge is proportional to the electric current. In the article, we compare the expression for the anomaly with the current generated by massless particles created in the adiabatic limit, and we verify that the creation of chiral charge indeed agrees with the source of the axial anomaly. In fact, this idea can be easily visualized in the 2-dimensional case since the chirality is related to the direction of motion, and changes the criterion between particles and antiparticles. Thus, for example, a massless particle moving to the right would have right-handed chirality, and a massless antiparticle moving to the left would also have right-handed chirality. Since the electric field creates particle-antiparticle pairs with opposite electric charge and momentum, it implies a net creation of chiral charge. This non-conservation of chiral charge is consistent with the axial anomaly. This anomaly persists regardless of the speed at which the background field changes, even in the adiabatic limit, and therefore, in that limit, there must always be a remnant creation of massless pairs.

Regarding the 4-dimensional case, the anomaly only arises for massless fermions and is given by

$$\left\langle \partial_\mu j_A^\mu \right\rangle_{ren} = -\frac{q^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \,. \tag{7.8}$$

For a time-dependent electric field and a constant magnetic field, this expression reduces to

$$\left\langle j_5^0 \right\rangle_{\rm ren} = -\frac{q^2}{2\pi^2} \int_{-\infty}^{t} dt' \vec{E}\left(t'\right) \vec{B} \,. \tag{7.9}$$

We can see that the chiral charge is only created when the fields $\vec{E}$ and $\vec{B}$ are not perpendicular, which occurs, precisely, when the adiabatic invariance is broken. Furthermore, we have also verified that the creation of chiral charge of massless fermions in the adiabatic limit coincides with the expression for the anomaly.

In conclusion, we infer that the breaking of adiabatic invariance occurs in those cases where axial anomaly emerges, meaning that these two phenomena are closely related. In other words, the breaking of adiabatic invariance is a necessary condition for the existence of the axial anomaly.

7.3 Article 3: Adiabatic renormalization method to Dirac fields in an electric background

The adiabatic renormalization method was introduced by L. Parker and S.A. Fulling to renormalize the stress-energy tensor in cosmological backgrounds [20, 21]. In Section 5.2, we reviewed this method for the case of a scalar field in an expanding universe. Although this method is usually applied in cosmology, it can also be used for a classical electric field background [22, 23, 24]. However, recent studies have shown that there is a drawback in these works [25, 26]. The vector potential A^μ is considered in [22, 23, 24] to be of adiabatic order 0, in analogy to the scale factor $a(t)$ in the cosmological case. This is consistent in the case of having (only) an electric background, but if we also add a gravitational field, the renormalized expressions that one obtains are inconsistent with the covariant conservation of the stress-energy tensor, as well as with the axial and trace anomalies. As shown in [25, 26], to recover the overall consistency of the method it is necessary to impose that A^μ is of adiabatic order 1 (the first derivative would be of order 2, the second of order 3...). Likewise, a new reformulation of the method is proposed with this assumption for the case of charged scalar fields and for Dirac fields in 2 dimensions. The extension from 2 to 4 dimensions for Dirac fields (with the new assumption) turns out to be non-trivial and requires a thorough analysis. This is the aim of Article 3 of this Thesis (shown in Section III).

The first result we obtain in Article 3 is a new argument that justifies the choice of A^μ as a quantity of adiabatic order 1. It is known that the adiabatic renormalization method for a gravitational background is consistent/equivalent with the well-known DeWitt-Schwinger point-splitting method [27, 28] (which we briefly explain in section 5.1). In the article we show that, in the presence of electric and gravitational backgrounds, this consistency is only maintained under the assumption that A^μ is of adiabatic order 1. Namely, we prove that, for both scalar and Dirac fields in 2 dimensions, the adiabatic expansion of the vacuum polarization $\langle\phi^2\rangle$ agrees exactly with the DeWitt-Schwinger expansion if we consider this assignment (we explicitly verify this up to adiabatic order 6).

Armed with those results in 2 dimensions we then face the main aim of Article 3: the extension of the adiabatic method for Dirac fields in 4 dimensions in presence of an electric background. The main problem that arises in this case is that the usual WKB ansatz for the field modes is not consistent. For this reason we propose a new ansatz, which we show to be fully consistent and allows us to proceed with the adiabatic regularization method.

Let us briefly see what our method consists of. We consider a Dirac field ψ in 4 dimensions, with mass m and charge q, coupled to an electric background with potential vector of the form $A_\mu = (0, 0, 0, -A(t))$. The Dirac equation for this system is given by

$$(i\gamma^\mu D_\mu - m)\,\psi = 0\,, \tag{7.10}$$

where $D_\mu \equiv \partial_\mu - iqA_\mu$ and γ^μ are the Dirac matrices. In order to construct the ansatz, it is necessary to apply a unitary transformation to the field of the form $\psi' = U\psi$, where

$$U = \frac{1}{\sqrt{2}}\gamma^0\left(I - \gamma^3\right)\,. \tag{7.11}$$

This has allowed us to express the Dirac field in terms of only two functions dependent on time, $h^I_{\vec{k}}(t)$ and $h^{II}_{\vec{k}}(t)$, which can be regarded as the field modes with momentum $\vec{k} = (k_1, k_2, k_3)$. It should be emphasized that the idea of applying this transformation has been crucial and it is important to point this out. We finally obtain that the Dirac equation is reduced to the following differential equations for the field modes

$$\dot{h}^I_{\vec{k}} - i\,(k_3 + qA)\,h^I_{\vec{k}} - i\kappa h^{II}_{\vec{k}} = 0\,, \tag{7.12}$$

$$\dot{h}^{II}_{\vec{k}} + i\,(k_3 + qA)\,h^{II}_{\vec{k}} - i\kappa h^{I}_{\vec{k}} = 0\,, \tag{7.13}$$

where $\kappa \equiv \sqrt{k_1^2 + k_2^2 + m^2}$. The main advantage of this procedure is that it has allowed us to write the Dirac equation in terms of two differential equations very similar to those of the same problem in 2 dimensions (see [26]). The only difference is that now κ plays the role of m. Finally, using these expressions, the field can be quantized in terms of creation and annihilation operators (see the article for more details). The anticommutation relations of those operators are guaranteed if the normalization condition

$$\left|h^I_{\vec{k}}\right|^2 + \left|h^{II}_{\vec{k}}\right|^2 = 1\ , \tag{7.14}$$

is satisfied.

With all these ingredients, the adiabatic expansion can be proposed. This is where our ansatz comes in. Inspired by the solution in two dimensions, we propose a similar ansatz, with the natural replacement $m \to \kappa$,

$$h^I_{\vec{k}} = \sqrt{\frac{\omega - k_3}{2\omega}}F(t)e^{-i\int^t \Omega(t')dt'}\,, \tag{7.15}$$

$$h^{II}_{\vec{k}} = -\sqrt{\frac{\omega + k_3}{2\omega}}G(t)e^{-i\int^t \Omega(t')dt'}\,, \tag{7.16}$$

where $\omega = \sqrt{k_3^2 + \kappa^2}$, F and G are complex functions and Ω is a real function. These functions are assumed to admit an adiabatic expansion, which is obtained by enforcing the field equations and the normalization condition order by order. In the Article 3 we give a set of recurrence relations that can be solved iteratively to determine the adiabatic expansion to any order. Using this expansion the expectation values of any other observable can also be expanded in adiabatic series. This allows us to identify and subtract the terms that generate UV divergences, that is, to apply adiabatic renormalization. To illustrate the power of this new method, we apply it to calculate the renormalized expectation value of the electric current, defined by $\langle j^\mu \rangle = -q \left\langle \bar{\psi}\gamma^\mu \psi \right\rangle$. For the relevant spacetime component, we obtain the expression

$$\begin{aligned} \left\langle j^3 \right\rangle_{\rm ren} &= \frac{q}{2\pi^2} \int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3 \Bigg[\left(\left| h^{II}_{\vec{k}} \right|^2 - \left| h^{I}_{\vec{k}} \right|^2 \right) - \frac{k_3}{\omega} \\ &- \frac{\kappa^2 q A}{\omega^3} + \frac{3\kappa^2 k_3 q^2 A^2}{2\omega^5} + \frac{\left(\kappa^2 - 4k_3^2\right)\kappa^2 q^3 A^3}{2\omega^7} + \frac{\kappa^2 q \ddot{A}}{4\omega^5} \Bigg] \end{aligned} \tag{7.17}$$

where $k_\perp = \sqrt{k_1^2 + k_2^2}$.

To check the robustness of this new proposed method we verify two non-trivial sanity checks. On the one hand, we calculate the renormalized trace of the stress-energy tensor, which is given by $\langle T^\mu_\mu \rangle = m \langle \bar{\psi}\psi \rangle$, and we verify that in the limit $m \to 0$ we recover the usual trace anomaly. For massless Dirac fields in the presence of an electromagnetic background, this anomaly is given by $\langle T^\mu_\mu \rangle_{\rm ren} = \frac{q^2}{24\pi^2} F_{\mu\nu} F^{\mu\nu}$ [29]. [In section 6.2 we explain this anomaly in detail]. On the other hand, we also verify that, as it occurs in all other cases where the adiabatic method is applied, the adiabatic expansion agrees with the DeWitt-Schwinger expansion, thus proving the equivalence between both methods. Finally, we also verify the equivalence with the Hadamard renormalization method [30].

The usual adiabatic formalism implicitly assumes that the renormalization scale μ equals the mass of the field. In our work we further extend the method for an arbitrary renormalization scale by noticing, as previous works did [31], that in the adiabatic method there is an intrinsic ambiguity in the choice for the zero adiabatic order when solving the recurrence relation. Instead of $\sqrt{\vec{k}^2+m^2}$, it is possible to define $\omega^{(0)} \equiv \omega = \sqrt{\vec{k}^2+\mu^2}$, where μ is an arbitrary mass scale. The method can be developed in a more or less straightforward manner and we eventually get a new expansion of the modes in terms of the mass scale μ. We applied this extension to renormalize again the electric current, obtaining an expression that depends on μ. The ambiguity in μ can be absorbed in the renormalized coupling constants in the effective action, in this case the elecric charge q. Following usual ideas in effective field theories, we obtain the effective charge as a function of the scale: $q^{-2}(\mu) - q^{-2}(\mu_0) = -\left(12\pi^2\right)^{-1} \ln \frac{\mu^2}{\mu_0^2}$. This result agrees with that obtained in perturbative QED using dimensional regularization [19].

Finally, in order to test the practical usefulness of the method, we applied it to a specific electric background. We considered a Sauter-type pulse given by $E(t) = E_0 \cosh^{-2}(t/\tau)$, where E_0 indicates the height of the pulse and τ the width. This also allowed us to study physical properties of the particle creation phenomenon. We numerically calculated the renormalized current as a function of time from the general expression obtained with our method (7.17). In the article, representations of the result can be found for different values of the parameters. We verify that the current tends to become constant in the limit $t \to \infty$, as expected for this background. This limit can be calculated analytically, specifically we obtain

$$\langle j^3 \rangle_{\rm ren} \sim -\frac{q}{\pi^2} \int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3 \frac{k_3 + qA_0}{\omega_{\rm out}} \left|\beta_{\vec{k}}\right|^2 , \tag{7.18}$$

where $\omega_{\rm out} = \sqrt{(k_3+qA_0)^2+\kappa^2}$ and $\left|\beta_{\vec{k}}\right|^2$ is the Bogoliubov coefficient that gives the density of particles created with momentum k at $t \to \infty$. Applying the expression for $\left|\beta_{\vec{k}}\right|^2$ corresponding to the Sauter pulse we obtain the result of the current. In addition, we have used this result to estimate the value of the electric current in the limit of a very intense electric field

($E_0 >> 0$). In this limit, we obtain the expression

$$\left\langle j^3 \right\rangle_{\text{ren}} \sim \frac{2}{3\pi^3} q^3 E_0^2 \tau \,. \tag{7.19}$$

Likewise, we have obtained the expression for the particle density in this same limit, obtaining $\langle N \rangle \sim \frac{2}{3\pi^3} q^2 E_0^2 \tau$.

7.4 Article 4: Pragmatic mode-sum regularization method in a cosmological background

The covariant DeWitt-Schwinger point-splitting renormalization method [124, 125] (which we explain in section 5.1), although fully satisfactory from a theoretical viewpoint, is not easy to implement in practical situations. This is specially problematic in physically important scenarios where the field modes are only available numerically, as for black holes. Recently, A. Levi and A. Ori have proposed a method that has proved to be very efficient for numerically implementing the DeWitt-Schwinger point-splitting procedure in different black hole frameworks, known as the *pragmatic mode-sum* regularization method [32, 33, 34]. It can be applied to spacetimes that possess some type of symmetry (such as static or stationary black holes) and can be understood as a method that completes what was initially proposed by Candelas in the 1980s [35]. [In section 5.3 we review the historical methods proposed for implementing point-splitting in black holes.] In the Article 4 of this Thesis (shown in part III), we review this method and extend it to accommodate spacetimes that have three-dimensional spatial symmetries, like FLRW metrics in cosmology. We show that the pragmatic mode-sum regularization method reduces to the conventional adiabatic regularization method.

In particular, we consider a scalar field coupled to a FLRW metric, $ds^2 = dt^2 - a^2(t)d\vec{x}^2$, and focus on the renormalization of the vacuum polarization $\langle\phi^2\rangle$. Following the point-splitting method, the renormalized vacuum polarization is given by

$$\left\langle \phi^2(x) \right\rangle_{\rm ren} = \lim_{x'\to x} \left[\left\langle \left\{ \phi(x), \phi\left(x'\right) \right\} \right\rangle - G^{(1)}_{\rm DS}\left(x, x'\right) \right] . \tag{7.20}$$

$G^{(1)}_{\rm DS}$ is the DeWitt-Schwinger counter-term for the two-point function (see Eq. (5.9)). Following the initial idea of Candelas, further developed by Levi and Ori, we separate the points based on the symmetry of the system. In this case, the translational symmetry of the spacetime suggests that it is convenient to choose spatially separated points, i.e., $x \equiv (t, \vec{x})$ and $x' \equiv (t, \vec{x} + \vec{\epsilon})$. Thus, the expectation value of the (symmetric) two-point function for these points is given by

$$\left\langle \left\{ \phi(x), \phi\left(x'\right) \right\} \right\rangle = \frac{1}{4\pi^2 a(t)^3} \int_0^\infty dk k^2 \left| h_k(t) \right|^2 \frac{\sin k\epsilon}{k\epsilon} , \tag{7.21}$$

where $\epsilon = |\vec{\epsilon}|$ and $h_k(t)$ are the field modes.

The counter-terms $G^{(1)}_{\rm DS}$ can be expanded in powers of ϵ as

$$\begin{aligned} G^{(1)}_{\rm DS}\left(x, x'\right) &= \frac{1}{4\pi^2} \left[\frac{1}{a^2\epsilon^2} + \frac{1}{2}\left(m^2 + (\xi - 1/6)R\right)\left(\gamma + \log\left(\frac{ma}{2}\epsilon\right)\right) \right. \\ &- \left. \frac{m^2}{4} + \frac{R}{72} \right] + \mathcal{O}(\epsilon) . \end{aligned} \tag{7.22}$$

Applying integral identities of the form $\int_0^\infty dk k \frac{\sin k\epsilon}{k\epsilon} = \frac{1}{\epsilon^2}$, we can express (7.22) as an integral in k and subtract it in (7.21). The divergence at $\epsilon \to 0$ is canceled, so we can take the coincident limit under the integral, obtaining

$$\left\langle \phi^2 \right\rangle_{\rm ren} = \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 \left[|h_k|^2 - \frac{1}{\omega} - \frac{\left(\frac{1}{6} - \xi\right) R}{2\omega^3} \right] - \frac{R}{288\pi^2} . \tag{7.23}$$

This is exactly the same result that one can obtain by using the adiabatic renormalization method (see equation (5.24) in section 5.2). Thus, we conclude that the extension of the Levi-Ori renormalization method for a time-dependent homogeneous background reduces to the adiabatic method. In the article, we also prove that the equivalence between the pragmatic mode-sum method and the adiabatic method also holds when computing the renormalized stress-energy tensor.

Finally, we have extended the pragmatic mode-sum renormalization method in this framework by including an arbitrary mass scale μ. This is necessary for the $m = 0$ case, since the subtraction term (7.22) is not well defined in this case. Following the technique proposed in [36], we apply a change of the form $m^2 \to m^2 + \mu^2$ at a specific point of the point-splitting method, and we eventually arrive at the following expression for the subtraction term (up to order $\mathcal{O}(\epsilon^0)$)

$$G^{(1)}_{\mathrm{DS}}(x, x') = \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 \frac{\sin(k\epsilon)}{k\epsilon} \left[\frac{1}{\omega_{\mathrm{eff}}} + \frac{\left(\frac{1}{6} - \xi\right) R}{2\omega_{\mathrm{eff}}^3} + \frac{\mu^2}{2\omega_{\mathrm{eff}}^3} \right] + \frac{R}{288\pi^2} \tag{7.24}$$

where $\omega_{\mathrm{eff}}^2 = \frac{k^2}{a^2} + m^2 + \mu^2$. Moreover, doing a similar analysis with the scale μ for the second-order subtraction term in adiabatic renormalization, we verify that it agrees with (7.24), manifesting the consistency between the two methods.

7.5 Article 5: Quantum vacuum corrections to the Schwarzschild metric

Recent progress in gravitational wave detections [37] as well as in very long baseline interferometry [38] are opening the door to the possibility of testing the existence of black hole horizons experimentally. In recent years this has sparked a growing interest in the study of exotic compact objects (ECOs) that mimic the physics of black holes, as well as the physical processes that would allow to distinguish them from black holes [39]. [In section 4.2, we provide a brief explanation of the types of ECOs proposed so far.] Classical General Relativity does not allow the existence of such objects due to the Buchdahl theorem, but the introduction of quantum effects may allow for the violation of this theorem, opening the possibility of formation channels of ECOs. There are several ways to construct these objects, and one of them may be through semiclassical effects generated by quantum fields. This is the path we explore in the Article 5 of this Thesis (shown in part III). In particular, we study the backreaction effects produced by the polarization of the quantum vacuum around a static, non-rotating black hole, obtaining quantum corrections to the Schwarzschild metric.

To achieve this, we looked for solutions of the semiclassical Einstein equations without matter

$$G_{ab} = 8\pi \langle T_{ab} \rangle \ . \tag{7.25}$$

The main issue that arises when facing this problem is that in 4 dimensions we do not have a renormalized expression of $\langle T_{ab} \rangle$ in closed analytical form for a general metric. But since the renormalized stress-energy tensor is known in exact form in $1 + 1$ dimensions, the authors of [40, 41, 42] proposed an approximation to solve the semiclassical Einstein's equations by freezing the angular degrees of freedom of the quantized field, and thereby reducing the problem to a 2-dimensional effective spacetime. This was later analyzed in more detail and studied for different cases by other authors [43, 44, 45, 46, 47]. In sharp contrast, in this article we propose an alternative approach to face this problem directly in 4 dimensions. One simplification will consist in restricting to quantum effects generated by conformal fields (more precisely, to a conformal scalar field). It is reasonable to think that the results for other types of fields will be qualitatively similar. For conformal fields, the well-known trace anomaly (which we explain in detail

,in section 6.2) univocally defines a relationship between the components of the stress-energy tensor, given by

$$-\langle \rho \rangle + \langle p_r \rangle + 2\, \langle p_t \rangle = \langle T^a_a \rangle \ , \tag{7.26}$$

where $\langle \rho \rangle$ is the density of the quantum vacuum, $\langle p_r \rangle$ and $\langle p_t \rangle$ are the radial and tangential pressures, and $\langle T^a_a \rangle$ is the expression of the trace anomaly, which depends on the metric. [Since we are looking for static and spherically symmetric solutions, we have also chosen a vacuum state with these symmetries, leading to a diagonal, time-independent renormalized stress-energy tensor.] Thus, our proposal consists of solving the semiclassical Einstein equations by adding (7.26) as an equation of state. Note that, with this procedure, it is not necessary to give an explicit expression for the stress-energy tensor as a functional of the metric (which was the main problem of the conventional approach), since now its components are introduced as unknowns of the system of differential equations.

To make the system fully solvable we need one last assumption, which consists in considering the radial pressure equal to the tangential pressure ($\langle p_r \rangle = \langle p_t \rangle$). This simplification is inspired from the result of the stress-energy tensor in a fixed Schwarzschild background [35]. Near the Schwarzschild horizon the pressures tend to equalize. It is reasonable to expect that the exact solution, including backreaction, behaves similarly ($\langle p_r \rangle \approx \langle p_t \rangle$) near $r = 2M$. In any case, we have subsequently verified that the results for other assumptions regarding the pressures are qualitatively similar.

For static and spherically symmetric solutions, the system of equations to be solved is analogous to the well-known TOV equations (but now with quantum density and pressure), adding the aforementioned equation of state (7.26). As a first approximation, we solve the system perturbatively in $\hbar$. Restricting ourselves to the region near the horizon (where quantum effects are more important), we obtain the following first-order correction in $\hbar$ to the Schwarzschild metric

$$\begin{aligned} ds^2 &= -\left(f(r) - \hbar\left(\frac{1}{13440\pi M^2 f(r)} + \mathcal{O}(\log f(r))\right) + \mathcal{O}\left(\hbar^2\right)\right) dt^2 \\ &+ \frac{dr^2}{f(r) - \hbar\left(\frac{1}{4480\pi M^2 f(r)} + \mathcal{O}(\log f(r))\right) + \mathcal{O}\left(\hbar^2\right)} + r^2 d\Omega^2\,, \end{aligned} \tag{7.27}$$

where $f(r) = 1 - 2M/r$. From this result, we can draw a main conclusion: the classical horizon of the Schwarzschild metric disappears. Indeed, for the value that makes $g_{rr}^{-1}(r) = 0$, which is

$$r_0 = 2M + \frac{\sqrt{\hbar}}{4\sqrt{70\pi}} + \mathcal{O}(\hbar)\,, \tag{7.28}$$

the $g_{tt}(r)$ component does not vanish, i.e. $g_{tt}(r_0) \neq 0$, unlike the classical Scharzschild metric. Thus, a wormhole-type metric is obtained (see section 4.2 for more details on these objects). However, this result may not be entirely reliable since the quantum pressure and density turn out to be of order $\hbar/f^2$, which near the throat of the wormhole ($r = r_0$) tends to be of order $\hbar^0$. Therefore, in the region near the throat, the perturbative assumption fails, and the problem must be studied exactly by numerical methods. After analyzing the exact numerical solution in great detail, we found results qualitatively similar to the perturbative case, except for numerical factors of order one. Specifically, we obtain that the throat is located at $r_0 \approx 2M + 0.01947\sqrt{\hbar}$, which differs slightly from the previous result.

In summary, we have obtained a coordinate singularity for a value of r separated from the classical value ($r = 2M$) by a distance of the order of the Planck length ($\sqrt{\hbar}$). The singularity represents the throat of a wormhole. The next logical step is to extend the metric beyond this coordinate singularity, as is done in the classical case. In the article, we propose a Morris-Thorne-type extension, suitable for a wormhole metric, defined by the change $l(r) \equiv \int_{r_0}^{r} \sqrt{g_{rr}(r')}dr'$. The throat of the wormhole is located at $l = 0$. The extension of the metric to the region $l < 0$ results in an asymmetric wormhole. Additionally, we find a new singularity located at $l_s \sim -0.278\hbar^{1/4}\sqrt{M}$. In the article we prove that this is a curvature singularity, and it is located on a null hypersurface. Figure 7.1 shows a qualitative Penrose diagram of this solution. We also demonstrate that this singularity is located at a geodesic distance of order $\mathcal{O}(\sqrt{\hbar})$ from the throat, so that an observer passing through the wormhole would encounter the singularity almost immediately. The shape of this solution (asymmetric wormhole with a null curvature singularity) agrees qualitatively with the conclusions obtained using the 2-dimensional approximation in [41], which reinforces the validity of this 2-dimensional approximation.

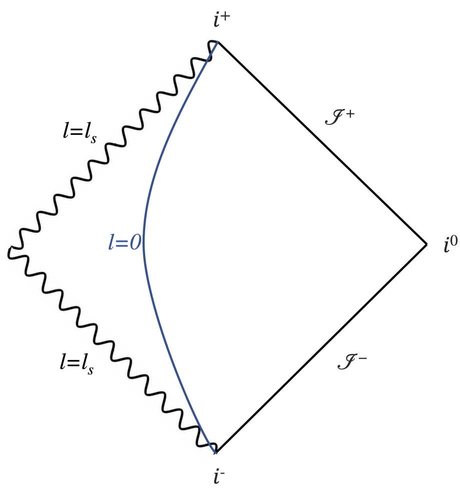

Figure 7.1: Penrose diagram showing the throat of the wormhole ($l = 0$) and the null curvature singularity ($l = l_s$).

This wormhole solution is the maximal extension of a purely (quantum) vacuum solution of the semiclassical Einstein's equations. Alternatively, one may think of matching the semiclassical solution with the metric describing the interior of a static, spherically symmetric star. The inclusion of matter may generate ultra-compact stellar objects [48, 44, 45, 46, 47]. If we match these solutions to our quantum vacuum Schwarzschild metric for the exterior of the star, our result imposes a maximum value for the compactness of these objects, given by the minimum of the radial function (the throat of the wormhole). To be more precise, we get that the maximum compactness (measured as $2M/r$) would be given by

$$\frac{2M}{r_0} \sim 1 - 0.01686 \frac{\sqrt{\hbar}}{2M} \,. \tag{7.29}$$

This is an important constraint for any exotic compact object that may be proposed in the literature.

On the other hand, we find that quantum corrections well away from the classical Schwarschild horizon are very much suppressed to be observed with current interferometers. In particular, as an example, we calculated the quantum correction (at first order in $\hbar$) to the frequencies of the so-called *light-ring* modes of scalar and electromagnetic perturbations. To do so we used the WKB analytical approximation [49, 50]. We obtained results of the form $\omega^2 = \omega^2_{Sch} + \mathcal{O}(\hbar)$, where ω^2_{Sch} are the frequencies for the classical Schwarzschild geometry. For instance, in the case of the electromagnetic perturbation we get

$$\omega^2 = \omega^2_{\mathrm{Sch}} + \frac{\hbar}{17010\pi M^2} \left(-13 \,\mathrm{Re}\left[\omega^2_{\mathrm{Sch}}\right] + 11 i \,\mathrm{Im}\left[\omega^2_{\mathrm{Sch}}\right] \right) \,. \tag{7.30}$$

We can see that quantum corrections to the frequencies of *light-ring* modes are negligible. This is what we expected, since the *light-ring* is located at around $r = 3M$, which is far enough away from the throat region where quantum corrections are expected to play a more important role. We therefore conclude that, although quantum effects imply drastic changes in the geometry of the black hole near the horizon, they do not seem to imply significant corrections in the exterior. As seen by distant exterior observers, these semiclassical solutions mimic a non-rotating black hole.

Part III

Articles

Article 1

Translational anomaly of chiral fermions in two dimensions

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It is well known that a quantized two-dimensional Weyl fermion coupled to gravity spoils general covariance and breaks the covariant conservation of the energy-momentum tensor. In this brief article, we point out that the quantum conservation of the momentum can also fail in flat spacetime, provided the Weyl fermion is coupled to a time-varying homogeneous electric field. This signals a quantum anomaly of the space-translation symmetry, which has not been highlighted in the literature so far.

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I. INTRODUCTION

Symmetries and their corresponding Noether conservation laws play a major role in classical physics. It was long thought that symmetries and conservation laws are preserved in the quantization of the classical system. For example, the momentum of a classical system possessing the space-translation invariance is a conserved quantity, and it is expected to be also conserved in the quantum theory. In the same way, invariance under phase transformations implies charge conservation, and it is also expected that, after quantization, the charge operator is conserved in time. In some special situations, a classical symmetry cannot be maintained in the procedure of quantization. This happens most frequently in field theory, in which one encounters intrinsic ultraviolet divergences. The removal of these infinities, through the process of renormalization, might produce finite and unambiguous results that may imply an unavoidable conflict with the symmetry of the classical theory.

This was first discovered in the analysis of a quantized Dirac field ψ in the presence of an electromagnetic background [1,2]. The classical action for a massless Dirac field is invariant under chiral transformations $\psi \to e^{-i\epsilon\gamma^5}\psi$. This implies, via Noether's theorem, that the axial current $j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is a conserved current $\partial_\mu j_A^\mu = 0$. However, in the quantized theory, this is no longer true. One finds the nonzero vacuum expectation value

$$\langle \partial_\mu j_A^\mu \rangle = -\frac{q^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}, \qquad (1)$$

where $F_{\mu\nu}$ is the electromagnetic field strength. This is a quantum breaking of the original symmetry, and it is usually referred to as an anomaly.

Equation (1) reflects an anomaly in a global symmetry, and it allows us to better understand the underlying physics. However, anomalies in currents coupled to gauge fields make the theory ill defined. They imply an unavoidable obstruction to constructing the quantized theory, and only their exact cancellation can restore the physical consistency. For example, in quantum electrodynamics with a single charged Weyl fermion, we have $\langle \partial_\mu j^\mu \rangle \neq 0$, and hence the theory is inconsistent. However, by adding a charged Weyl fermion of opposite chirality, consistency is restored. This type of anomaly can only occur in even-dimensional spacetimes.

A different class of gauge anomalies involves the breaking of general covariance, reflected in the nonzero expectation values in the divergence of the energy-momentum tensor $\langle \nabla_\mu T^{\mu\nu} \rangle \neq 0$. They are called gravitational anomalies [3]. These anomalies can occur in theories with chiral fields coupled to gravity and in spacetimes of dimension $4k + 2 = 2, 6, \ldots$, where k is an integer (for a review on anomalies, see Ref. [4]).

In two dimensions, one can construct very simple examples of quantum anomalies. A Dirac field interacting with an external electromagnetic field has a chiral anomaly,

$$\langle \partial_\mu j_A^\mu \rangle = -\frac{q}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \qquad (2)$$

This implies that a (right-handed) Weyl field interacting with an external electromagnetic field possesses a harmful anomaly in the source current to which the gauge field is coupled. The classical $U(1)$ local gauge symmetry is broken at the quantum level. A chiral field in two dimensions also possesses a gravitational anomaly [3–5],

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$$\langle \nabla_\mu T^\mu_\nu \rangle = \frac{1}{96\pi\sqrt{-g}} \epsilon^{\alpha\beta} \partial_\beta \partial_\rho \Gamma^\rho_{\nu\alpha}. \tag{3}$$

It signals the breaking of the spacetime coordinate reparametrization group.

The purpose of this paper is to point out that the breaking of a relevant spacetime symmetry could also happen in the quantization of a two-dimensional Weyl field, not in the presence of gravity but in the presence of a homogeneous electric background $E = E(t)$. In this case, a charged Weyl field possesses translation invariance in the spatial direction. In the classical theory, one has conservation of the 01 component of the canonical stress-energy tensor $\partial_\mu T^{\mu 1} = 0$. However, in the quantized theory, we find the anomalous result

$$\partial_\mu \langle T^{\mu 1}_{R,L} \rangle = \mp \frac{q^2 A \dot{A}}{2\pi}, \tag{4}$$

for right-/left-handed Weyl fields, respectively, where $A(t)$ is the vector potential for the electric field $E(t) = -\dot{A}(t)$.

The result (4) has not been stressed in the previous literature, and it can be easily obtained using the method of adiabatic regularization. The adiabatic subtraction method was originally introduced to deal with ultraviolet divergences of quantized scalar fields in a homogeneous expanding universe [6–8]. It has been extended to quantized Dirac fields in the presence of a homogenous electromagnetic background in Refs. [9–11].

II. TRANSLATIONAL ANOMALIES AND ADIABATIC REGULARIZATION

Let us consider a quantized Dirac field interacting with an external homogeneous electric field $E(t)$. The classical action for the Dirac field is given by

$$\mathcal{S} = \int d^2x \left(\frac{1}{2} \bar{\psi} i \gamma^\mu \overleftrightarrow{D}_\mu \psi - m \bar{\psi} \psi \right), \tag{5}$$

where $D_\mu \equiv \partial_\mu - iqA_\mu$ and γ^μ are the Dirac matrices satisfying the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The corresponding Dirac equation reads

$$(i\gamma^\mu D_\mu - m)\psi = 0. \tag{6}$$

For our purposes, it is very convenient to express the electric field in terms of a homogeneous vector potential $E(t) = -\dot{A}(t)$. The Dirac equation (6), with $A_\mu = (0, -A(t))$, becomes (we follow here Refs. [9,10])

$$(i\gamma^0 \partial_0 + (i\partial_x - qA)\gamma^1 - m)\psi = 0. \tag{7}$$

From now on, we will use the Weyl representation (with $\gamma^5 \equiv \gamma^0 \gamma^1$)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We expand the field in momentum modes

$$\psi(t,x) = \int_{-\infty}^{\infty} dk [B_k u_k(t,x) + D_k^\dagger v_k(t,x)], \tag{8}$$

where the two independent spinor solutions are

$$u_k(t,x) = \frac{e^{ikx}}{\sqrt{2\pi}} \begin{pmatrix} h^I_k(t) \\ -h^{II}_k(t) \end{pmatrix}$$
$$v_k(t,x) = \frac{e^{-ikx}}{\sqrt{2\pi}} \begin{pmatrix} h^{II*}_{-k}(t) \\ h^{I*}_{-k}(t) \end{pmatrix}.$$

B_k and D_k are the creation and annihilation operators, which fulfill the usual anticommutation relations. Equation (7) is converted into

$$\dot{h}^I_k - i(k + qA) h^I_k - im h^{II}_k = 0 \tag{9}$$

$$\dot{h}^{II}_k + i(k + qA) h^{II}_k - im h^I_k = 0, \tag{10}$$

where we assume the normalization condition $|h^I_k|^2 + |h^{II}_k|^2 = 1$, ensuring the usual anticommutation relation between creation and annihilation operators. In the massless case, we have a decoupled system, and it can be solved analytically.

In the presence of an external homogeneous electric field, the theory (5) possesses the translational invariance in the space coordinate: $x^1 \to x^1 + \epsilon$. Therefore, Noether's theorem ensures that the classical energy-momentum tensor $T^{\mu\nu}$ obeys the conservation law $\partial_\mu T^{\mu 1} = 0$. This happens for every value of the mass. For the study of chiral conservation laws, we can use the simplest (canonical) form of the energy-momentum tensor

$$T^{\mu\nu} = \frac{i}{2} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial^\nu} \psi \tag{11}$$

and split it into the two chiral components $T^{\mu\nu} = T^{\mu\nu}_R + T^{\mu\nu}_L$:

$$T^{\mu\nu}_R = \frac{i}{2} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial^\nu} \frac{I + \gamma^5}{2} \psi \tag{12}$$

$$T^{\mu\nu}_L = \frac{i}{2} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial^\nu} \frac{I - \gamma^5}{2} \psi. \tag{13}$$

We note that, for a massless theory and as a consequence of the underlying symmetry, the $\mu 1$ component of each chiral Weyl sector is separately conserved,

$$\partial_\mu T^{\mu 1}_{R,L} = 0. \tag{14}$$

We will show that this is no longer true in the quantum theory. We recall that, when $A = 0$, the chiral fields $\psi_{R,L} = \frac{I\pm\gamma^5}{2}\psi$ obey the equations $\partial_+\psi_R = 0 = \partial_-\psi_L$, with $x^\pm = x^0 \pm x^1$. The quantized fields $\psi_{R,L}$ describe particles and antiparticles traveling to the right/left, with positive/negative spatial momentum, respectively.

The formal vacuum expectation values of these currents, for a generic value of the mass, take the form

$$\langle T^{01}_R\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dkk|h^I_k|^2, \tag{15}$$

$$\langle T^{01}_L\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dkk|h^{II}_k|^2. \tag{16}$$

These expressions are divergent, and we have to add appropriate subtractions. Since we are working with a homogeneous background, it is very convenient to use the adiabatic regularization method. The method works with subtractions derived from the adiabatic expansion of the modes [6,7]. Following Refs. [9–11], one can univocally determine the subtractions required in the renormalization of the above chiral currents. $A(t)$ is considered of adiabatic order 1, as explained in Refs. [9,11]. For an arbitrary mass, and assuming that at early times $A(t)$ vanishes, the renormalized expression for $\langle T^{01}_{R,L}\rangle_{\rm ren}$ is given by

$$\langle T^{01}_{R,L}\rangle_{\rm ren} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dkk\left(|h^{I,II}_k|^2 - \frac{\omega\mp k}{2\omega} \mp \frac{3km^2q^2A^2}{4\omega^5}\right), \tag{17}$$

with $\omega = \sqrt{k^2+m^2}$. We can now evaluate the time derivative of the above expressions,

$$\partial_t\langle T^{01}_{R,L}\rangle_{\rm ren} = \pm\frac{m}{\pi}\int_{-\infty}^{\infty} k{\rm Im}(h^I_k h^{II*}_k)dk \mp \frac{q^2A\dot{A}}{2\pi}, \tag{18}$$

where we have used the equations for the modes (9) and (10). In the massless limit, the first term in (18) vanishes, and we are left with

$$\partial_\mu\langle T^{\mu 1}_{R,L}\rangle_{\rm ren} = \mp\frac{q^2A\dot{A}}{2\pi}. \tag{19}$$

This nonvanishing result shows the existence of an anomaly in the classical translational symmetry for each chiral sector. Furthermore, this anomaly is accompanied by the well-known anomaly for the R/L currents,

$$\partial_\mu\langle j^\mu_{R,L}\rangle_{\rm ren} = \pm\frac{q\dot{A}}{2\pi} = \mp\frac{q}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}, \tag{20}$$

where $j^\mu_{R,L} = \bar{\psi}_{R,L}\gamma^\mu\psi_{R,L}$, which can also be derived in a similar way from the adiabatic subtractions. For a massless Dirac field, the anomalies cancel out, and one restores the translational invariance $\partial_\mu(\langle T^{\mu 1}_R\rangle_{\rm ren} + \langle T^{\mu 1}_L\rangle_{\rm ren}) = 0$, as well as the phase invariance $\partial_\mu(\langle j^\mu_R\rangle_{\rm ren} + \langle j^\mu_L\rangle_{\rm ren}) = 0$.

A. Symmetric stress-energy tensor and translational anomaly

The anomaly (19) in the translational symmetry can also be realized in terms of the (symmetric) Belinfante stress-energy tensor $\Theta^{\mu\nu}$, constructed as

$$\Theta^{\mu\nu} = \frac{i}{4}(\bar{\psi}\gamma^\mu \overleftrightarrow{D}^\nu\psi + \bar{\psi}\gamma^\nu\overleftrightarrow{D}^\mu\psi). \tag{21}$$

We have to remark that, although the canonical stress-energy tensor is more appropriate to show the existence of the translational anomaly, it is the Belinfante stress-energy tensor the right one to understand the anomaly in terms of the underlying process of particle creation.

The symmetric tensor $\Theta^{\mu\nu}$ is related to the canonical one $T^{\mu\nu}$ by

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\alpha B^{\alpha\mu\nu} + q\bar{\psi}\gamma^\mu\psi A^\nu, \tag{22}$$

where the antisymmetric tensor $B^{\alpha\mu\nu}$ is defined as $B^{\alpha\mu\nu} = \frac{1}{8}\bar{\psi}\{\gamma^\alpha,\sigma^{\mu\nu}\}\psi$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$. The divergence of the vacuum expectation values $\langle\Theta^{\mu 1}_{R,L}\rangle$ can be read from (22)

$$\partial_\mu\langle\Theta^{\mu 1}_{R,L}\rangle = \partial_\mu\langle T^{\mu 1}_{R,L}\rangle + q(\partial_\mu A^1)\langle j^\mu_{R,L}\rangle + qA^1\partial_\mu\langle j^\mu_{R,L}\rangle. \tag{23}$$

Now, taking into account (20), and the facts that $A^1 = A(t)$ and $A(t=-\infty) = 0$, we obtain

$$\langle j^0_{R,L}\rangle_{\rm ren} = \pm\frac{qA}{2\pi}. \tag{24}$$

Using the result for the translational anomaly (19), the anomalies for the R/L currents, and Eq. (24), we get immediately

$$\partial_\mu\langle\Theta^{\mu 1}_{R,L}\rangle_{\rm ren} = \pm\frac{q^2A\dot{A}}{2\pi} = \pm\frac{q^2}{2\pi}E(t)\int_{-\infty}^{t} E(t')dt', \tag{25}$$

which can be regarded as parallel to the result (19). Note, however, the important change of sign, as compared to (19).

It is also interesting to evaluate the rate of the 00 component of the stress-energy tensor. Using adiabatic regularization we find

$$\partial_\mu \langle \Theta^{\mu 0}\rangle_{\rm ren} = \frac{q^2 A\dot{A}}{\pi} = -qF^0{}_1\langle j^1\rangle_{\rm ren}. \qquad (26)$$

The above results can be re-expressed in null coordinates $x^\pm$ maintaining locality, Lorentz-covariance, and gauge-invariance. It is not difficult to get

$$\partial_+\langle \Theta_{--}\rangle_{\rm ren} = -qF_{-+}\langle j_-\rangle_{\rm ren} \qquad (27)$$

$$\partial_-\langle \Theta_{++}\rangle_{\rm ren} = -qF_{+-}\langle j_+\rangle_{\rm ren}. \qquad (28)$$

To visualize the anomalous behavior we have to take second derivatives of the stress-energy tensor. We find $\partial_+^2\langle\Theta_{--}\rangle_{\rm ren} = -q\partial_+F_{-+}\langle j_-\rangle_{\rm ren} + \frac{q^2}{2\pi}F_{-+}^2$ and a similar relation for $\langle\Theta_{++}\rangle_{\rm ren}$. The anomalous c-number terms in the second derivatives of the stress-energy tensor components are linked to the standard anomalous behavior of the chiral currents $\partial_\pm\langle j_\mp\rangle_{\rm ren} = \pm q(2\pi)^{-1}F_{+-}$. Note that Θ_{++} and Θ_{--} are related to the energy flux of the left and right moving sectors, respectively, of the Dirac field. Note also that, in flat space, the trace of the two-dimensional stress tensor is zero, $\langle\Theta_{+-}\rangle_{\rm ren} = 0$. Summing up the second-order equations, one also gets

$$\partial_\mu\partial_\nu\langle\Theta^{\mu\nu}\rangle_{\rm ren} = -q\partial_\nu F^\nu{}_\rho\langle j^\rho\rangle_{\rm ren} - \frac{q^2}{2\pi}F_{\mu\nu}F^{\mu\nu}. \qquad (29)$$

The quantum theory, mainly due to the above c-number terms, breaks the conservation of the chiral fluxes of momenta in a way compatible with the anomalous behavior of the chiral currents. The underlying reason for all the above anomalies finds its origin in a particle creation phenomenon.

B. Relation to particle creation

The result (25) can be understood in terms of the well-known process of particle creation. Following the Bogoliubov transformation method [7], the field modes h_k^I and h_k^{II} for a pulsed electric field can be related, at late times, to the number density of created particles n_k. After some calculations, one obtains the following relations in the massless limit:

$$\langle\Theta_R^{01}\rangle_{\rm ren} = \int_0^\infty dk k n_k, \qquad \langle\Theta_L^{01}\rangle_{\rm ren} = \int_{-\infty}^0 dk k n_k. \qquad (30)$$

It is clear that the R(L) part of the symmetric tensor gives the total momentum of the created quanta with positive (negative) momentum. Assuming $A = 0$ at early times, the number density n_k in the massless case is $(2\pi)^{-1}$ into the interval $(-qA(t), qA(t))$ and 0 for any other k [2,12]. Integrating (30) between these limits, one obtains a result in full agreement with (25).

The physical picture of the underlying particle production process is significantly modified by the mass. Let us consider a positive electric pulse $E(t) > 0$. Massless particles with positive charge are always created with positive momentum in the interval $(0, |qA(t)|)$, while antiparticles with negative charge are created with momentum in the interval $(-|qA(t)|, 0)$. For massive fermions, a fraction of particles with positive charge can be created with negative momentum, while antiparticles with negative charge can also be created with positive momentum.

Finally, we remark that a somewhat similar result can also be obtained for each chiral sector of quantized massless scalar fields. However, the result (25) is only valid in the adiabatic limit, for an infinitely slow evolution of $A(t)$. In contrast, the result for fermions is completely general, valid for arbitrary $A(t)$.

C. Relation to backreaction equations

Another way to illustrate the translational anomaly (19) is by solving the semiclassical backreaction equations for the quantized Dirac field $\psi = \psi_R + \psi_L$ obeying the Maxwell equation $\dot{E} = -q\langle j^1\rangle_{\rm ren}$. According to the adiabatic subtraction method, we have

$$\langle j^1\rangle_{\rm ren} = \frac{1}{2\pi}\int_{-\infty}^\infty dk\left(|h_k^I|^2 - |h_k^{II}|^2 - \frac{k}{\omega} + \frac{qm^2}{\omega^3}A\right). \qquad (31)$$

In the massless limit, the system can be solved analytically, finding harmonic oscillations with frequency $\frac{|q|}{\sqrt{\pi}}$. In Fig. 1(a), we show the solution for the electric field.

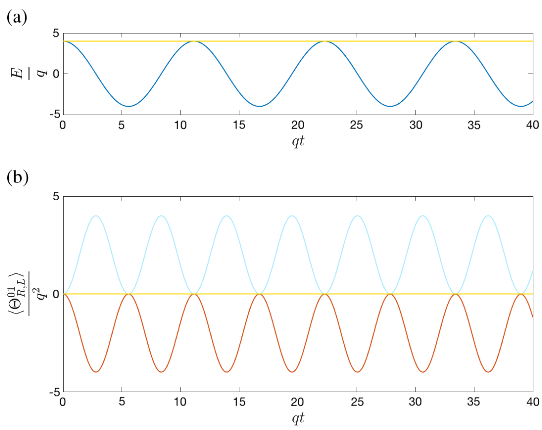

FIG. 1. Solution for the electric field (a) and the chiral projections of $\langle\Theta_R^{01}\rangle_{\rm ren}$ (light blue line) and $\langle\Theta_L^{01}\rangle_{\rm ren}$ (dark orange line) (b) for $m = 0$. We have chosen $E_0 = 4q$ as the initial condition for the electric field. The initial state for the matter field is the vacuum. The solution for the classical limit is also plotted (yellow line).

It is very illuminating to see the time evolution of the fluxes $\langle\Theta^{01}_{R,L}\rangle_{\rm ren}$, since they represent the created chiral momentum in the massless case. As we can see in Fig. 1(b), for each set of massless right-handed fermions/antifermions created with total momentum $P_R > 0$, there is a set of massless left-handed antifermions/fermions with momentum $P_L = -P_R$. The required energy to create particles is extracted from the electric field, generating a continuous energy exchange between the electric and the fermionic fields. Particles can also be destroyed, returning energy to the electric field.

In the massive case $m \neq 0$, the backreaction equations also induce electric oscillations, which can be regarded as perturbations of the oscillations at $m = 0$.

III. CONCLUSIONS

In this brief article, we have pointed out that quantized chiral fields in two dimensions coupled to a homogeneous time-varying electric field break the classical conservation of the canonical stress-energy tensor $\partial_\mu\langle T^{\mu 1}_{R,L}\rangle_{\rm ren} = \mp\frac{q^2 A\dot{A}}{2\pi}$. This quantum anomaly has not been stressed in the previous literature. This result can be reexpressed in terms of the symmetric stress-energy tensor of the left- or right-moving sectors of the Dirac field $\partial_\mu\langle\Theta^{\mu 1}_{R,L}\rangle_{\rm ren} = \pm\frac{q^2 A\dot{A}}{2\pi}$. Furthermore, our results have a direct physical interpretation in terms of particle creation in a way compatible with the axial anomaly.

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Article 2

Breaking of adiabatic invariance in the creation of particles by electromagnetic backgrounds

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Particles are spontaneously created from the vacuum by time-varying gravitational or electromagnetic backgrounds. It has been proven that the particle number operator in an expanding universe is an adiabatic invariant. In this paper we show that, in some special cases, the expected adiabatic invariance of the particle number fails in presence of electromagnetic backgrounds. In order to do this, we consider as a prototype a Sauter-type electric pulse. Furthermore, we also show a close relation between the breaking of the adiabatic invariance and the emergence of the axial anomaly.

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I. INTRODUCTION

The understanding of particle creation phenomena in terms of Bogolubov transformations was pioneered in the analysis of quantized fields in an isotropically expanding universe [1–3] (for a retrospective analysis see [4]). A fundamental issue in the study of particle creation in an expanding universe was the adiabatic invariance of the number of created particles. The particle number of a quantized field, in the limit of an infinitely slow and smooth expansion of the universe, that is, an adiabatic expansion, does not change with time [4], even if the quantized field is massless. In other words, the density of created particles by the cosmic expansion approaches zero when the Hubble rate $\dot{a}/a$ is each time negligible even if the final amount of expansion $a(t_{\rm final})/a(t_{\rm initial})$ is large. Hence, we say that the particle number is an adiabatic invariant. Moreover, pair production can also take place in time-varying electric [5,6] or scalar backgrounds, and it can be regarded as a very important nonperturbative process in quantum field theory [7]. It is also fundamental to understand the reheating epoch in cosmology [8], nonequilibrium processes induced by strong fields [9,10], and astrophysical phenomena [11].

The main purpose of this work is to analyze the adiabatic invariance of the particle number observable in the presence of an electromagnetic background. We find that for massive fields adiabatic invariance is, as expected, preserved. For slowly varying electromagnetic potentials no quanta is being produced, even if the change in the electromagnetic potential over a long period is very large. However, in some cases and only for massless fields, the particle number is not an adiabatic invariant. In other words, particles are still created in the adiabatic limit. We analyze the problem in detail in a two-dimensional scenario, for both scalar and Dirac fields. As a by-product of our analysis, we point out a connection between the (anomalous) breaking of the adiabatic invariance of the particle number operator and the emergence of a quantum anomaly in the chiral symmetry. We will show that the breaking of adiabatic invariance and its connection to the axial anomaly can be easily translated to four dimensions.

Conservation laws and symmetries play a fundamental role in the understanding of a physical system. Anomalies are symmetries of a classical theory that fail to survive upon quantization. This happens, typically, in field theory because of the need for regularization and renormalization of ultraviolet divergences. A very illustrative example occurs in quantum electrodynamics in the limit of massless Dirac fermions. The classical theory is invariant under chiral transformations, and this implies the conservation of the axial current j_5^μ. However, this symmetry is broken in the quantum theory. The chiral anomaly opens the possibility of having processes violating the conservation of chirality. Nevertheless, all elementary processes of quantum electrodynamics, based on the perturbative expansion of the S-matrix, preserve chirality [12]. One has to resort to a nonperturbative phenomena, i.e., the spontaneous pair production by electromagnetic fields, to unveil conservation-law violation of chirality of massless fermions. The nonconservation of chirality seems to be directly

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related to the breaking of adiabaticity in the particle number observable.

The paper is organized as follows. Section II is devoted to briefly illustrate the problem within the conventional cosmological scenario, as described in [3,4]. In Sec. III we will analyze the case of a two-dimensional complex scalar field coupled to an external electric pulse. The role of the mass is analyzed in detail, and we will show explicitly that adiabatic invariance of the particle number is broken for massless fields. In Sec. IV we generalize the result to Dirac fields, showing a connection between the breaking of adiabatic invariance and the emergence of the chiral anomaly. The next step is to extend our result to four dimensions. This will be done in Sec. V. We will find again that adiabatic invariance requires a nonvanishing effective mass, as happens for two-dimensional quantized fields coupled to an electric field. However, a zero effective mass can only be achieved for Dirac (not for scalar) fields coupled to both electric and magnetic fields. The breaking of adiabatic invariance also emerges in parallel to the emergence of the chiral anomaly. In Sec. VI we summarize the main conclusions.

II. A BRIEF ORIENTATION: ADIABATIC INVARIANCE IN THE EXPANDING UNIVERSE

The adiabatic invariance of the particle number operator in an expanding universe can be easily illustrated with the simple example (borrowed from [3]) of a scalar field with mass m in the presence of a two-dimensional bounded expanding universe. This example, although well-known, will serve to better clarify the main idea of the next sections. Consider the following metric:

$$ds^2 = dt^2 - a^2(t)dx^2 = C(\eta)(d\eta^2 - dx^2), \qquad (1)$$

where $d\eta = a^{-1}(t)dt$ and the conformal scale factor is given by the function $C(\eta) = 1 + B(1 + \tanh \rho\eta)$, with B a positive constant. This represents a smooth expansion bounded by asymptotically static and flat spacetime regions. The expansion factor has smoothly shifted from $a_{\rm in} \equiv a(-\infty) = 1$ to $a_{\rm out} \equiv a(+\infty) = \sqrt{1+2B}$. In Fig. 1 it is shown the behavior of the conformal scale factor $C(\eta)$ as well as the Hubble rate $H(\eta) = \frac{C'(\eta)}{2C^{3/2}}$ for different values of the adiabatic parameter ρ in terms of dimensionless variables.

The equation for the modes of the scalar field in the background metric (1) is given by

$$\frac{d^2}{d\eta^2} h_k(\eta) + (m^2 C(\eta) + k^2) h_k(\eta) = 0. \qquad (2)$$

In the remote past the normalized modes are assumed to behave as the positive frequency modes in Minkowski space,

$$\frac{1}{\sqrt{2(2\pi)\omega_{\rm in}}} e^{ikx} e^{-i\omega_{\rm in} t}, \qquad (3)$$

with $\omega_{\rm in} = \sqrt{k^2 + m^2}$. As time evolves these modes behave, in the remote future, as a mixture of positive and negative frequency modes of the form,

$$\frac{\alpha_k}{\sqrt{2(2\pi)\omega_{\rm out}}} e^{ikx} e^{-i\omega_{\rm out} t} + \frac{\beta_k}{\sqrt{2(2\pi)\omega_{\rm out}}} e^{ikx} e^{+i\omega_{\rm out} t}, \qquad (4)$$

with $\omega_{\rm out} = \sqrt{(\frac{k}{a_{\rm out}})^2 + m^2}$. α_k and β_k are the so-called Bogolubov coefficients. The annihilation operators for physical particles at late times a_k are related to the annihilation and creation operators at early times (A_k and $A_k^\dagger$) by the relations,

$$a_k = \alpha_k A_k + \beta_k^* A_{-k}^\dagger. \qquad (5)$$

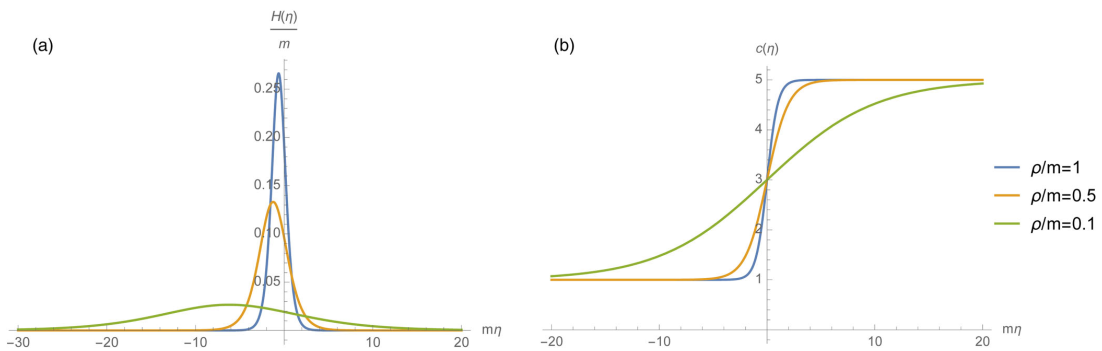

FIG. 1. Conformal scale factor for $B = 2$. Figure (a) shows the Hubble rate H/m for different values of the dimensionless "slowness" parameter ρ/m. Figure (b) shows the dependence of the conformal scale factor $C(\eta)$ on ρ/m. The adiabatic limit corresponds to $\rho \to 0$. Note that the area defined by the curves $H(\eta)$ in Fig. 1(a) does not depend on ρ/m.

The average density number of created particles n_k, with momentum k, is given by

$$n_k = |\beta_k|^2 = \frac{\sinh^2(\pi \frac{\omega_-}{\rho})}{\sinh(\pi \frac{\omega_{\rm in}}{\rho}) \sinh(\pi \frac{a_{\rm out}\omega_{\rm out}}{\rho})}, \qquad (6)$$

where $\omega_- = \frac{1}{2}(a_{\rm out}\omega_{\rm out} - \omega_{\rm in})$. It is very easy to check that in the adiabatic limit, that is, for an extremely slow expansion $\rho \to 0$, the density number of created particles goes to $n_k \sim e^{-2\pi\omega_{\rm in}/\rho} \to 0$. This shows the fact that the particle number is an adiabatic invariant. This behavior of the particle number observable is generic, and it can be extended to isotropically expanding universes in four dimensions, irrespective of the value of the mass [1,4].

Furthermore, one can reinforce this idea by looking at a gravitational collapse producing a black hole. An adiabatic collapse can be thought as the (physically inaccessible) limit of a collapse approaching to a black hole with a very large mass $M \to \infty$ (and zero surface gravity) in an infinite amount of advanced time [13]. It is well-known that the late-time particle creation of a gravitational collapse is encapsulated by the surface gravity parameter. The produced radiation is thermal [2,3,14,15], with a temperature proportional to the surface gravity. In the adiabatic limit the production of scalar particles is expected to vanish, in agreement with Hawking's result.

III. BREAKING OF ADIABATIC INVARIANCE IN SCALAR PAIR PRODUCTION BY ELECTRIC FIELDS IN TWO DIMENSIONS

We will now analyze the same question for the phenomena of particle creation in electric fields. We will consider a classical and homogeneous electric field $E(t)$ interacting with a quantum, two-dimensional charged scalar field ϕ obeying the field equation,

$$(D_\mu D^\mu + m^2)\phi = 0, \qquad (7)$$

where $D_\mu\phi = (\partial_\mu + iqA_\mu)\phi$. We can expand the field in Fourier modes as

$$\phi(t,x) = \frac{1}{\sqrt{2(2\pi)}} \int dk[A_k e^{ikx} h_k(t) + B_k^\dagger e^{-ikx} h_{-k}^*(t)], \qquad (8)$$

where $A_k^\dagger, B_k^\dagger$, and A_k, B_k are the usual creation and annihilation operators. The mode functions $h_k(t)$ must obey the Wronskian consistency condition,

$$h_k \dot{h}_k^* - h_k^* \dot{h}_k = 2i, \qquad (9)$$

to ensure the usual commutation relations. Substituting (8) into (7) we get the equation,

$$\ddot{h}_k(t) + (m^2 + (k - qA(t))^2) h_k(t) = 0, \qquad (10)$$

where we have chosen an homogeneous time dependent potential $A_\mu = (0, -A(t))$ in the appropriate gauge. In order to study the adiabatic limit for the electric pair production, in a way similar to the gravitational case explained above, we need to consider a *bounded potential* $A(t)$. At an heuristic level, $A(t)$ will play a somewhat similar role to the conformal factor $C(\eta)$ for the expanding spacetime. Note by comparing (2) and (10) that the time dependence of the mode equation is encoded in $C(\eta)$ for the gravitational example and, analogously, it is in $A(t)$ in the electric case (see, for instance, [5] for a general discussion). We choose for convenience a Sauter-type electric pulse [16] of the form,

$$E(t) = -\frac{\rho A_0}{2} \cosh^{-2}(\rho t), \qquad (11)$$

which can be described by the potential $[E(t) = -\dot{A}(t)]$,

$$A(t) = \frac{1}{2} A_0 (\tanh(\rho t) + 1). \qquad (12)$$

This potential is bounded both at early and late times, as shown explicitly in Fig. 2(b). Note that ρ plays the role of a slowness parameter. It is very illustrative to compare Fig. 2 with Fig. 1.

We have chosen the above Sauter-type pulse [16] for convenience. Note that this potential is bounded both at early and late times (see Fig. 2). Note also that for all the figures we work with dimensionless variables. The adiabatic limit is an extremely slow evolution of the potential, obtained when $\rho \to 0$. We have to remark that the adiabatic limit is not the limit of a vanishing electric field. If the electric field had support in a bounded period of time, there would not be production of particles when $E(t) \to 0$. But the adiabatic limit is a more subtle limit, in which the electric field varies very slowly. Although $E \to 0$ when $\rho \to 0$, the width of the pulse is also very large maintaining constant and nonvanishing the integral,

$$\int_{-\infty}^{+\infty} E_{\rho_1}(t) dt = \int_{-\infty}^{+\infty} E_{\rho_2}(t) dt = \text{constant} = -qA_0. \qquad (13)$$

To clarify things we remark that a different scenario is given by the alternative choice $E(t) = -E_0 \cosh^{-2}(\rho t)$, with E_0 a constant value, independent of ρ. The limit $\rho \to 0$ corresponds then to a constant electric field, with an *unbounded potential* $A(t)$. This produces, as expected, the Schwinger-type rate of pair creation by a constant electric field [17]. In this paper we focus our analysis in the adiabatic limit $\rho \to 0$ in (11) and (12), as it produces a bounded potential and a completely analogous situation to that considered in the cosmological scenario. As we will

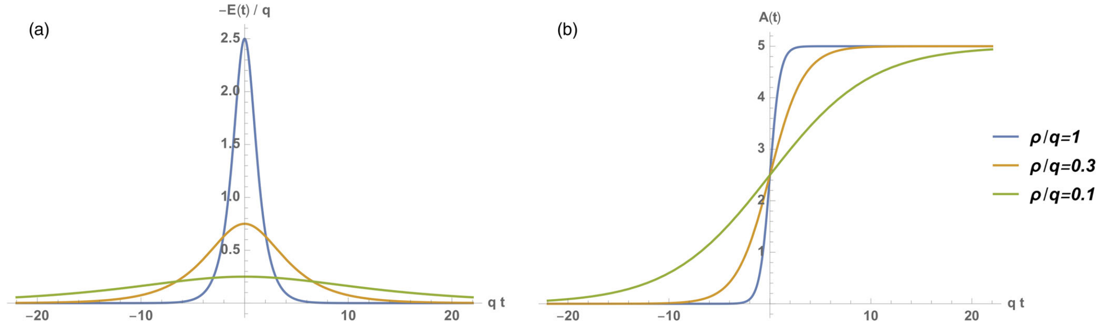

FIG. 2. Sauter-type electric pulse for $A_0 = 5$. Figure (a) shows the electric field E/q for different values of ρ/q. Figure (b) shows the dependence of the potential $A(t)$ on the dimensionless "slowness" parameter ρ/q. The adiabatic limit corresponds to $\rho \to 0$. Note that the area defined by the curves $-E(t)/q$ in Fig. 2(a) is A_0, irrespective of the value of ρ/q.

show later on, in this case, it is indeed possible to produce particles by the electric field if extra conditions are met (i.e., a massless field or the presence of magnetic fields in the four-dimensional case with fermions).

Inserting the potential (12) in (10) we obtain the physical solution in terms of the usual hypergeometric functions,

$$h_k(t) = \frac{1}{\sqrt{\omega_{\rm in}}} e^{-i\omega_{\rm in}t}(1+e^{2\rho t})^{(\frac{1}{2}-i\frac{\kappa}{\rho})} F\left(\frac{1}{2} - i\frac{\omega_+ + \kappa}{\rho}, \frac{1}{2} + i\frac{\omega_- - \kappa}{\rho}, 1 - i\frac{\omega_{\rm in}}{\rho}; -e^{2\rho t}\right), \tag{14}$$

where $\kappa = \frac{1}{2}\sqrt{(qA_0)^2 - \rho^2}$, $\omega_{\rm in} = \sqrt{k^2+m^2}$, $\omega_{\rm out} = \sqrt{(k-qA_0)^2+m^2}$, and $\omega_\pm = \frac{1}{2}(\omega_{\rm out} \pm \omega_{\rm in})$. We have fixed this solution by demanding that at early times the modes behave as the Minkowskian modes for a free scalar field,

$$h_k(t) \sim \frac{1}{\sqrt{\omega_{\rm in}}} e^{-i\omega_{\rm in}t}. \tag{15}$$

At late times the modes behave as

$$h_k(t) \sim \frac{\alpha_k}{\sqrt{\omega_{\rm out}}} e^{-i\omega_{\rm out}t} + \frac{\beta_k}{\sqrt{\omega_{\rm out}}} e^{+i\omega_{\rm out}t}, \tag{16}$$

where α_k and β_k are the Bogoliubov coefficients. They satisfy the relation $|\alpha_k|^2 - |\beta_k|^2 = 1$ according to the normalization condition (9). These coefficients serve to relate the early time creation and annihilation operators A_k, B_k, defining the initial Fock space, with the late time operators a_k, b_k,

$$a_k = \alpha_k A_k + \beta_k^* B_{-k}^\dagger \tag{17}$$

$$b_k = \alpha_{-k} B_k + \beta_{-k}^* A_{-k}^\dagger. \tag{18}$$

Therefore, we can define the number operator as

$$\langle N \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk N_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk(|\beta_k|^2 + |\beta_{-k}|^2), \tag{19}$$

where $N_k = n_k + \bar{n}_k = \langle 0|a_k^\dagger a_k|0\rangle + \langle 0|b_k^\dagger b_k|0\rangle$ is the number density of quanta (i.e., $n_k = |\beta_k|^2$ particles and $\bar{n}_k = |\beta_{-k}|^2$ antiparticles). Taking the late time limit $t \to \infty$ in (14) and matching with (16) we obtain

$$\alpha_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}} \frac{\Gamma(1 - i\frac{\omega_{\rm in}}{\rho})\Gamma(-i\frac{\omega_{\rm out}}{\rho})}{\Gamma(\frac{1}{2} - i\frac{\omega_+ + \kappa}{\rho})\Gamma(\frac{1}{2} - i\frac{\omega_+ - \kappa}{\rho})} \tag{20}$$

$$\beta_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}} \frac{\Gamma(1 - i\frac{\omega_{\rm in}}{\rho})\Gamma(i\frac{\omega_{\rm out}}{\rho})}{\Gamma(\frac{1}{2} + i\frac{\omega_- + \kappa}{\rho})\Gamma(\frac{1}{2} + i\frac{\omega_- - \kappa}{\rho})}, \tag{21}$$

where we have used the usual properties of the hypergeometric function [18]. Finally we get

$$|\beta_k|^2 = \frac{\cosh(2\pi\frac{\omega_-}{\rho}) + \cosh(2\pi\frac{\kappa}{\rho})}{2\sinh(\pi\frac{\omega_{\rm in}}{\rho})\sinh(\pi\frac{\omega_{\rm out}}{\rho})}. \tag{22}$$

Figure 3 shows a representation of this expression for different values of m and ρ, which can be interpreted as the momentum distribution of the created particles (the spectra of antiparticles would be obtained by making the shift $k \to -k$). We easily observe that $|\beta_k|^2$ decreases as $\rho \to 0$, for fixed $m \neq 0$. In the same way, the particle density also decreases for large m with ρ fixed. Note in passing that for a sudden electric pulse ($\rho \gg 0$) the momentum distribution of the particles is concentrated in the characteristic values $k = 0$ and $k = qA_0$.

To see whether $|\beta_k|^2$ vanishes in the adiabatic limit we analyze in detail the behavior $\rho \to 0$ on (22). We get

$$|\beta_k|^2 \sim e^{-2\pi\omega_{\rm in}/\rho} + e^{-2\pi\omega_{\rm out}/\rho} + e^{-\frac{\pi}{\rho}\delta}, \tag{23}$$

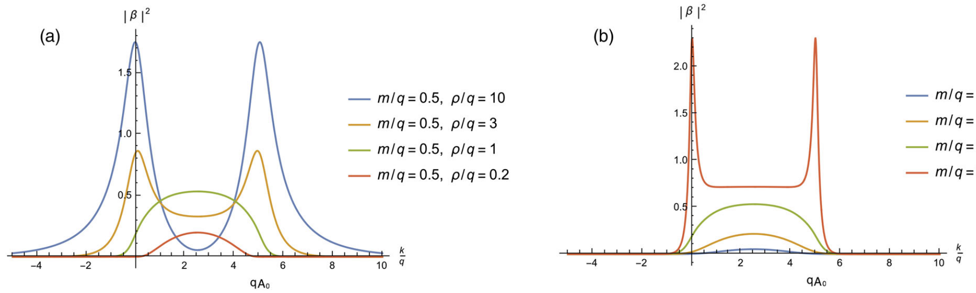

FIG. 3. Momentum distribution of the created scalar particles with positive charge at late times ($|\beta_k|^2$) by an electric pulse with $A_0 = 5$ and different values of m/q and ρ/q. In Fig. 3(a) the mass is fixed, while in (b) the dimensionless parameter of adiabaticity ρ/q is fixed.

where $\delta = 2(\omega_+ - \kappa)$ and $\kappa \to \frac{|qA_0|}{2}$. Since $\omega_{\rm in}, \omega_{\rm out} > 0$, the first two terms vanish as $\rho \to 0$. For $m \neq 0$, the function $\delta(k)$ has a minimum at $k = \frac{qA_0}{2}$, with a value $\delta_{\rm min} = \sqrt{(qA_0)^2 + 4m^2} - |qA_0| > 0$. It means that $\delta > 0$, and hence $|\beta_k|^2 \to 0$ when $\rho \to 0$, as in the case of a gravitational field. According with that, in Fig. 3(a) one can realize how the number of particles decreases with the adiabatic parameter ρ, vanishing in the limit $\rho \to 0$. However, for $m = 0$ this is no longer valid since $\delta = 0$ for $k \in (0, qA_0)$, and hence $|\beta_k|^2 \to 1$, meaning that particles are being produced even in the adiabatic limit. In short, we have obtained, when $\rho \to 0$,

$$N_k \to \begin{cases} 0 & \text{for } m \neq 0 \quad \text{or} \quad m = 0 \quad \text{and} \quad k \notin (-qA_0, qA_0) \\ 1 & \text{for } m = 0 \quad \text{and} \quad k \in (-qA_0, qA_0). \end{cases} \tag{24}$$

In order to visualize this behavior, we represent in Fig. 4 the dependence of the total density of created particles $\langle N \rangle$ [given by (19)] on the parameter ρ. One can see how in the adiabatic limit the density of quanta tends to vanish, except in the case $m = 0$, for which it tends to a nonzero value. This value is given by

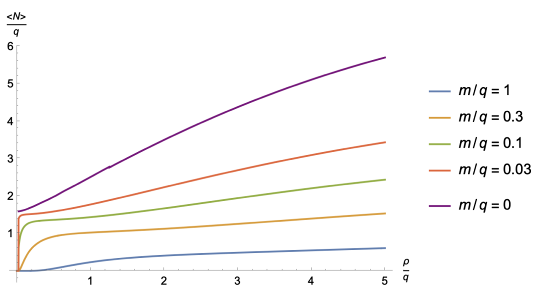

FIG. 4. Number of late-time created scalar particles as a function of the dimensionless adiabaticity parameter ρ/q, for $A_0 = 5$ and for different values of the mass.

$$\langle N \rangle = \frac{1}{2\pi} \int_{-|qA_0|}^{|qA_0|} dk\, N_k = \frac{|qA_0|}{\pi}. \tag{25}$$

This implies that the particle number is not an adiabatic invariant for the massless case. Furthermore, as we will see in the next section, the above result for the density number of created particles in the adiabatic limit coincides exactly with the analogous result for massless Dirac particles.

For completeness we will study now the vacuum expectation values of the electric current and the energy density induced by the underlying particle creation process. This will also serve to test the adiabatic invariance, or the breaking of it, in terms of the current and the energy density.

A. Electric current

For a two-dimensional charged scalar field, the electric current is given by $j^\mu = iq[\phi^\dagger D^\mu \phi - (D^\mu \phi)^\dagger \phi]$. The vacuum expectation of this observable is UV-divergent and has to be renormalized. In the context of an homogeneous and time dependent background it is very convenient to use the adiabatic regularization/renormalization method described in [17,19]. After performing the appropriated subtractions, one obtains

$$\langle j^x \rangle_{\rm ren} = q \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[(k - qA)|h_k|^2 - \frac{k}{\omega} + \frac{m^2 qA}{\omega^3} \right], \tag{26}$$

where $h_k(t)$ are the mode functions of the scalar field satisfying the equation of motion (10) and $\omega = \sqrt{k^2 + m^2}$. For more details on the original adiabatic method for scalar fields see [20].

Let us focus on the late-time behavior of the electric current, for which we can relate (26) to the Bogoliubov coefficients computed in the last section. We restrict again the analysis to an electric-pulse configuration (12) with bounded asymptotic states. Introducing (16) in (26) we have

$$\langle j^x\rangle_{\rm ren} \sim q\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[\frac{k-qA_0}{\omega_{\rm out}}(2|\beta_k|^2+2{\rm Re}(\alpha_k\beta_k^* e^{-i2\omega_{\rm out}t})) + \frac{k-qA_0}{\omega_{\rm out}} - \frac{k}{\omega} + \frac{m^2 qA_0}{\omega^3}\right]. \quad (27)$$

It is easy to see that the terms which are independent of the Bogoliubov coefficients do not contribute to the electric current. One can derive this result by realizing that the first two terms correspond to linearly divergent integrals, differing by a constant shift,

$$q\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[\frac{k-qA_0}{\sqrt{(k-qA_0)^2+m^2}} - \frac{k}{\sqrt{k^2+m^2}}\right] = -\frac{q^2A_0}{\pi}, \quad (28)$$

while the last term in (27) is a finite integral, which cancels with (28). The second term in (27) depends on time and produces oscillations of the form $\cos(2\omega_{\rm out}t)$. In the limit $t\to\infty$ the Riemann-Lebesgue lemma ensures that the integral in dk of this term vanishes. With the above considerations, and using the symmetry properties of $|\beta_k|^2$ [reflection under $k\to-(k-qA_0)$], one can rewrite the expression of the electric current as follows:

$$\langle j^x\rangle_{\rm ren} \sim -q\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{k}{\omega}(|\beta_k|^2 - |\beta_{-k}|^2). \quad (29)$$

This equation shows explicitly the close relation between the density of created quanta and the electric current. The first term accounts for particles and the second one for antiparticles. In the adiabatic limit, and for massive particles, the renormalized electric current also vanishes since $|\beta_k|^2\to 0$. However, the last result changes completely if $m=0$. As we have shown, in the adiabatic limit$|\beta_k|^2\to 1$ for $k\in(0,qA_0)$. Therefore, the current at late times for massless particles in the adiabatic limit is given by

$$\langle j^x\rangle_{\rm ren} \sim -\frac{q^2A_0}{\pi}. \quad (30)$$

As expected, a nonvanishing particle number $\langle N\rangle$, even in the adiabatic limit, induces an electric current different from zero.

B. Energy density

The renormalized vacuum expectation value of the energy density of a two-dimensional scalar field interacting with an electric field is given by

$$\langle T_{00}\rangle_{\rm ren} = \int_{-\infty}^{\infty}\frac{dk}{4\pi}\left[|\dot h_k|^2 + (m^2+(k-qA)^2)|h_k|^2 - 2\omega + \frac{2kqA}{\omega} - \frac{m^2q^2A^2}{\omega^3}\right], \quad (31)$$

where $h_k(t)$ are again the mode functions of the scalar field and the three last terms account for the adiabatic subtractions required by renormalization [19]. As for the electric current, we will we focus on the late time behavior. Plugging (16) in (31) and using the asymptotic expansion for the functions $\dot h(t)$,

$$\dot h_k(t) \sim -i\sqrt{\omega_{\rm out}}\alpha_k e^{-i\omega_{\rm out}t} + i\sqrt{\omega_{\rm out}}\beta_k e^{+i\omega_{\rm out}t}, \quad (32)$$

we finally obtain

$$\langle T_{00}\rangle_{\rm ren} \sim \int_{-\infty}^{\infty}\frac{dk}{4\pi}\left[4\omega_{\rm out}|\beta_k|^2 + 2\omega_{\rm out} - 2\omega + \frac{2kqA_0}{\omega} - \frac{m^2q^2A_0^2}{\omega^3}\right]. \quad (33)$$

Using the same arguments as in Sec. III A, it is easy to see that the only term contributing to the energy density is the one proportional to $|\beta_k|^2$. After some simplifications, we get the relation between the energy density and the particle number,

$$\langle T_{00}\rangle_{\rm ren} \sim \int_{-\infty}^{\infty}\frac{dk}{2\pi}\omega N_k, \quad (34)$$

where $N_k = |\beta_{-k}|^2 + |\beta_k|^2$. In the adiabatic limit, we get $|\beta_k|^2\to 0$, and therefore $\langle T_{00}\rangle_{\rm ren}\to 0$. Nevertheless, for $m=0$ there is indeed creation of energy. As we said, the adiabatic limit for the massless case gives us a nonvanishing $|\beta_k|^2$ for $k\in(0,qA_0)$. In this region, $|\beta_k|^2=1$, and therefore the created energy density is

$$\langle T_{00}\rangle_{\rm ren} \sim \frac{q^2A_0^2}{2\pi}. \quad (35)$$

IV. BREAKING OF ADIABATIC INVARIANCE IN FERMIONIC PAIR PRODUCTION BY ELECTRIC FIELDS IN TWO DIMENSIONS

Let us consider now a two-dimensional charged Dirac field ψ interacting with an homogeneous, time-dependent electric field. The corresponding Dirac equation is

$$(i\gamma^\mu D_\mu - m)\psi = 0, \quad (36)$$

where γ^μ are the Dirac matrices satisfying the anticommutation relations $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ and $D_\mu \equiv \partial_\mu - iqA_\mu$. [We follow here the convention that the electric charge of the fermion is $-q$]. The electromagnetic field is assumed to be an external classical field, while ψ is a quantized field interacting with the classical electric background. Assuming also that the electric field is described by the potential $A_\mu = (0,-A(t))$ in the appropriate gauge, the Dirac equation (36) becomes

$$(i\gamma^0\partial_0 + (i\partial_x - qA)\gamma^1 - m)\psi = 0. \tag{37}$$

From now on we will use the Weyl representation (with $\gamma^5 \equiv \gamma^0\gamma^1$),

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We expand the field in momentum modes,

$$\psi(t,x) = \int_{-\infty}^{\infty} dk[B_k u_k(t,x) + D_k^\dagger v_k(t,x)], \tag{38}$$

where the two independent and normalized spinor solutions are

$$u_k(t,x) = \frac{e^{ikx}}{\sqrt{2\pi}} \begin{pmatrix} h_k^I(t) \\ -h_k^{II}(t) \end{pmatrix} \tag{39}$$

$$v_k(t,x) = \frac{e^{-ikx}}{\sqrt{2\pi}} \begin{pmatrix} h_{-k}^{II*}(t) \\ h_{-k}^{I*}(t) \end{pmatrix}. \tag{40}$$

B_k and D_k are the creation and annihilation operators which fulfill the usual anticommutation relations. The field equation (37) is converted into

$$\dot{h}_k^I - i(k+qA)h_k^I - imh_k^{II} = 0 \tag{41}$$

$$\dot{h}_k^{II} + i(k+qA)h_k^{II} - imh_k^I = 0, \tag{42}$$

and we have assumed the normalization condition $|h_k^I|^2 + |h_k^{II}|^2 = 1$. Let us consider, as in the scalar case, the electric pulse $A(t) = \frac{1}{2}A_0(\tanh(\rho t) + 1)$. With this input the mode equations (41) and (42) can be solved exactly in terms of hypergeometric functions,

$$h_k^I(t) = \sqrt{\frac{\omega_{\rm in} - k}{2\omega_{\rm in}}} \left(\frac{A(t)}{A_0}\right)^{-i\frac{\omega_{\rm in}}{2\rho}} \left(1 - \frac{A(t)}{A_0}\right)^{i\frac{\omega_{\rm out}}{2\rho}} \times F\left(i\frac{\omega_- + qA_0/2}{\rho}, 1 + i\frac{\omega_- - qA_0/2}{\rho}, 1 - i\frac{\omega_{\rm in}}{\rho}; \frac{A(t)}{A_0}\right) \tag{43}$$

$$h_k^{II}(t) = -\sqrt{\frac{\omega_{\rm in} + k}{2\omega_{\rm in}}} \left(\frac{A(t)}{A_0}\right)^{-i\frac{\omega_{\rm in}}{2\rho}} \left(1 - \frac{A(t)}{A_0}\right)^{i\frac{\omega_{\rm out}}{2\rho}} \times F\left(i\frac{\omega_- - qA_0/2}{\rho}, 1 + i\frac{\omega_- + qA_0/2}{\rho}, 1 - i\frac{\omega_{\rm in}}{\rho}; \frac{A(t)}{A_0}\right), \tag{44}$$

where $\omega_{\rm in} = \sqrt{k^2 + m^2}$, $\omega_{\rm out} = \sqrt{(k + qA_0)^2 + m^2}$ and $\omega_\pm = \frac{1}{2}(\omega_{\rm out} \pm \omega_{\rm in})$. We have fixed the initial condition in order to recover the positive frequency solution for a free field at early times $t \to -\infty$,

$$h_k^{I/II}(t) \sim \pm\sqrt{\frac{\omega_{\rm in} \mp k}{2\omega_{\rm in}}} e^{-i\omega_{\rm in} t}. \tag{45}$$

At late times $t \to +\infty$ the modes can be written as

$$h_k^{I/II}(t) \sim \pm\sqrt{\frac{\omega_{\rm out} \mp (k + qA_0)}{2\omega_{\rm out}}} \alpha_k e^{-i\omega_{\rm out} t} + \sqrt{\frac{\omega_{\rm out} \pm (k + qA_0)}{2\omega_{\rm out}}} \beta_k e^{i\omega_{\rm out} t}. \tag{46}$$

α_k and β_k are the Bogoliubov coefficients satisfying the relation $|\alpha_k|^2 + |\beta_k|^2 = 1$. These coefficients relate the early time creation and annihilation operators B_k, D_k with the late time operators b_k, d_k as follows:

$$b_k = \alpha_k B_k + \beta_k^* D_{-k}^\dagger \tag{47}$$

$$d_k = \alpha_{-k} D_k - \beta_{-k}^* B_{-k}^\dagger. \tag{48}$$

The density of created quanta is given by $N_k = \langle 0|b_k^\dagger b_k|0\rangle + \langle 0|d_k^\dagger d_k|0\rangle \equiv n_k + \bar{n}_k$, where $n_k = |\beta_k|^2$ and $\bar{n}_k = |\beta_{-k}|^2$. Therefore, the particle number is also

$$\langle N\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, N_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk(|\beta_k|^2 + |\beta_{-k}|^2). \tag{49}$$

The matching of (43)–(44) with (46) at late times determines the Bogoliubov coefficients. For the beta coefficients we get

$$\beta_k = \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}} \frac{\omega_{\rm in} - k}{\omega_{\rm out} + k + qA_0}} \frac{\Gamma(1 - i\frac{\omega_{\rm in}}{\rho})\Gamma(-i\frac{\omega_{\rm out}}{\rho})}{\Gamma(1 + i\frac{\omega_- + qA_0/2}{\rho})\Gamma(1 + i\frac{\omega_- - qA_0/2}{\rho})}. \tag{50}$$

And after simplifying, we obtain

$$|\beta_k|^2 = \frac{\cosh(2\pi\frac{\omega_-}{\rho}) - \cosh(\pi\frac{qA_0}{\rho})}{2\sinh(\pi\frac{\omega_{\rm in}}{\rho})\sinh(\pi\frac{\omega_{\rm out}}{\rho})}. \tag{51}$$

Some representations of this expression are shown in Fig. 5. As in the scalar case, the number of particles decreases as $\rho \to 0$ and increases as $m \to 0$. For fermions, the relation $|\alpha_k|^2 + |\beta_k|^2 = 1$ implies that $|\beta_k|^2 \le 1$ for any value of k, according to Pauli's exclusion principle.

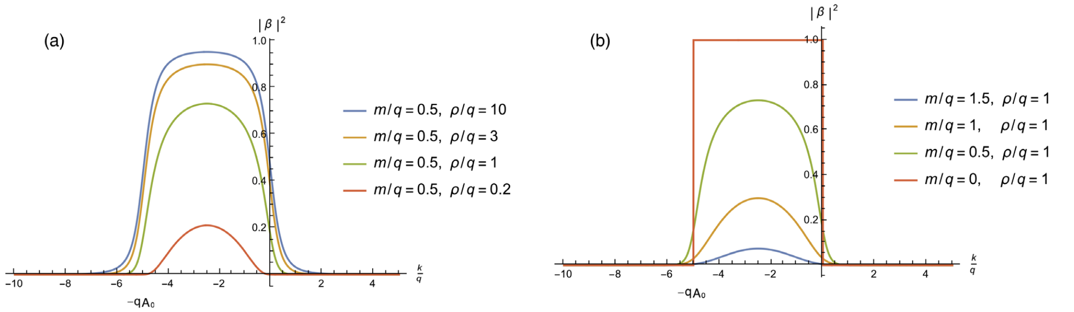

FIG. 5. Momentum distribution of the created fermions with positive charge at late times $|\beta_k|^2$ by an electric pulse with $A_0 = 5$ and different values of m/q and ρ/q. In Fig. 5(a) the mass is fixed, while in (b) the parameter of adiabaticity is fixed. Note that the relative position of the of the curves $|\beta_k|^2$ with respect to the vertical axis is different to the scalar plots because of the opposite convention for the electric charge, as explained in the main text.

In the massless case, irrespective of the value of ρ, one obtains [see Fig. 5(b)]

$$\lim_{m\to 0} |\beta_k|^2 = 1 \tag{52}$$

for $k \in (0, qA_0)$, and hence,

$$N_k = \begin{cases} 0 & \text{for } k \notin (-qA_0, qA_0) \\ 1 & \text{for } k \in (-qA_0, qA_0) \end{cases}. \tag{53}$$

The total density of created quanta is

$$\langle N \rangle = \frac{1}{2\pi} \int_{-|qA_0|}^{|qA_0|} dk\, N_k = \frac{|qA_0|}{\pi}. \tag{54}$$

Note that the same result is obtained by performing the adiabatic limit $\rho \to 0$ in the scalar case. In contrast, this result is valid for any value of ρ, which means that the number of created massless fermions does not depend on the history of $A(t)$, but only on its final value. This nonvanishing result of the particle number implies again the breaking of the adiabatic invariance.

For massive fermions and in the limit $\rho \to 0$, expression (51) behaves essentially as

$$|\beta_k|^2 \sim e^{-\frac{\pi}{\rho}\delta}, \tag{55}$$

where $\delta = 2\omega_+ - |qA_0|$. For $m \neq 0$, the former has a minimum at $k = -\frac{qA_0}{2}$, with a value $\delta_{\text{min}} = \sqrt{(qA_0)^2 + 4m^2} - |qA_0| > 0$. Hence, $\delta > 0$ and $|\beta_k|^2 \to 0$, as we can see in Fig. 5(a). Therefore we can conclude that the particle number is an adiabatic invariant for massive fermions, as in the scalar case. To visualize this behavior, we have depicted in Fig. 6 the dependence of the total density of created particles on the parameter ρ. We can also observe that the density of quanta in the massless case does not vanish and, in contrast to the scalar case, it remains constant, according to the above calculations.

A. Electric current

Using the renormalization method described in [17,21,22] for a Dirac field interacting with an homogeneous time-dependent electric field, the vacuum expectation value of the electric current $j^\mu = -q\bar{\psi}\gamma^\mu\psi$ is given by

$$\langle j^x \rangle_{\text{ren}} = \frac{q}{2\pi} \int dk \left(|h_k^{II}|^2 - |h_k^I|^2 - \frac{k}{\omega} - \frac{qm^2}{\omega^3} A \right). \tag{56}$$

To study the explicit dependence of the electric current $\langle j^x \rangle$ with the mass, we can compute their time derivative,

$$\partial_t \langle j^x \rangle_{\text{ren}} = \frac{2qm}{\pi} \left(\int \text{Im}(h_k^{II} h_k^{I*}) dk \right) - \frac{q^2}{\pi} \dot{A}. \tag{57}$$

It is immediate to see that in the massless limit the first term vanishes, and the equation below can be easily integrated. With $A(-\infty) = 0$ as initial condition one obtains

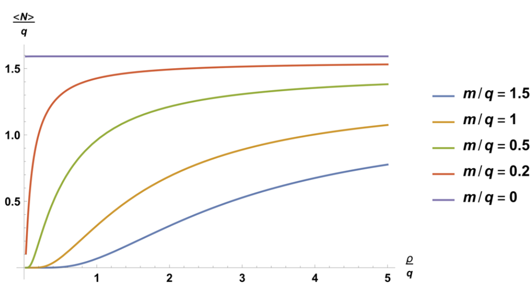

FIG. 6. Number of late-time created fermions as a function of the dimensionless adiabaticity parameter ρ/q, for $A_0 = 5$ and for different values of the mass.

$$\langle j^x \rangle_{\rm ren} = -\frac{q^2 A(t)}{\pi}. \tag{58}$$

Note again that at each instant t, the value of the electric current depends only on the value of the potential vector $A(t)$ at t, and not on its history. This contrasts with the behavior of the created current for a massive field, since h_k^I and h_k^{II} depend on the particular shape of the electric pulse. In the latter case, and assuming the electric pulse configuration given in (12), it can be proven by using the Bogoliubov coefficient (51) and repeating the same calculations done in Sec. III A that the electric current vanishes in the adiabatic limit. The analysis of the renormalized energy density can be carried out analogously, and it leads to a similar physical conclusion.

As a final comment, and for completeness, we remark that in the massless case, one can directly solve the semiclassical Maxwell equations for the electric field $\dot{E} = -\langle j^x \rangle_{\rm ren}$. Assuming, for instance, the initial conditions $A(0) = 0$ and $\dot{A}(0) = -E_0$ the previous equation can be easily integrated, with solution $E(t) = E_0 \cos(\frac{|q|}{\sqrt{\pi}} t)$. We find harmonic oscillations with frequency $\frac{|q|}{\sqrt{\pi}}$. This is consistent with the well-known fact that radiative corrections to the Schwinger model induce a mass for the "photon", with a value $m_\gamma^2 = q^2/\pi$ [12,23].

B. Relation with the axial anomaly

We have found that the expected adiabatic invariance of the particle number observable fails for a massless Dirac field. This is accompanied with a nonvanishing electric current, even in the adiabatic limit, as can be read from (58). Furthermore, this result brings about a creation of chirality as a consequence of the fact that, in two-dimensions, the axial current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is related by duality to the electric current $\langle j_\mu \rangle_{\rm ren} = q\epsilon_{\mu\nu}\langle j_5^\nu \rangle_{\rm ren}$. Hence, the result (58) implies the axial anomaly [24],

$$\partial_\mu \langle j_5^\mu \rangle_{\rm ren} = -\frac{q}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{59}$$

In fact, one can also interpret the breaking of the adiabatic invariance as a natural and necessary consequence required by the axial anomaly. We remark that the loss of the adiabatic invariance of the particle number for a scalar field in two-dimensions, which coincides quantitatively with the result for fermions, can also be naturally interpreted in the language of anomalies. In two-dimensions, a massless scalar field inherits a classical chiral-type symmetry, in the sense that the classical wave equation splits into two disconnected sectors: right and left-moving degrees of freedom, as the fermionic two-dimensional field. The corresponding right and left electric currents are, in the adiabatic limit, separately conserved in the classical theory. However, in the quantum theory these currents also cease to be conserved. The creation of right and left electric currents in the quantum theory is exactly the same for massless scalar and Dirac fields in the adiabatic limit, as can be easily observed from (58) and (30).

V. GENERALIZATION OF PREVIOUS RESULTS TO 4D

In the previous sections we have shown that the particle number operator is not an adiabatic invariant for two-dimensional massless fields. Here, we extend our analysis to four dimensions for both scalar and fermionic fields. We briefly study whether the breaking of the adiabatic invariance could also happen in electric and magnetic backgrounds.

A. Scalar field

Consider now a charged scalar field obeying the wave equation $(D_\mu D^\mu + m^2)\phi = 0$, where we assume an homogeneous electric pulse defined by the vector potential $A_\mu = (0, 0, 0, -A(t))$ with $A(t)$ given again by (12). The Fourier expansion of the quantized field is

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{2(2\pi)^3}} \int d^3k [A_{\vec{k}} e^{i\vec{k}\vec{x}} h_{\vec{k}}(t) + B_{\vec{k}}^\dagger e^{-i\vec{k}\vec{x}} h_{-\vec{k}}^*(t)]. \tag{60}$$

The mode functions $h_{\vec{k}}(t)$ satisfy the normalization condition $h_{\vec{k}}\dot{h}_{\vec{k}}^* - h_{\vec{k}}^*\dot{h}_{\vec{k}} = 2i$, and their time evolution is given by

$$\ddot{h}_{\vec{k}}(t) + (m^2 + k_1^2 + k_2^2 + (k_3 - qA(t))^2) h_{\vec{k}}(t) = 0. \tag{61}$$

This equation is very similar to the one found in the two-dimensional case (10). It allows us to partially reduce the four-dimensional problem to a two-dimensional one, by introducing an effective mass $m_{\rm eff}^2 = m^2 + k_1^2 + k_2^2$. Therefore, the beta coefficients can be obtained from Eq. (22) replacing k by k_3 and m by $m_{\rm eff}$.

According to our previous results for scalar fields, only for $m = 0$ and $k_1 = 0 = k_2$ one can have a nonvanishing $|\beta_k|^2$ in the adiabatic limit. However, since k_1 and k_2 are continuous quantum numbers characterizing the modes, the amount of created particles $\langle N \rangle \sim \int d^3k(|\beta_k|^2 + |\beta_{-k}|^2)$ is diluted into the infinite-volume of the unbounded three-dimensional space. Therefore, the total number density of produced particles turns out to be an adiabatic invariant.

This result cannot be altered by the introduction of a magnetic field. Adding a constant magnetic field $\vec{B}$ in the z-direction and choosing $A_\mu = (0, 0, -Bx^1, -A(t))$, the Fourier expansion for the scalar field is

$$\phi(t,\vec{x}) = \frac{1}{\sqrt{2(2\pi)^2}}\sum_n \int\int dk_2 dk_3 [A_{n,k_2,k_3} e^{i(k_2x^2+k_3x^3)}\Phi_{n,k_2}(x^1)h_{n,k_3}(t) + B^\dagger_{n,k_2,k_3} e^{-i(k_2x^2+k_3x^3)}\Phi_{n,-k_2}(x^1)h^*_{n,-k_3}(t)], \quad (62)$$

where

$$\Phi_{n,k_2}(x^1) = \left(\frac{qB}{\pi}\right)^{1/4} \frac{1}{2^{\frac{n}{2}}\sqrt{n!}} e^{-\xi^2/2} H_n(\xi), \quad (63)$$

$\xi = \sqrt{qB}(x^1 - k_2/qB)$, and $H_n(\xi)$ are the Hermite polynomials with $n = 0, 1, 2, \ldots$. For simplicity, and without loss of generality, we have assumed $qB > 0$. The time evolution is given by

$$\ddot{h}_{n,k_3} + (m^2 + (2n+1)qB + (k_3 - qA(t))^2)h_k(t) = 0. \quad (64)$$

From the two-dimensional viewpoint, the effective value of the mass, given now by $m^2_{\rm eff} = m^2 + (2n+1)qB$, is a positive quantity, even for $m = 0$ and $n = 0$. Using again the result of Sec. III we can similarly conclude that the particle number, defined now as

$$\langle N\rangle = \frac{qB}{4\pi^2}\sum_{n=0}^\infty \int_{-\infty}^\infty dk_3 N_{n,k_3} = \frac{qB}{4\pi^2}\sum_{n=0}^\infty \int_{-\infty}^\infty dk_3(|\beta_{n,k_3}|^2 + |\beta_{n,-k_3}|^2), \quad (65)$$

is also an adiabatic invariant for a scalar field in four dimensions, regardless of the value of the mass, given that $|\beta_{n,k_3}|^2 \to 0$. This is in sharp contrast with the result obtained for a massless scalar field in two dimensions. Note that for a scalar field in four dimensions there is no analog of the axial anomaly.

B. Dirac field

We can repeat the analysis for Dirac fermions. For massive fermions adiabatic invariance is preserved. Therefore we will focus on the massless case. In the latter, one can split the Dirac spinor in two independent chiral parts $\psi = \binom{\psi_L}{\psi_R}$. For the left sector the Weyl equation reads $\partial_0\psi_L - \vec{\sigma}\vec{D}\psi_L = 0$. Considering an homogeneous electric pulse with vector potential $A_\mu = (0,0,0,-A(t))$ given by (12), the Fourier expansion of the quantized field is

$$\psi_L(t,\vec{x}) = \int d^3k[B_{\vec{k}}u_{\vec{k}}(t,\vec{x}) + D^\dagger_{\vec{k}}v_{\vec{k}}(t,\vec{x})]. \quad (66)$$

The two independent and normalized spinor solutions can be expressed as

$$u_{\vec{k}}(t,\vec{x}) = \frac{e^{i\vec{k}\vec{x}}}{(2\pi)^{3/2}k_\perp}\begin{pmatrix}(k_1 - ik_2)h^I_{\vec{k}}(t)\\ k_\perp h^{II}_{\vec{k}}(t)\end{pmatrix} \quad (67)$$

$$v_{\vec{k}}(t,\vec{x}) = \frac{e^{-i\vec{k}\vec{x}}}{(2\pi)^{3/2}k_\perp}\begin{pmatrix}(k_1 - ik_2)h^{II*}_{-\vec{k}}(t)\\ k_\perp h^{I*}_{-\vec{k}}(t)\end{pmatrix}, \quad (68)$$

where $k_\perp = \sqrt{k_1^2 + k_2^2}$. The equations for the modes are

$$\dot{h}^I_{\vec{k}} - i(k_3 + qA)h^I_{\vec{k}} - ik_\perp h^{II}_{\vec{k}} = 0$$
$$\dot{h}^{II}_{\vec{k}} + i(k_3 + qA)h^{II}_{\vec{k}} - ik_\perp h^I_{\vec{k}} = 0. \quad (69)$$

These equations are similar to the ones found in the two-dimensional case (41) and (42), with an effective mass $m_{\rm eff} = k_\perp$. Hence, the beta coefficients are given by Eq. (51) with the obvious replacements. As in the scalar case, only for $k_1 = k_2 = 0$ one can have a nonvanishing beta coefficient in the adiabatic limit; therefore the amount of created particles is diluted and the total number density of produced particles is an adiabatic invariant. However, this is no longer true in the presence of a magnetic field. Adding a constant magnetic field $\vec{B}$ in the z-direction and choosing $A_\mu = (0,0,Bx^1,-A(t))$, the generic form of the modes for a massless field is

$$u_{n,k_2,k_3}(t,\vec{x}) = \frac{e^{i(k_2x^2+k_3x^3)}}{2\pi}\begin{pmatrix}h^I_{n,k_3}(t)\Phi_{n,k_2}(x^1)\\ -ih^{II}_{n,k_3}(t)\Phi_{n-1,k_2}(x^1)\end{pmatrix} \quad (70)$$

$$v_{n,k_2,k_3}(t,\vec{x}) = \frac{e^{-i(k_2x^2+k_3x^3)}}{2\pi}\begin{pmatrix}h^{II*}_{n,-k_3}(t)\Phi_{n,-k_2}(x^1)\\ ih^{I*}_{n,-k_3}(t)\Phi_{n-1,-k_2}(x^1)\end{pmatrix}, \quad (71)$$

where Φ_{n,k_2} is defined as in the scalar case (63). The time evolution of the modes is given by

$$\dot{h}^I_{n,k_3} - i(k_3 + qA)h^I_{n,k_3} - i\sqrt{2nqB}h^{II}_{n,k_3} = 0$$
$$\dot{h}^{II}_{n,k_3} + i(k_3 + qA)h^{II}_{n,k_3} - i\sqrt{2nqB}h^I_{n,k_3} = 0. \quad (72)$$

In this case, we can identify the effective mass as $m^2_{\rm eff} = 2nqB$, which vanishes at $n = 0$. Therefore, in the adiabatic limit, the beta coefficients $|\beta_{n,k_3}|^2$ [we recall they can also be obtained from the two-dimensional analog (51)] vanish for any value of n except for $n = 0$. Since $\langle N\rangle \sim \sum_n \int dk_3(|\beta_{n,k_3}|^2 + |\beta_{n,-k_3}|^2)$, the particle number tends to a nonzero value because the discrete state $n = 0$

survives after summation. This contrasts with the previous case in which the mode $k_1 = k_2 = 0$ was diluted after integration. Hence, the particle number $\langle N \rangle$ is no longer adiabatic invariant.

This result is also linked to the axial anomaly, as happens in two dimensions. Note that in four dimensions the anomaly is only nonzero when both electric and magnetic fields are present. As in the two-dimensional case, the adiabatic anomaly must be reflected in the electric current $\langle j^z \rangle = -q\langle \bar{\psi}\gamma^3\psi \rangle$ and also in the chiral charge density $\langle j_5^0 \rangle = \langle \bar{\psi}\gamma^0\gamma^5\psi \rangle$. Repeating the previous analysis for the right part ψ_R and computing the formal vacuum expectation value $\langle j^z \rangle$ one finds

$$\langle j^z \rangle = \frac{q^2 B}{4\pi^2}\int_{-\infty}^{\infty} dk_3 (|h^{II}_{0,k_3}|^2 - |h^{I}_{0,k_3}|^2) + \frac{q^2 B}{2\pi^2}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} dk_3 (|h^{II}_{n,k_3}|^2 - |h^{I}_{n,k_3}|^2). \tag{73}$$

From this result one can easily see the special role of the $n = 0$ modes, which are the only ones contributing to the breaking of the adiabatic invariance. Although in the most general case all the modes contribute to the electric current, in the adiabatic limit the contribution of the modes with $n > 0$, for which $m_{\rm eff} \neq 0$, will vanish, as happens in the two-dimensional case. This gives us a lower bound for the current. On the other hand, by looking at the chiral charge,

$$\langle j_5^0 \rangle = \frac{qB}{4\pi^2}\int_{-\infty}^{\infty} dk_3 (|h^{I}_{0,k_3}|^2 - |h^{II}_{0,k_3}|^2), \tag{74}$$

one realizes that only the mode with $n = 0$ creates chirality, even in a nonadiabatic regime. Furthermore, it is immediate to see that the lower bound of the electric current is given by $\langle j^z \rangle_{\rm min} = -q\langle j_5^0 \rangle$.

Note that (74) can be renormalized using the adiabatic prescription in two dimensions [see Eq. (56)] and the result is compatible with the axial anomaly $\langle j_5^0 \rangle_{\rm ren}(t) = -\frac{q^2}{2\pi^2}\int_{-\infty}^{t} dt' \vec{E}(t')\vec{B}$. It can be easily argued that a similar result can also be obtained for a time-dependent magnetic field.

VI. CONCLUSIONS

We have reexamined the adiabatic invariance of the particle number operator of quantized fields in two dimensions coupled to a background electric field with bounded vector potential. We have pointed out that, for massless fields, the expected adiabatic invariance fails. This fact is accompanied by the emergence of the axial anomaly in two dimensions. In other words, the breaking of the adiabatic invariance (pair creation even in the limit $\rho \to 0$) is required to keep physical consistency with the axial anomaly. We have also shown that the breaking of the adiabatic invariance is also reproduced for a massless Dirac field in four dimensions, but requiring the presence of electric and magnetic fields, showing up again a deep connection with the axial anomaly.

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Article 3

Adiabatic regularization for Dirac fields in time-varying electric backgrounds

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The adiabatic regularization method was originally proposed by Parker and Fulling to renormalize the energy-momentum tensor of scalar fields in expanding universes. It can be extended to renormalize the electric current induced by quantized scalar fields in a time-varying electric background. This can be done in a way consistent with gravity if the vector potential is considered as a variable of adiabatic order one. Assuming this, we further extend the method to deal with Dirac fields in four spacetime dimensions. This requires a self-consistent ansatz for the adiabatic expansion, in presence of a prescribed time-dependent electric field, which is different from the conventional expansion used for scalar fields. Our proposal is consistent, in the massless limit, with the conformal anomaly. We also provide evidence that our proposed adiabatic expansion for the fermionic modes parallels the Schwinger-DeWitt adiabatic expansion of the two-point function. We give the renormalized expression of the electric current and analyze, using numerical and analytical tools, the pair production induced by a Sauter-type electric pulse. We also analyze the scaling properties of the current for a large field strength.

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I. INTRODUCTION

The landmark work of Heisenberg and Euler [1], motivated by earlier work of Sauter [2], established the instability of the quantum vacuum under the influence of a prescribed (slowly varying) electric field. If the field is sufficiently strong, real electron-positron pairs can be created. This result was reobtained by Schwinger in the modern language of quantum electrodynamics by finding a positive imaginary contribution to the effective action W. The quantity e^{-2ImW} represents then the probability that no actual pair creation occurs during the history of the field [3].

The quantum mechanism driving the spontaneous creation of particles by a gravitational field was discovered by Parker in the early sixties by studying quantized fields in an expanding universe. The crucial fact is as follows [4]: creation and annihilation operators evolve, under the influence of the expansion of the universe (or a generic time-varying gravitational field), into a superposition of creation and annihilation operators. During a cosmic period when the expansion factor is almost constant one can interpret the effect of the gravitational field on the particle number and unambiguously establish the spontaneous creation of real particles by the evolving gravitational field. Major applications of this remarkable phenomena occurs in the very early universe [5,6] and in the vicinity of a collapsing star forming a black hole [7]. These pioneer works on particle creation launched the theory of quantum fields in curved spacetime, as a first step to merge gravity and quantum mechanics within a self-consistent and successful framework [8–11]. The underlying machinery was also employed to study time-varying electromagnetic fields [12,13]. In the limit of a slowly varying electric field the Schwinger result can be recovered.

In the gravitational scenario, the most relevant physical observable is the energy-momentum tensor. Its vacuum expectation value $\langle T_{\mu\nu} \rangle$ possesses ultraviolet (UV) divergences and has to be regularized and renormalized. In the seventies many methods were proposed to this end, as explained in the monographs [8–11]. For homogeneous, time-dependent spacetimes a generic expression for $\langle T_{\mu\nu} \rangle$ was obtained for scalar fields within the so-called adiabatic regularization scheme [14–18]. The adiabatic method uses a mode by mode subtraction process, naturally suggested by the definition of a single-particle state in an expanding universe, and in such a way that preserves the basic symmetries of the theory. Furthermore, the adiabatic method has been proved to be equivalent to the point-splitting Schwinger-DeWitt renormalization scheme [15,16,18].

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The adiabatic expansion of the field modes parallels the Schwinger-DeWitt adiabatic expansion of the Feynman propagator in Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetimes. The advantage of the adiabatic regularization method is that it very efficient to implement numerical computations, and it is widely used in cosmology. It has been recently extended to spin-$1/2$ fields in FLRW universes [19,20].

As mentioned above, the analysis of particle creation by time-varying electric fields can be carried out using the techniques first proposed to treat curved backgrounds. The electromagnetic field is considered as an external, unquantized background, while the created particles are excitations of the quantized matter field. From the experimental side, this particle production effect is also of special interest since it may not be far from being experimentally detected in high intensity lasers [21], and in beam-beam collisions [22]. This effect is also very important in astrophysical [23,24] and cosmological scenarios [25–27], and in nonequilibrium processes induced by strong fields [28]. In this context, the most important physical local expectation value is the electric current $\langle j_\mu \rangle$, which also possesses ultraviolet divergences and has to be renormalized in a proper way. Recent discussions on foundational issues related to the particle number density of the created particles, adiabatic invariance, and unitary evolution can be seen in [29–31].

Due to the similarities with the gravitational case, it is a good strategy to readapt the adiabatic regularization scheme to the case in which the external background is an electric field. This program was initiated in [32,33] to study backreaction problems when the matter field is a charged scalar field. It was further extended to treat charged Dirac fermions [34]. It was assumed that the adiabatic order of the vector potential A_μ is 0. The problem was reconsidered for a charged scalar field in [35] by assuming that the adiabatic order of A_μ is 1, instead of zero. This new reassignment of the adiabatic order for A_μ is an unavoidable requirement in presence of a gravitational background. The argument was reinforced in [36,37] on the basis of the covariant conservation of the energy-momentum tensor. The adiabatic regularization of two-dimensional fermions incorporating the adiabatic order assignment 1 for A_μ has been further reanalyzed in [31,35,38]. Other renormalization methods have been generalized to incorporate an electromagnetic background, as for instance the Hadamard point-splitting method for complex scalar fields [39,40], with results in agreement with [35].

Within the above context, it seems natural to extend the adiabatic regularization/renormalization method, with the assumption that A_μ is of adiabatic order 1, to Dirac fields in presence of an electric field background in four spacetime dimensions. This is the main aim of this work. As stressed above, previous studies in the literature on this problem [34] assumed that A_μ is of adiabatic order 0. This extension requires a self-consistent ansatz for the adiabatic expansion of the field modes. We give a proper ansatz, which cannot be fitted within the Wentzel-Kramers-Brillouin (WKB)-type expansion used for scalar fields [8,10,11]. Our extension of the adiabatic method is in agreement with the trace anomaly. Even more, we provide strong evidence that our adiabatic expansion of the field modes parallels the adiabatic Schwinger-DeWitt expansion of the propagator. In addition to the trace anomaly, our adiabatic expansion also reproduces the DeWitt coefficient E_3, at sixth adiabatic order. We carry out the adiabatic renormalization and provide a general expression for the renormalized electric current. We illustrate the power of the method by studying with detail a Sauter-type electric pulse.

The paper is organized as follows. In Sec. II we will describe the status of adiabatic regularization when a time-varying electric field is part of the background. We will give strong reasons for adopting a new viewpoint and reprehend the problem of the adiabatic regularization of charged $4d$ fermions in time-dependent electric fields. In Sec. III we introduce the basic ingredients of our ansatz to construct the adiabatic expansion of the four-dimensional fermionic modes coupled to a prescribed time-dependent electric field. Section IV is devoted to explain the details of the adiabatic renormalization procedure in this context. In particular, we give a generic and explicit expression of the renormalized electric current. We also test the consistency of the method and discuss some intrinsic renormalization ambiguities. In Sec. V we study the particular case in which the background field is a Sauter-type electric pulse. We analyze the particle production phenomena in terms of the renormalized electric current. We also discuss the scaling properties of the created current. In Sec. VI we state our main conclusions. Our work is complemented with a series of appendices where we give technical details. We also discuss in the Appendix B the connection between the adiabatic method and the Hadamard renormalization scheme for charged scalar fields.

II. BACKGROUND AND MOTIVATION

To motivate the main idea of this work it is very convenient to present the status of the adiabatic regularization method for a charged 4-dimensional scalar field interacting with a classical, homogeneous, time-dependent electric background. We will assume that the electric field is of the form $\vec{E} = (0, 0, E(t))$ with potential vector $A_\mu = (0, 0, 0, -A(t))$. We will also assume that the spacetime is described by a FLRW metric of the form $ds^2 = dt^2 - a^2(t) d\vec{x}^2$. The Klein Gordon equation reads

$$(D_\mu D^\mu + m^2 + \xi R)\phi = 0, \tag{1}$$

where $D_\mu \phi = (\nabla_\mu + iqA_\mu)\phi$ and R is the Ricci scalar. Since the potential vector A_μ is homogeneous, one can expand the scalar field in modes as $\phi = \frac{1}{\sqrt{2(2\pi a)^3}} \int d^3k (A_{\vec{k}} e^{i\vec{k}\vec{x}} h_{\vec{k}} + B_{\vec{k}} e^{-i\vec{k}\vec{x}} h^*_{-\vec{k}})$, where the mode functions $h_{\vec{k}}(t)$ satisfy

$$\ddot{h}_{\vec{k}} + (a^{-2}(k_3 + qA)^2 + a^{-2}k_\perp^2 + m^2 + \sigma)h_{\vec{k}} = 0, \quad (2)$$

with $\sigma = (6\xi - 3/4)\dot{a}^2/a^2 + (6\xi - 3/2)\ddot{a}/a$. Once we have obtained the mode equation (2), we can make an adiabatic expansion of the field modes. To this end, one can propose the usual WKB ansatz

$$h_{\vec{k}} = \frac{1}{\sqrt{\Omega_{\vec{k}}}} e^{-i\int \Omega_{\vec{k}}(t)dt}, \quad (3)$$

where W_k can be expanded adiabatically, in powers of derivatives of $a(t)$ and $A(t)$, as $\Omega_{\vec{k}} = \sum_{n=0}^{\infty} \omega^{(n)}$.

The choice of the leading terms $\omega^{(0)}$ is a crucial ingredient to define the adiabatic expansion. For $A = 0$ the proper choice for $\omega^{(0)}$ is $\omega^{(0)} = \omega = \sqrt{\vec{k}^2/a^2 + m^2}$. This defines the conventional adiabatic expansion for a scalar field, as first introduced in the pioneer works [14]. When the background spacetime is Minkowski $a = 1$, the choice proposed in [32] was $\omega^{(0)} = \omega = \sqrt{(\vec{k} - q\vec{A})^2 + m^2}$. This choice assumes that $A(t)$ should be treated as a variable of zero adiabatic order, like $a(t)$. As noted in [35,36], this choice runs into difficulties in presence of a gravitational background. It was proposed in [35,36] that the leading term should be maintained as $\omega^{(0)} = \omega = \sqrt{\vec{k}^2/a^2 + m^2}$, even in the presence of an electromagnetic field. This means that $A(t)$ must be considered as a variable of adiabatic order 1, like $\dot{a}$. Hence $\dot{A}$ is of adiabatic order 2, etc.

Next to leading order terms can be obtained recursively from (2). The adiabatic expansion allows us to regularize the observables performing adiabatic subtractions. Since the first terms of the adiabatic expansion capture all potential ultraviolet divergences, one can subtract them, obtaining finite and meaningful results. With this method, we obtain the following vacuum expectation value of the two point function

$$\langle \phi^\dagger \phi \rangle_{\rm ren} = \frac{1}{2(2\pi a(t))^3} \int d^3k [|h_{\vec{k}}|^2 - \langle \phi^\dagger \phi \rangle_{\vec{k}}^{(0-2)}], \quad (4)$$

where $\langle \phi^\dagger \phi \rangle_{\vec{k}}^{(0-2)} = \sum_{n=0}^{2} (\Omega_{\vec{k}}^{-1})^{(n)}$.

As stressed in the introduction, the adiabatic expansion of the field modes translates into an adiabatic expansion of the two-point function. The latter turns out to be equivalent to the Schwinger-DeWitt expansion of the Feynman propagator $\langle T\phi^\dagger(x)\phi(x')\rangle$. We will show this explicitly at the coincident limit $x' \to x$ at fourth and sixth adiabatic orders. At fourth adiabatic order the corresponding momentum integral is finite and we get

$$\begin{aligned} \langle \phi^\dagger \phi \rangle^{(4)} &= \frac{1}{2(2\pi a)^3} \int d^3k \langle \phi^\dagger \phi \rangle_{\vec{k}}^{(4)} \\ &= \frac{1}{16\pi^2 m^2} \left(\frac{36\dot{a}^2\xi^2\ddot{a}}{a^3} - \frac{17\dot{a}^2\xi\ddot{a}}{a^3} + \frac{29\dot{a}^2\ddot{a}}{15a^3} \right. \\ &\quad + \frac{18\xi^2\ddot{a}^2}{a^2} - \frac{5\xi\ddot{a}^2}{a^2} + \frac{3\ddot{a}^2}{10a^2} + \frac{18\dot{a}^4\xi^2}{a^4} - \frac{6\dot{a}^4\xi}{a^4} \\ &\quad \left. + \frac{a^{(4)}\xi}{a} + \frac{\dot{a}^4}{2a^4} - \frac{a^{(4)}}{5a} + \frac{\dot{A}^2q^2}{6a^2} + \frac{3a^{(3)}\dot{a}\xi}{a^2} - \frac{3a^{(3)}\dot{a}}{5a^2} \right). \end{aligned} \quad (5)$$

It is not difficult to check that the above result can be reexpressed in the following covariant form

$$\langle \phi^\dagger \phi \rangle^{(4)} = \frac{E_2}{16\pi^2 m^2}, \quad (6)$$

where E_2 matches exactly the DeWitt coefficient [41]

$$\begin{aligned} E_2 = &-\frac{1}{30}\Box R + \frac{1}{72}R^2 - \frac{1}{180}R^{\mu\nu}R_{\mu\nu} + \frac{1}{180}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\ &- \frac{q^2}{12}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}Q^2 - \frac{1}{6}RQ + \frac{1}{6}\Box Q. \end{aligned} \quad (7)$$

In the above expression Q is given by $Q = \xi R$. We note that, to obtain equivalence with the Schwinger-DeWitt proper-time method it has been essential to assume that A_μ is of adiabatic order 1, $F_{\mu\nu}$ of adiabatic order 2, etc. With the zero adiabatic order assignment for A_μ one obtains a noncovariant and ill-defined expression for $\langle \phi^\dagger \phi \rangle^{(4)}$. With our proposed leading order choice for $\omega^{(0)} = \omega = \sqrt{\vec{k}^2/a^2 + m^2}$ we find equivalence with the Schwinger-DeWitt expansion at very nontrivial higher orders. For instance, our calculation for $\langle \phi^\dagger \phi \rangle^{(6)}$ gives

$$\begin{aligned}\langle\phi^\dagger\phi\rangle^{(6)} &= \frac{1}{2(2\pi a)^3}\int d^3k\langle\phi^\dagger\phi\rangle^{(6)}_{\vec{k}}\\ &= -\frac{108\dot{a}^4\xi^3\ddot{a}}{a^5}+\frac{96\dot{a}^4\xi^2\ddot{a}}{a^5}-\frac{217\dot{a}^4\xi\ddot{a}}{10a^5}+\frac{197\dot{a}^4\ddot{a}}{140a^5}-\frac{108\dot{a}^2\xi^3\ddot{a}^2}{a^4}+\frac{75\dot{a}^2\xi^2\ddot{a}^2}{a^4}-\frac{221\dot{a}^2\xi\ddot{a}^2}{10a^4}+\frac{159\dot{a}^2\ddot{a}^2}{70a^4}-\frac{\dot{A}^2\xi q^2\ddot{a}}{a^3}\\ &+\frac{\dot{a}\dot{A}q^2\ddot{a}}{90a^3}+\frac{23\dot{A}^2q^2\ddot{a}}{90a^3}-\frac{36\xi^3\ddot{a}^3}{a^3}-\frac{24a^{(3)}\dot{a}\xi^2\ddot{a}}{a^3}+\frac{12\xi^2\ddot{a}^3}{a^3}+\frac{133a^{(3)}\dot{a}\xi\ddot{a}}{10a^3}+\frac{3\xi\ddot{a}^3}{5a^3}-\frac{103a^{(3)}\dot{a}\ddot{a}}{60a^3}-\frac{14\ddot{a}^3}{45a^3}-\frac{q^2\ddot{a}^2}{20a^2}\\ &-\frac{6a^{(4)}\xi^2\ddot{a}}{a^2}+\frac{17a^{(4)}\xi\ddot{a}}{10a^2}-\frac{43a^{(4)}\ddot{a}}{420a^2}-\frac{36\dot{a}^6\xi^3}{a^6}+\frac{6\dot{a}^6\xi^2}{a^6}+\frac{\dot{a}^6\xi}{a^6}-\frac{a^{(6)}\xi}{10a}-\frac{\dot{a}^6}{6a^6}+\frac{3a^{(6)}}{140a}-\frac{\dot{a}^2\dot{A}^2\xi q^2}{a^4}+\frac{37\dot{a}^2\dot{A}^2q^2}{180a^4}\\ &-\frac{\dot{A}A^{(3)}q^2}{15a^2}-\frac{2a^{(5)}\dot{a}\xi}{5a^2}+\frac{3a^{(5)}\dot{a}}{35a^2}-\frac{6a^{(3)}\dot{a}^3\xi^2}{a^4}-\frac{6a^{(4)}\dot{a}^2\xi^2}{a^3}+\frac{7a^{(3)}\dot{a}^3\xi}{10a^4}+\frac{31a^{(4)}\dot{a}^2\xi}{10a^3}+\frac{13a^{(3)}\dot{a}^3}{140a^4}-\frac{23a^{(4)}\dot{a}^2}{60a^3}\\ &-\frac{3(a^{(3)})^2\xi^2}{a^2}+\frac{7(a^{(3)})^2\xi}{10a^2}-\frac{(a^{(3)})^2}{42a^2},\end{aligned} \tag{8}$$

where $a^{(n)}$ refers to d^na/dt^n. The result turns out to be proportional to the corresponding DeWitt coefficient of sixth adiabatic order E_3 [42,43]. The covariant expression is given in the Appendix A.

We stress again that it has been crucial for obtaining the above results the choice $\omega^{(0)}=\omega=\sqrt{\vec{k}^2/a^2+m^2}$, instead of $\omega^{(0)}=\omega=\sqrt{(\vec{k}-q\vec{A})^2/a^2+m^2}$. For completeness, a comparison of the above formulation of the adiabatic regularization with the Hadamard renormalization scheme is given in the Appendix B.

A. Adiabatic regularization for fermions in two-dimensions

To reinforce the previous analysis, and prior to face the adiabatic regularization of charged fermions in four spacetime dimensions, it is also convenient to consider the problem for a charged Dirac field in two-dimensions. We will follow [35,36] and compare the results with the pioneer analysis in [34]. The comparison will allow us to understand why it has been necessary to reprehend the problem, as already stressed above.

The quantum field satisfies the Dirac equation $(i\underline{\gamma}^\mu D_\mu - m)\psi = 0$, where $D_\mu \equiv \nabla_\mu - \Gamma_\mu - iqA_\mu$ and Γ_μ is the spin connection. The curved space Dirac matrices satisfy the anticommutation relations $\{\underline{\gamma}^\mu,\underline{\gamma}^\mu\} = 2g^{\mu\nu}$. We assume a homogeneous, time-dependent electric background $E(t)$, with associated potential vector $A_\mu = (0,-A(t))$. The metric is also assumed of the FLRW form $ds^2 = dt^2 - a^2(t)dx^2$. One can expand the Dirac field as $\psi = \int_{-\infty}^{\infty} dk[B_k u_k(t,x) + D^\dagger v_k(x,t)]$, where the two independent spinor solutions can be written as

$$u_k(t,x)=\frac{e^{ikx}}{\sqrt{2\pi a}}\begin{pmatrix} h_k^I(t)\\ -h_k^{II}(t)\end{pmatrix},\quad v_k(t,x)=\frac{e^{-ikx}}{\sqrt{2\pi a}}\begin{pmatrix} h_{-k}^{II*}(t)\\ h_{-k}^{I*}(t)\end{pmatrix}. \tag{9}$$

The classical electric field satisfies the semiclassical Maxwell equations $\nabla_\mu F^{\mu\nu} = -q\langle\bar{\psi}\underline{\gamma}^\nu\psi\rangle_{\rm ren} = \langle j^\nu\rangle_{\rm ren}$, which in our system turns out to be a single equation $\dot{E} = -\langle j^x\rangle_{\rm ren}$. In this scenario the adiabatic rules are univocally fixed: $a(t)$ has to be considered of adiabatic order 0, the energy-momentum tensor must be regularized up to the second adiabatic order and the electric current must to be regularized up to the first adiabatic order. The adiabatic subtractions required to regularize the electric current $\langle j^x\rangle$ will be different depending on the adiabatic order that we choose for the background field $A(t)$, i.e.,

$$\langle j^x\rangle_{\rm ren}^{A\sim O(0)} = q\int\frac{dk}{2\pi a}\left(|h_k^{II}|^2 - |h_k^I|^2 - \frac{k+qA}{a\sqrt{(k+qA)^2/a^2+m^2}}\right), \tag{10}$$

$$\langle j^x\rangle_{\rm ren}^{A\sim O(1)} = q\int\frac{dk}{2\pi a}\left(|h_k^{II}|^2 - |h_k^I|^2 - \frac{k}{a\omega} - \frac{m^2qA}{a\omega^3}\right), \tag{11}$$

where $\omega = \sqrt{k^2/a^2+m^2}$. In (10) we have considered A of adiabatic order zero, while in (11) we have considered it of adiabatic order one. One can check that the subtractions obtained in the first case are the same to the ones obtained in [34] for $a=1$. Although it can be proven [37] that these two choices are equivalent when $a=1$, in the sense that $\triangle\langle j^x\rangle_{\rm ren} = \langle j^x\rangle_{\rm ren}^{A\sim O(0)} - \langle j^x\rangle_{\rm ren}^{A\sim O(1)} = 0$, they are in general nonequivalent. We can see how gravity breaks this equivalence. In the second case we can easily see that the energy density is covariantly conserved

$$\nabla_\mu\langle T^{\mu 0}\rangle_{\rm ren} + \nabla_\mu T^{\mu 0}_{\rm elec} = E(\dot{E} + \langle j^x\rangle_{\rm ren}) = 0. \tag{12}$$

But, when we consider $A(t)$ of adiabatic order 0, the conservation does not hold any more, and one finds $\nabla_\mu\langle T^{\mu 0}\rangle_{\rm ren} + \nabla_\mu T^{\mu 0}_{\rm elec} \sim E(t)\langle j^x\rangle^{(2)}$, where $\langle j^x\rangle^{(2)}$ is the subtraction term of adiabatic order two, which cannot be properly absorbed into the definition of the electric current.

Moreover, only when A is considered of adiabatic order 1 the adiabatic expansion of the field modes turns out to be equivalent to the Schwinger-DeWitt expansion of the two-point function. For instance, the adiabatic expansion of the two-point function at coincidence is found to be (at second and fourth adiabatic order)

$$\langle\bar{\psi}\psi\rangle^{(2)} = \frac{1}{4\pi m}\left(\frac{\ddot{a}}{3a}\right) = -\frac{\mathrm{tr}E_1}{4\pi m}, \tag{13}$$

$$\langle\bar{\psi}\psi\rangle^{(4)} = \frac{1}{4\pi m^3}\left(-\frac{\dot{a}^2\ddot{a}}{30a^3}+\frac{\ddot{a}^2}{15a^2}-\frac{a^{(4)}}{30a}+\frac{2\dot{A}^2q^2}{3a^2}+\frac{a^{(3)}\dot{a}}{30a^2}\right) = -\frac{\mathrm{tr}E_2}{4\pi m^3}, \tag{14}$$

where E_1 and E_2 are the corresponding DeWitt coefficients. They are given, in the covariant form, by [8,42,43]

$$E_1 = \frac{1}{6}RI - Q, \tag{15}$$

$$E_2 = \left(-\frac{1}{30}\Box R + \frac{1}{72}R^2 - \frac{1}{180}R^{\mu\nu}R_{\mu\nu} + \frac{1}{180}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right)I + \frac{1}{12}W^{\mu\nu}W_{\mu\nu} + \frac{1}{2}Q^2 - \frac{1}{6}RQ + \frac{1}{6}\Box Q, \tag{16}$$

where $Q=\frac{1}{4}RI-\frac{i}{2}qF_{\mu\nu}\underline{\gamma}^{\mu}\underline{\gamma}^{\nu}$ and $W_{\mu\nu}=-iqF_{\mu\nu}I-\frac{1}{4}R_{\mu\nu\rho\sigma}\underline{\gamma}^{\rho}\underline{\gamma}^{\sigma}$.

The above arguments make it necessary to reconsider the problem of adiabatic regularization for fermions in time-varying electric backgrounds in four dimensions. We will adopt the view of considering A_μ of adiabatic order 1, as advocated in [31,35,36,38], and in contrast to the view adopted in [34]. The main reasons, as exposed above, are (i) expected agreement with the Schwinger-DeWitt adiabatic expansion of the two-point function at coincidence; (ii) consistency with the covariant conservation of the energy-momentum tensor when gravity is turned on. We think these are convincing arguments to go further with our proposed approach. For simplicity we will restrict our analysis to Minkowski spacetime.

III. 4D DIRAC FIELDS: MODE EQUATIONS, ANSATZ AND ADIABATIC EXPANSION

Let us consider a massive 4-dimensional spinor field ψ interacting with a prescribed electric field. The corresponding Dirac equation reads

$$(i\gamma^\mu D_\mu - m)\psi = 0, \tag{17}$$

where $D_\mu \equiv \partial_\mu - iqA_\mu$ and γ^μ are the (flat-space) Dirac matrices satisfying the anticommutation relations $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$. We consider ψ as a quantized Dirac field, while the electromagnetic field is assumed to be a classical and spatially homogeneous field $\vec{E}(t) = (0,0,E(t))$. It is very convenient to choose a gauge such that only the z-component of the vector potential is nonvanishing: $A_\mu=(0,0,0,-A(t))$, where $E(t) = -\dot{A}(t)$.

To prepare things to propose a consistent ansatz for the adiabatic expansion of the field modes it is very important to transform the Dirac field as $\psi' = U\psi$, where U is the unitary operator $U = \frac{1}{\sqrt{2}}\gamma^0(I-\gamma^3)$, which verifies $U = U^\dagger = U^{-1}$. This transformation will allow us to express the Dirac field in terms of only two time-dependent functions [see (20)]. The field ψ' obeys the Dirac equation for the transformed matrices $\gamma'^\mu = U\gamma^\mu U^\dagger$, namely: $\gamma'^0 = \gamma^3\gamma^0$, $\gamma'^1 = -\gamma^3\gamma^1$, $\gamma'^2 = -\gamma^3\gamma^2$, $\gamma'^3 = -\gamma^3$. Substituting them in the Dirac equation we easily get

$$[\gamma^0\partial_0 - \gamma^1\partial_1 - \gamma^2\partial_2 - \partial_3 - iqA(t) - im\gamma^3]\psi' = 0. \tag{18}$$

Expanding the field in Fourier modes, $\psi'(t,\vec{x}) = \int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\psi'_{\vec{k}}(t)e^{i\vec{k}\vec{x}}$, we obtain the following equation

$$[\partial_0 - i\gamma^0(k_1\gamma^1 + k_2\gamma^2 + m\gamma^3) - i(k_3 + qA(t))\gamma^0]\psi'_{\vec{k}}(t) = 0, \tag{19}$$

where $\vec{k} \equiv (k_1,k_2,k_3)$. The form of the above equation allows us to reexpress the field in terms of two-component spinors as follows

$$\psi'_{\vec{k},\lambda}(t) = \begin{pmatrix} h^I_{\vec{k}}(t)\eta_\lambda(\vec{k}) \\ h^{II}_{\vec{k}}(t)\lambda\eta_\lambda(\vec{k}) \end{pmatrix}, \tag{20}$$

where η_λ with $\lambda = \pm 1$ form an orthonormal basis of two-spinors ($\eta^\dagger_\lambda\eta_{\lambda'} = \delta_{\lambda,\lambda'}$) verifying $\frac{k_1\sigma^1+k_2\sigma^2+m\sigma^3}{\sqrt{k_1^2+k_2^2+m^2}}\eta_\lambda = \lambda\eta_\lambda$. Their explicit expressions are

$$\eta_{+1}(\vec{k}) = \frac{1}{\sqrt{2\kappa(\kappa+m)}}\begin{pmatrix} \kappa+m \\ k_1+ik_2 \end{pmatrix}, \quad \eta_{-1}(\vec{k}) = \frac{1}{\sqrt{2\kappa(\kappa+m)}}\begin{pmatrix} -k_1+ik_2 \\ \kappa+m \end{pmatrix}, \tag{21}$$

where $\kappa \equiv \sqrt{k_1^2+k_2^2+m^2}$. Substituting (20) in (19) and using the Dirac representation for the matrices γ^μ, one obtains the following differential equations for the functions $h^I_{\vec{k}}$ and $h^{II}_{\vec{k}}$

$$\dot{h}^I_{\vec{k}} - i(k_3+qA)h^I_{\vec{k}} - i\kappa h^{II}_{\vec{k}} = 0, \tag{22}$$

$$\dot{h}^{II}_{\vec{k}} + i(k_3+qA)h^{II}_{\vec{k}} - i\kappa h^I_{\vec{k}} = 0. \tag{23}$$

These equations are exactly the same as those obtained in the two-dimensional case [35], where κ plays here the role of the mass. With the solutions of these equations we can construct the u-type field modes (assumed to be of positive frequency at early times) as follows

$$u_{\vec{k},\lambda}(x) = \frac{e^{i\vec{k}\vec{x}}}{(2\pi)^{3/2}} \begin{pmatrix} h^I_{\vec{k}}(t)\eta_\lambda(\vec{k}) \\ h^{II}_{\vec{k}}(t)\lambda\eta_\lambda(\vec{k}) \end{pmatrix}. \tag{24}$$

Similarly, one can construct the orthogonal v-type field modes (of negative frequency at early times) as

$$v_{\vec{k},\lambda}(x) = \frac{e^{-i\vec{k}\vec{x}}}{(2\pi)^{3/2}} \begin{pmatrix} -h^{II*}_{-\vec{k}}(t)\eta_{-\lambda}(-\vec{k}) \\ -h^{I*}_{-\vec{k}}(t)\lambda\eta_{-\lambda}(-\vec{k}) \end{pmatrix}. \tag{25}$$

The normalization conditions for this set of spinors, $(u_{\vec{k},\lambda}, v_{\vec{k}',\lambda'}) = 0$, $(u_{\vec{k},\lambda}, u_{\vec{k}',\lambda'}) = (v_{\vec{k},\lambda}, v_{\vec{k}',\lambda'}) = \delta^{(3)}(\vec{k}-\vec{k}')\delta_{\lambda\lambda'}$, where (,) is the Dirac inner product, are ensured with the normalization condition

$$|h^I_{\vec{k}}|^2 + |h^{II}_{\vec{k}}|^2 = 1, \tag{26}$$

which will be preserved on time. With this set of basic spinor solutions one can construct the Fourier expansion of the Dirac field operator

$$\psi'(x) = \sum_\lambda \int d^3\vec{k}[B_{\vec{k},\lambda}u_{\vec{k},\lambda}(x) + D^\dagger_{\vec{k},\lambda}v_{\vec{k},\lambda}(x)], \tag{27}$$

where $B_{\vec{k}\lambda}$ and $D_{\vec{k}\lambda}$ are the annihilation operators for particles and antiparticles respectively. The normalization condition (26) guaranties the usual anticommutation relations for these operators: $\{B_{\vec{k},\lambda}, B^\dagger_{\vec{k}',\lambda'}\} = \{D_{\vec{k},\lambda}, D^\dagger_{\vec{k}',\lambda'}\} = \delta^3(\vec{k}-\vec{k}')\delta_{\lambda,\lambda'}$, and all other combinations are 0.

A. Adiabatic expansion

Armed with the above results we can determine a consistent adiabatic expansion of the four dimensional Dirac field modes interacting with the prescribed electric background. Based on the two dimensional expansion given in [35], and taking into account that the positive-frequency solution with vanishing electric field, in the representation associated to ψ', is given by

$$h^{I(0)}_{\vec{k}} = \sqrt{\frac{\omega - k_3}{2\omega}} e^{-i\omega t}, \tag{28}$$

$$h^{II(0)}_{\vec{k}} = -\sqrt{\frac{\omega + k_3}{2\omega}} e^{-i\omega t}, \tag{29}$$

with $\omega = \sqrt{k_3^2 + \kappa^2}$, we propose the following *ansatz* for the field modes:

$$h^I_{\vec{k}} = \sqrt{\frac{\omega - k_3}{2\omega}} F(t) e^{-i\int^t \Omega(t')dt'}, \qquad h^{II}_{\vec{k}} = -\sqrt{\frac{\omega + k_3}{2\omega}} G(t) e^{-i\int^t \Omega(t')dt'}, \tag{30}$$

where the complex functions $F(t)$ and $G(t)$ and the real function $\Omega(t)$ are expanded adiabatically

$$\Omega(t) = \sum_{n=0}^{\infty} \omega^{(n)}(t), \qquad F(t) = \sum_{n=0}^{\infty} F^{(n)}(t), \qquad G(t) = \sum_{n=0}^{\infty} G^{(n)}(t). \tag{31}$$

Here, $\Omega^{(n)}$, $F^{(n)}$ and $G^{(n)}$ are functions of adiabatic order n. The adiabatic order of a given function will be determined by its dependence on the potential vector $A(t)$ and its derivatives. In order to recover at leading order the exact solution with vanishing electric field $A(t) = 0$ we demand $F^{(0)} = G^{(0)} = 1$ and $\omega^{(0)} = \omega$. With this condition we are implicitly fixing the adiabatic order of the potential vector $A(t)$ to 1, hence, $\dot{A}(t)$ and $A(t)^2$ will be of order 2, $\ddot{A}(t)$, $A(t)\dot{A}(t)$ and $A(t)^3$ of order three and so on. For a detailed discussion on the adiabatic order assignment see [36].

Plugging the ansatz (30) in the mode equations (22) and (23) and also in the normalization condition (26) we get a system of equations for the functions $F(t)$, $G(t)$ and $\Omega(t)$

$$(\omega - k_3)(\dot{F} - i\Omega F - i(k_3 + qA)F) + i\kappa^2 G = 0, \tag{32}$$

$$(\omega + k_3)(\dot{G} - i\Omega G + i(k_3 + qA)G) + i\kappa^2 F = 0, \tag{33}$$

$$\frac{\omega - k_3}{2\omega}|F|^2 + \frac{\omega + k_3}{2\omega}|G|^2 = 1. \tag{34}$$

In order to obtain the expressions of the adiabatic terms $\omega^{(n)}$, $F^{(n)}$ and $G^{(n)}$, we introduce the expansion (31) into Eqs. (32), (33) and (34) and solve them recursively, order by order. Note that $G(k_3, qA)$ satisfies the same equations as $F(-k_3, -qA)$, hence we take $G(k_3, qA) = F(-k_3, -qA)$. The system can be solved algebraically by iteration and the general solution is given by

$$\omega^{(n)} = \frac{(\omega - k_3)}{2\omega}\left[\dot{F}^{(n-1)}_y - \sum_{i=1}^{n-1}\omega^{(n-i)}F^{(i)}_x - qAF^{(n-1)}_x\right] + \frac{(\omega + k_3)}{2\omega}\left[\dot{G}^{(n-1)}_y - \sum_{i=1}^{n-1}\omega^{(n-i)}G^{(i)}_x + qAG^{(n-1)}_x\right], \tag{35}$$

$$F_x^{(n)} = \frac{(\omega+k_3)}{4\omega^2}\left[\dot{F}_y^{(n-1)} - \sum_{i=1}^{n-1}\omega^{(n-i)}F_x^{(i)} - qAF_x^{(n-1)} - \dot{G}_y^{(n-1)} + \sum_{i=1}^{n-1}\omega^{(n-i)}G_x^{(i)} - qAG_x^{(n-1)}\right] - \frac{(\omega-k_3)}{4\omega}\sum_{i=1}^{n-1}(F_x^{(i)}F_x^{(n-i)} + F_y^{(i)}F_y^{(n-i)}) - \frac{(\omega+k_3)}{4\omega}\sum_{i=1}^{n-1}(G_x^{(i)}G_x^{(n-i)} + G_y^{(i)}G_y^{(n-i)}), \quad (36)$$

$$F_y^{(n)} = G_y^{(n)} - \frac{(\omega-k_3)}{\kappa^2}\left[\dot{F}_x^{(n-1)} + \sum_{i=1}^{n-1}\omega^{(n-i)}F_y^{(i)} + qAF_y^{(n-1)}\right], \quad (37)$$

where we have parametrized F and G in terms of real functions as $F = F_x + iF_y$ and $G = G_x + iG_y$. Note that there is an ambiguity in the imaginary part (37). However, it disappears when computing physical observables. Further discussions on this issue are given in [20]. For simplicity we choose

$$F_y^{(n)} = -G_y^{(n)} = -\frac{(\omega-k_3)}{2\kappa^2}\left[\dot{F}_x^{(n-1)} + \sum_{i=1}^{n-1}\omega^{(n-i)}F_y^{(i)} + qAF_y^{(n-1)}\right]. \quad (38)$$

With the initial conditions $F_x^{(0)} = G_x^{(0)} = 1$, $F_y^{(0)} = G_y^{(0)} = 0$ and $\omega^{(0)} = \omega$ and by fixing the ambiguity according to (38), the solutions for the adiabatic functions $F^{(n)}$, $G^{(n)}$ and $\omega^{(n)}$ are univocally determined. In Appendix C we give the four first terms of the adiabatic expansion.

IV. 4D DIRAC FIELDS: ADIABATIC REGULARIZATION/RENORMALIZATION

In this section we will carry out the detailed renormalization of the vacuum expectation value of the electric current $\langle j^\mu\rangle = -q\langle\bar{\psi}\gamma^\mu\psi\rangle$, which constitutes the most important physical quantity in the context of strong electrodynamics [44]. The only non-vanishing component of the electric current is the one parallel to the electric field. With the results of Sec. III A we can obtain the formal expression of the z-component of the mean electric current

$$\langle j^3\rangle = \frac{2q}{(2\pi)^3}\int d^3k(|h_{\vec{k}}^{II}|^2 - |h_{\vec{k}}^{I}|^2) = \frac{q}{2\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3(|h_{\vec{k}}^{II}|^2 - |h_{\vec{k}}^{I}|^2), \quad (39)$$

where $k_\perp = \sqrt{k_1^2 + k_2^2}$. This expression is UV divergent and we have to renormalize it. The current has scaling dimension 3, meaning that the divergences could appear up to third adiabatic order, so we have to perform adiabatic subtractions until and including the third order (note that the energy-momentum tensor requires adiabatic subtractions of order 4) [8]. Therefore, the renormalized form of the electric current is

$$\langle j^3\rangle_{\rm ren} = \frac{q}{2\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3(|h_{\vec{k}}^{II}|^2 - |h_{\vec{k}}^{I}|^2 - \langle j^3\rangle_{\vec{k}}^{(0-3)}), \quad (40)$$

with $\langle j^3\rangle_{\vec{k}}^{(n)} = (|h_{\vec{k}}^{II}|^2 - |h_{\vec{k}}^{I}|^2)^{(n)} = -\frac{\omega-k_3}{2\omega}\sum_{i=0}^n F^{(i)}F^{*(n-i)} + \frac{\omega+k_3}{2\omega}\sum_{i=0}^n G^{(i)}G^{*(n-i)}$. These subtraction terms contain all the divergences of the electric current, giving us a finite and meaningful result for $\langle j^3\rangle_{\rm ren}$. The other components give a vanishing result. After computing the subtraction terms, we finally obtain

$$\langle j^3\rangle_{\rm ren} = \frac{q}{2\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3\left[(|h_{\vec{k}}^{II}|^2 - |h_{\vec{k}}^{I}|^2) - \frac{k_3}{\omega} - \frac{\kappa^2 qA}{\omega^3} + \frac{3\kappa^2 k_3 q^2A^2}{2\omega^5} + \frac{(\kappa^2 - 4k_3^2)\kappa^2 q^3A^3}{2\omega^7} + \frac{\kappa^2 q\ddot{A}}{4\omega^5}\right]. \quad (41)$$

A. Conformal anomaly

An important test of any proposed renormalization method is the necessary agreement with the conformal anomaly. Here we compute the trace anomaly with our proposed extended adiabatic method. The trace of the energy-momentum tensor is proportional to the mass of the field $\langle T_\mu^\mu\rangle = m\langle\bar{\psi}\psi\rangle$. Although the two point function has to be renormalized until the third adiabatic order, the trace of the energy momentum tensor must be regularized up to fourth order, i.e.,

$$\langle T_\mu^\mu\rangle_{\rm ren} = m(\langle\bar{\psi}\psi\rangle_{\rm ren} - \langle\bar{\psi}\psi\rangle^{(4)}). \quad (42)$$

In the massless limit the first term vanishes, so the anomaly should appear in the subtractions of adiabatic order 4, that is

$$\langle T_\mu^\mu\rangle_{\rm ren} = -\lim_{m\to 0} m\langle\bar{\psi}\psi\rangle^{(4)}. \quad (43)$$

The vacuum expectation value of the two-point function $\langle\bar{\psi}\psi\rangle$ is given by

$$\langle\bar{\psi}\psi\rangle = \frac{1}{2\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3 \frac{m}{\kappa}(h_{\vec{k}}^{I*}h_{\vec{k}}^{II} + h_{\vec{k}}^{II*}h_{\vec{k}}^{I}). \quad (44)$$

By using the adiabatic regularization method, one can find the 4th order subtraction terms. Hence, in the massless limit we get

$$\langle T^\mu_\mu\rangle_{\rm ren} = \lim_{m\to 0}\frac{m^2}{2\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^{\infty} dk_3 \frac{1}{\kappa}(h^{I*}_{\vec{k}} h^{II}_{\vec{k}} + h^{II*}_{\vec{k}} h^{I}_{\vec{k}})^{(4)} = -\frac{q^2\dot{A}^2}{12\pi^2}. \quad (45)$$

One can easily rewrite this result in a covariant way, obtaining the result

$$\langle T^\mu_\mu\rangle_{\rm ren} = \frac{q^2}{24\pi^2}F_{\mu\nu}F^{\mu\nu}. \quad (46)$$

It fully agrees with the well-known result for the trace anomaly induced by an electromagnetic field for a Dirac field [45].

B. Relation with the DeWitt coefficients

We will briefly see that the proposed adiabatic expansion for the fermionic modes agrees with the Schwinger-DeWitt adiabatic expansion for the Feynman propagator. We have proved this for the adiabatic expansion of the two-dimensional theory in Sec. II. In the previous subsection we have implicitly obtained the 4th adiabatic order, given by

$$\langle\bar{\psi}\psi\rangle^{(4)} = -\frac{1}{16\pi^2 m}\left(\frac{2}{3}q^2 F_{\mu\nu}F^{\mu\nu}\right) = -\frac{{\rm tr}E_2}{16\pi^2 m} \quad (47)$$

where E_2 coincides with the corresponding DeWitt coefficient at coincidence. Note that the numerical coefficient in the denominator is $(4\pi)^{d/2}$, where d is the spacetime dimension. Moreover, at 6th adiabatic order we obtain

$$\langle\bar{\psi}\psi\rangle^{(6)} = -\frac{1}{16\pi^2 m}\left(\frac{2q^2\ddot{A}^2}{15a^2} + \frac{2q^2 A^{(3)}\dot{A}}{5a^2}\right) \quad (48)$$

We can rewrite the above expression in a covariant form. It can be checked that it also fits with the DeWitt coefficient E_3

$$\langle\bar{\psi}\psi\rangle^{(6)} = -\frac{{\rm tr}E_3}{16\pi^2 m^3} \quad (49)$$

The general expression for E_3 is given in Appendix A. Here only the flat space terms are relevant

$$E_3 = -\frac{1}{360}(8W_{\mu\nu;\rho}W^{\mu\nu;\rho} + 2W_{\mu\nu}{}^{;\nu}W_{;\rho}{}^{\mu\rho} + 12W_{\mu\nu;\rho}{}^{\rho}W^{\mu\nu} - 12W_{\mu\nu}W^{\nu\rho}W_{\rho}{}^{\mu} + 6Q_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu} + 60QQ_{;\mu}{}^{\mu} + 30Q_{;\mu}Q^{;\mu} + 60Q^3 + 30QW_{\mu\nu}W^{\mu\nu}) \quad (50)$$

where $Q = -\frac{i}{2}qF_{\mu\nu}\underline{\gamma}^\mu\underline{\gamma}^\nu$ and $W_{\mu\nu} = -iqF_{\mu\nu}I$. We reinforce that the adiabatic order assignment 1 for A_μ is a basic ingredient for achieving the above equivalence.

C. Introduction of a mass scale and renormalization ambiguities

A crucial point in the adiabatic regularization method is to fix the leading order of the adiabatic expansion, namely $\omega^{(0)}$. It seems very natural to define it as $\omega^{(0)} \equiv \omega = \sqrt{\vec{k}^2 + m^2}$, as we did in Sec. III A. However, there exist an inherent ambiguity in the method [46]. It is possible to choose a slightly different expression for the leading term $\omega^{(0)} \equiv \omega_\mu = \sqrt{\vec{k}^2 + \mu^2}$, where μ corresponds to an arbitrary mass scale. In order to obtain the new adiabatic subtractions with this new choice of the leading order, one has to rewrite the mode equations as

$$i\partial_t h^I_{\vec{k}} = -(k_3 + qA(t))h^I_{\vec{k}} - (\kappa_\mu + \sigma)h^{II}_{\vec{k}}$$
$$i\partial_t h^{II}_{\vec{k}} = (k_3 + qA(t))h^{II}_{\vec{k}} - (\kappa_\mu + \sigma)h^{I}_{\vec{k}}, \quad (51)$$

where $\sigma = \kappa - \kappa_\mu \equiv \sqrt{k_1^2 + k_2^2 + m^2} - \sqrt{k_1^2 + k_2^2 + \mu^2}$ is assumed of adiabatic order 1. Note that we recover the original adiabatic subtraction method by choosing $\mu = m$, and hence $\sigma = 0$.

In this context, the ansatz of the adiabatic expansion will take the form

$$h^I_{\vec{k}} = \sqrt{\frac{\omega_\mu - k_3}{2\omega_\mu}}F_\mu(t)e^{-i\int^t \Omega_\mu(t')dt'},$$
$$h^{II}_{\vec{k}} = -\sqrt{\frac{\omega_\mu + k_3}{2\omega_\mu}}G_\mu(t)e^{-i\int^t \Omega_\mu(t')dt'}, \quad (52)$$

where the functions $F_\mu(t)$, $G_\mu(t)$ and $\Omega_\mu(t)$ are expanded adiabatically as in (31). In order to recover at order 0 the limit of vanishing electric field (and also the limit $\sigma \to 0$, since σ is now assumed of adiabatic order 1) we demand as initial conditions $F^{(0)}_\mu = 1$, $G^{(0)}_\mu = 1$ and $\omega^{(0)}_\mu = \omega_\mu$. With this new choice we can obtain the expressions of the adiabatic terms $\omega^{(n)}_\mu$, $F^{(n)}_\mu$ and $G^{(n)}_\mu$ as before: introducing the ansatz (52) in the mode equations (51) and in the normalization condition (26), expanding the functions $F_\mu(t)$, $G_\mu(t)$ and $\Omega_\mu(t)$ adiabatically, and finally, solving them recursively, order by order. In Appendix D we give the details of the computation and also the expression of the adiabatic renormalization subtractions for the electric current. We remark that the introduction of a mass scale μ causes an unavoidable ambiguity in the renormalization procedure: it allows us to perform different adiabatic subtractions to render finite the physical observables, depending on the scale μ we choose. For instance, concerning the renormalized current $\langle\bar{\psi}\gamma^\nu\psi\rangle$ one can compare it at two different scales. Using the results given in the Appendix D we easily obtain

$$\langle\bar{\psi}\gamma^\nu\psi\rangle_{\rm ren}(\mu)-\langle\bar{\psi}\gamma^\nu\psi\rangle_{\rm ren}(\mu_0)=\frac{q}{12\pi^2}\ln\left(\frac{\mu^2}{\mu_0^2}\right)\nabla_\sigma F^{\sigma\nu}. \quad (53)$$

This ambiguity can be absorbed in the renormalization of the coupling constant. To this end it is convenient to scale the field as $\tilde{A}^\nu \equiv qA^\nu$ and rewrite the semiclassical Maxwell equations as

$$\frac{1}{q^2}\nabla_\alpha \tilde{F}^{\alpha\beta} = -\langle\bar{\psi}\gamma^\nu\psi\rangle_{\rm ren}. \quad (54)$$

The above relation for the current (53), reexpressed in terms of $\tilde{F}^{\alpha\beta}$, translates into the well-known shift: $q^{-2}(\mu) - q^{-2}(\mu_0) = -(12\pi^2)^{-1}\ln\frac{\mu^2}{\mu_0^2}$, obtained within perturbative QED using minimal subtraction in dimensional regularization [47]. The renormalized current given in (41) should be understood as defined at the natural scale of the problem, defined by the physical mass of the charged field, i.e., $\mu = m$ and hence $q \equiv q(m)$.

V. PHYSICAL APPLICATION: THE SAUTER ELECTRIC PULSE

As mentioned in the Introduction, one of the main advantages of the adiabatic renormalization method is its proficiency to perform numerical computations and analytical approximations. We will devote this section to study the properties of the renormalized expression of the current (41) for the case of a pulsed electric field in a $1+3$ dimensional setting.

Let us consider the well-known Sauter-type pulse $E(t) = E_0\cosh^{-2}(t/\tau)$ with $\tau > 0$, and its corresponding potential $A(t) = -E_0\tau\tanh(t/\tau)$, which is bounded at early and late times, $A(\pm\infty) = \mp E_0\tau$. This kind of pulse produces a number of particles, and then also a current, which tends to be constant when $t\to\infty$. In Fig. 1 we represent the evolution of the current induced by this pulse for different values of E_0 and τ. These figures have been obtained by solving numerically the differential equations for the modes and integrating the expression of the renormalized current (41).

A. Late times behavior of the electric current

We can obtain an expression of the current at late times for an electric background that vanishes at early and late times. Let us consider a pulse such that in the early and late time limits the potential is bounded as $A(-\infty) = -A_0$, $A(\infty) = A_0$, and its derivatives vanish. From Eqs. (22) and (23), one can see that at late times $t\to+\infty$ the modes behave as [31]

$$h^{I/II}_{\vec{k}}(t) \sim \pm\sqrt{\frac{\omega_{\rm out}\mp(k_3+qA_0)}{2\omega_{\rm out}}}\alpha_{\vec{k}}e^{-i\omega_{\rm out}t} + \sqrt{\frac{\omega_{\rm out}\pm(k_3+qA_0)}{2\omega_{\rm out}}}\beta_{\vec{k}}e^{i\omega_{\rm out}t}, \quad (55)$$

where $\omega_{\rm in/out} = \sqrt{(k_3\mp qA_0)^2+\kappa^2}$, and $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ are the usual Bogoliubov coefficients satisfying the relation $|\alpha_{\vec{k}}|^2+|\beta_{\vec{k}}|^2=1$, that ensures the normalization condition (26). The coefficient $|\beta_{\vec{k}}|^2$ gives the density number of created particles at any value of $\vec{k}$.

The renormalized electric current at late times induced by an electric pulse in terms of the coefficient $|\beta_{\vec{k}}|^2$ can be obtained by introducing the expression of the modes at late

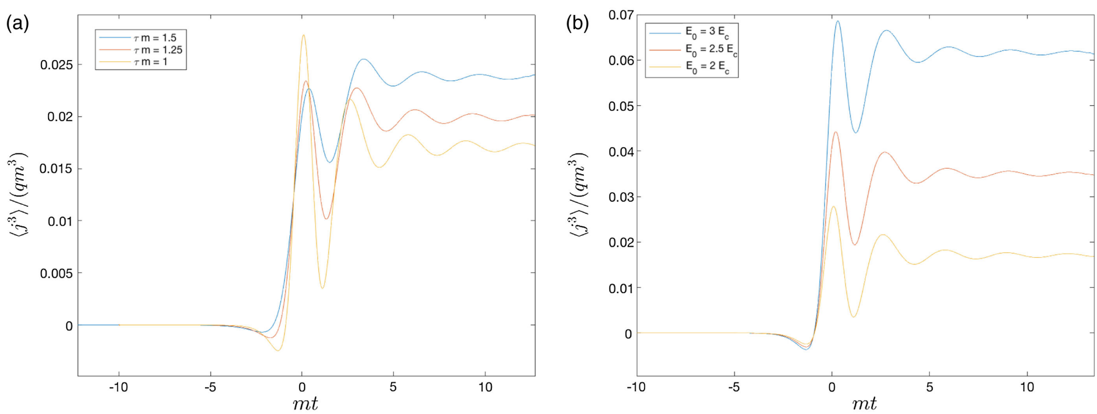

FIG. 1. Evolution of the renormalized current induced by a Sauter-type electric pulse for different values of the parameters. In figure (a) the field strength is fixed ($E_0 = 2E_c$), where $E_c = m^2/q$ is the critical electric field (or Schwinger limit), that is the scale above which the electric field can produce particles. In figure (b) the width of the pulse is fixed ($\tau = 1/m$). We have used dimensionless variables, in terms of the mass and the charge.

times (55) in the expression of the current (41). We obtain, for large t,

$$\langle j^3\rangle_{\rm ren} \sim -\frac{q}{\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^{\infty} dk_3 \frac{k_3+qA_0}{\omega_{\rm out}}|\beta_{\vec{k}}|^2 + \frac{q}{2\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^{\infty} dk_3 \left[\frac{k_3+qA_0}{\omega_{\rm out}} - \frac{k_3}{\omega} - \frac{\kappa^2 qA_0}{\omega^3} + \frac{3\kappa^2 k_3 q^2 A_0^2}{2\omega^5} + \frac{(\kappa^2-4k_3^2)\kappa^2 q^3 A_0^3}{2\omega^7}\right]. \quad (56)$$

In Appendix E we prove that the second integral of this expression vanishes, so the current at late times is given by the simple expression

$$\langle j^3\rangle_{\rm ren} \sim -\frac{q}{\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^{\infty} dk_3 \frac{k_3+qA_0}{\omega_{\rm out}}|\beta_{\vec{k}}|^2. \quad (57)$$

As expected, the final current is related to the number density of particles. The analytic expression of $|\beta_{\vec{k}}|^2$ depends on the form of the background.

B. Scaling behavior for large field strength

It is interesting to study the behavior of the current in the limit of large field strength. To this end, we consider again the example of the Sauter pulse, for which the coefficient $|\beta_{\vec{k}}|^2$ is given by (see [31] for more details)

$$|\beta_{\vec{k}}|^2 = \frac{\cosh(2\pi qE_0\tau^2) - \cosh(\pi(\omega_{\rm out}-\omega_{\rm in})\tau)}{2\sinh(\pi\omega_{\rm in}\tau)\sinh(\pi\omega_{\rm out}\tau)}. \quad (58)$$

Plugging it into (57) we can obtain the current at late times induced by the pulse. As a test, one can compare the results given by (57) with the ones given by the exact expression (41) for large t, which are represented in Fig. 1.

For this pulse, assuming $qE_0 > 0$, the large field strength limit corresponds to $qE_0 \gg 0$. A numerical analysis of the expression (58) shows that the relevant values of κ and k_3 are of the order of $\sqrt{qE_0}$ and $qE_0\tau$, respectively. Therefore, in order to study properly the limit of large E_0, it is convenient to introduce the following set of dimensionless variables

$$\tilde{k}_3 = \frac{k_3}{qE_0\tau}, \qquad \tilde{\kappa} = \frac{\kappa}{\sqrt{qE_0}}, \qquad x = qE_0\tau^2, \quad (59)$$

and study the limit $x\to\infty$ maintaining $\tilde{k}_3$ and $\tilde{\kappa}$ constant. Then, we rewrite $|\beta_{\vec{k}}|^2$ as

$$|\beta_{\vec{k}}|^2 = \frac{\cosh(2\pi x) - \cosh(\pi(\tilde{\omega}_{\rm out}(x)-\tilde{\omega}_{\rm in}(x)))}{2\sinh(\pi\tilde{\omega}_{\rm out}(x))\sinh(\pi\tilde{\omega}_{\rm in}(x))}, \quad (60)$$

where $\tilde{\omega}_{\rm in/out}(x) = \sqrt{x^2(\tilde{k}_3\pm 1)^2 + x\tilde{\kappa}^2}$. In the limit $x\to\infty$ the above expression for $|\beta_{\vec{k}}|^2$ is independent of x, and it is given by

$$|\beta_{\vec{k}}|^2 \sim e^{-\pi\frac{\tilde{\kappa}^2}{1-\tilde{k}_3^2}}\Theta(1-|\tilde{k}_3|) = e^{-\pi\frac{k_\perp^2+m^2}{qE_0}\left(\frac{1}{1-(\frac{k_3}{qE_0\tau})^2}\right)}\Theta(qE_0\tau-|k_3|). \quad (61)$$

Substituting the expression (61) into (57) and taking into account that $\frac{k_3-qE_0\tau}{\omega_{\rm out}} \sim -1$ for large E_0, we obtain the behavior of the current at late times created by a high intensity pulse

$$\langle j^3\rangle_{\rm ren} \sim \frac{q^3E_0^2\tau}{2\pi^3}\int_{-1}^{1} ds(1-s^2)e^{-\pi\frac{m^2}{qE_0}(\frac{1}{1-s^2})}. \quad (62)$$

Assuming now that $qE_0 \gg m^2$, the above integral (62) can be done exactly and we finally obtain

$$\langle j^3\rangle_{\rm ren} \sim \frac{2}{3\pi^3}q^3E_0^2\tau, \quad (63)$$

which is the predicted expression of the current in the limit of large field strength E_0. We can also obtain the total number density of created quanta for the Sauter pulse in this limit

$$\langle N\rangle = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}(|\beta_{\vec{k}}|^2 + |\beta_{-\vec{k}}|^2) \sim \frac{2}{3\pi^3}q^2E_0^2\tau. \quad (64)$$

It is interesting to compare the result (63) with the one obtained for a scalar field. The coefficient $|\beta_{\vec{k}}|^2$ in this case has a different expression, but it tends to the same limit for large E_0 (61). Therefore the scaling behavior of the current at late times ($\langle j^3\rangle_{\rm ren}^{\rm scalar} \sim \frac{1}{3\pi^3}q^3E_0^2\tau$) will be the same as in the fermionic case, except for the factor 2, on account of the absence of the spin degree of freedom.

For completeness, it is worth to see how the above results can also serve to describe the Schwinger limit, i.e, a constant electric field. Note that the expression (61) has been obtained for the limit $E_0\tau^2 \gg 0$, so it should also be valid for the limit of large τ, keeping E_0 constant, which describes a pulse with a large width. Bringing this limit to the extreme case $\tau\to\infty$, we get $|\beta_{\vec{k}}|^2 \sim \exp(-\pi\frac{k_\perp^2+m^2}{qE_0})$, which is the well-known expression for the beta coefficients of a constant electric field [12] leading to the Schwinger formula for the vacuum persistence amplitude.

VI. CONCLUSIONS

In this work we have extended the adiabatic regularization method for 4-dimensional Dirac fields interacting with a time-varying electric background. Our approach can be distinguished from previous analysis in the literature in the adiabatic order assignment for the vector potential, which is chosen to be of order 1. This choice is required to fit it with the expected equivalence with the

Schwinger-DeWitt adiabatic expansion. Our proposal has required to introduce a nontrivial ansatz, Eq. (30), to generate a self-consistent adiabatic expansion of the fermionic modes. The given expansion turns out to be different from the WKB-type expansion used for scalar fields. With this extension we have obtained a well-defined prediction, Eq. (41), for the renormalized electric current induced by the created particles. Our proposal is consistent, in the massless limit, with the conformal anomaly. The expected equivalence with the Schwinger-DeWitt expansion is explicitly realized. In parallel we have also explored the physical consequences of the introduction of an arbitrary mass scale on the adiabatic regularization scheme, finding consistency with the behavior of the effective scaling of the electric coupling constant. To illustrate the power of the method we have analyzed the pair production phenomenon in the particular case of a Sauter-type electric pulse. In particular, we have obtained the scaling behavior of the current in the strong field regime [Eq. (63)].

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APPENDIX A: DEWITT COEFFICIENT E_3

The expression for the DeWitt coefficient of sixth adiabatic order is [42,43]

$$
\begin{aligned}
E_3 = -\frac{1}{7!}\Big(&-18R_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu} + 17R_{;\mu}R^{;\mu} - 2R_{\mu\nu;\rho}R^{\mu\nu;\rho} - 4R_{\mu\nu;\rho}R^{\mu\rho;\nu} + 9R_{\mu\nu\rho\sigma;\alpha}R^{\mu\nu\rho\sigma;\alpha} + 28RR_{;\mu}{}^{\mu} - 8R_{\mu\nu}R^{\mu\nu}{}_{;\rho}{}^{\rho} \\
&+ 24R_{\mu\nu}R^{\mu\rho;\nu}{}_{\rho} + 12R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma;\alpha}{}_{\alpha} - \frac{35}{9}R^3 + \frac{14}{3}RR_{\mu\nu}R^{\mu\nu} - \frac{14}{3}RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{208}{9}R_{\mu\nu}R^{\mu\rho}R^{\nu}{}_{\rho} \\
&- \frac{64}{3}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} + \frac{16}{3}R_{\mu}{}^{\nu}R^{\mu\rho\sigma\alpha}R_{\nu\rho\sigma\alpha} - \frac{44}{9}R_{\mu\nu\rho\sigma}R^{\mu\nu\alpha\beta}R^{\rho\sigma}{}_{\alpha\beta} - \frac{80}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^{\nu}{}_{\alpha}{}^{\sigma}{}_{\beta}\Big)I \\
-\frac{1}{360}\Big(&8W_{\mu\nu;\rho}W^{\mu\nu;\rho} + 2W_{\mu\nu}{}^{;\nu}W^{\mu\rho}{}_{;\rho} + 12W_{\mu\nu;\rho}{}^{\rho}W^{\mu\nu} - 12W_{\mu\nu}W^{\nu\rho}W_{\rho}{}^{\mu} - 6R_{\mu\nu\rho\sigma}W^{\mu\nu}W^{\rho\sigma} + 4R_{\mu\nu}W^{\mu\rho}W^{\nu}{}_{\rho} \\
&- 5RW_{\mu\nu}W^{\mu\nu} + 6Q_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu} + 60QQ_{;\mu}{}^{\mu} + 30Q_{;\mu}Q^{;\mu} + 60Q^3 + 30QW_{\mu\nu}W^{\mu\nu} - 10RQ_{;\mu}{}^{\mu} - 4R_{\mu\nu}Q^{;\mu\nu} - 12R_{;\mu}Q^{;\mu} \\
&- 30Q^2R - 12QR_{;\mu}{}^{\mu} + 5QR^2 - 2QR_{\mu\nu}R^{\mu\nu} + 2QR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big) \qquad \text{(A1)}
\end{aligned}
$$

where, for scalar fields $Q = \xi R$, $W_{\mu\nu} = iqF_{\mu\nu}$ and $I = 1$, while for Dirac fields $Q = \frac{1}{4}RI - \frac{i}{2}qF_{\mu\nu}\underline{\gamma}^{\mu}\underline{\gamma}^{\nu}$, $W_{\mu\nu} = -iqF_{\mu\nu}I - \frac{1}{4}R_{\mu\nu\rho\sigma}\underline{\gamma}^{\rho}\underline{\gamma}^{\sigma}$ and I is the identity matrix.

APPENDIX B: MATCHING HADAMARD COEFFICIENTS WITH SCALAR ADIABATIC REGULARIZATION

In this appendix, we relate the adiabatic regularization method with Hadamard renormalization for charged scalar fields. To simplify the comparison we will restrict the analysis to Minkowski spacetime. In Sec. II we have introduced the basics of the adiabatic regularization method for 4-dimensional charged scalar fields interacting with an electromagnetic background. The renormalized vacuum expectation value on the two point function was given in (4). For the electric current, defined as $j_\mu = iq[\phi^\dagger D^\mu\phi - (D^\mu\phi)^\dagger\phi]$, we obtain

$$\langle j^3\rangle_{\mathrm{ren}} = \frac{q}{(2\pi)^3}\int d^3k[(k_3 - qA)|h_{\vec{k}}|^2 - \langle j^3\rangle_{\vec{k}}^{(0-3)}], \qquad \text{(B1)}$$

with $\langle j^3\rangle_{\vec{k}}^{(0-3)} = \sum_{n=0}^{3} k_3(\Omega_{\vec{k}}^{-1})^{(n)} - qA\langle\phi^\dagger\phi\rangle_{\vec{k}}^{(0-2)}$. To compute these subtraction terms, we usually fix the leading order of the adiabatic expansion as $\omega^{(0)} = \omega = \sqrt{\vec{k}^2 + m^2}$.

As explained in the main text, the choice of the leading term is crucial to define the adiabatic expansion. We have argued that to properly fix the leading therm we have to choose the vector potential A of adiabatic order 1. However, the choice of the leading term $\omega^{(0)}$ is not yet completely fixed, and one can make a more general choice defining $\omega^{(0)} = \omega_\mu = \sqrt{\vec{k}^2 + \mu^2}$, where μ is an arbitrary mass scale. With this new choice the adiabatic expansion can be recalculated, giving us slightly different subtraction terms. An exhaustive analysis of this ambiguity can be found in

[46]. The ambiguity on the subtractions, leads to an ambiguity on the physical observables. For the two point function the ambiguity manifests as

$$\langle\phi^\dagger\phi\rangle_{\rm ren}(\mu) = \langle\phi^\dagger\phi\rangle_{\rm ren}(\mu_0) - \frac{\alpha}{2}\left[m^2\ln\left(\frac{\mu^2}{\mu_0^2}\right) - \mu^2 + \mu_0^2\right], \tag{B2}$$

where $\alpha = \frac{1}{2(2\pi)^2}$, and for the electric current we find

$$\langle j^3\rangle_{\rm ren}(\mu) = \langle j^3\rangle_{\rm ren}(\mu_0) - \frac{\alpha}{6}\ln\left(\frac{\mu^2}{\mu_0^2}\right)q^2\ddot{A}. \tag{B3}$$

Rewriting the equation above in a covariant way, we get

$$\langle j^\nu\rangle_{\rm ren}(\mu) = \langle j^\nu\rangle_{\rm ren}(\mu_0) - \frac{q^2\alpha}{6}\ln\left(\frac{\mu^2}{\mu_0^2}\right)\nabla_\sigma F^{\sigma\nu}. \tag{B4}$$

1. Matching with Hadamard renormalization

We can compare the results summarized in Sec. II with the results given by Hadamard renormalization, particularizing for the case in which $A_\mu = (0,0,0,-A(t))$. Adopting the notation given in [39], the expectation value of the two point function can be expressed as

$$\langle\phi\phi^\dagger\rangle_{\rm ren} = \alpha w_0(x), \tag{B5}$$

and the electric current is given by

$$\langle j_\mu\rangle = -2q\alpha(qA_\mu w_0(x) + \Im[w_{1\mu}(x)]), \tag{B6}$$

where $\alpha = \frac{1}{2(2\pi)^2}$ and the functions w_0 and $w_{1\mu}$ are the first terms of the covariant Taylor series expansion of the Hadamard biscalar $W(x,x')$.

Comparing (B5) with (4) we immediately get

$$\alpha w_0 = \frac{1}{2(2\pi)^3}\int d^3k\left[|h_{\vec{k}}|^2 - \sum_{n=0}^{2}(\Omega_{\vec{k}}^{-1})^{(n)}\right], \tag{B7}$$

and hence, by using the previous result and Eqs. (B1) and (B6) we directly find

$$\alpha\Im(w_{13}) = \frac{q}{2(2\pi)^3}\int d^3k\left[k_3|h_{\vec{k}}|^2 - \sum_{n=0}^{3}k_3(\Omega_{\vec{k}}^{-1})^{(n)}\right]. \tag{B8}$$

Hadamard renormalization scheme also presents a renormalization ambiguity in even space-time dimensions, due to the choice of the renormalization lenght scale ℓ. The ambiguity is manifested in the physical observables as

$$\langle\phi\phi^\dagger\rangle_{\rm ren} \to \langle\phi\phi^\dagger\rangle_{\rm ren} + \frac{\alpha}{2}m^2\ln\ell^2, \tag{B9}$$

$$\langle j_\mu\rangle_{\rm ren} \to \langle j_\mu\rangle_{\rm ren} + \frac{\alpha q^2}{6}(\nabla^\rho F_{\rho\mu})\ln\ell^2. \tag{B10}$$

Note that the length scale ℓ is inversely proportional to the mass scale μ. Comparing these results with the ones obtained with adiabatic regularization [Eqs. (B2) and (B4)] we find that the logarithmic part of the ambiguity is exactly the same. However, with adiabatic regularization we also find a quadratic term in the ambiguity of the two point function.

APPENDIX C: SUBTRACTION TERMS

In this appendix we give the explicit expressions of the adiabatic expansion of the fermionic field modes up to and including the fourth adiabatic order. We remind that $G^{(n)}(k_3, qA) = F^{(n)}(-k_3, -qA)$.

Order 0

$$\omega^{(0)} = \omega, \quad F_x^{(0)} = G_x^{(0)} = 1, \quad F_y^{(0)} = G_y^{(0)} = 0. \tag{C1}$$

Order 1

$$\omega^{(1)} = \frac{qAk_3}{\omega}, \quad F_x^{(1)} = -\frac{qA(\omega+k_3)}{2\omega^2}, \quad F_y^{(1)} = G_y^{(1)} = 0. \tag{C2}$$

Order 2

$$\omega^{(2)} = \frac{q^2A^2\kappa^2}{2\omega^4}, \quad F_x^{(2)} = -\frac{5q^2A^2\kappa^2}{8\omega^4} + \frac{q^2A^2(\omega+k_3)}{2\omega^3},$$
$$F_y^{(2)} = -G_y^{(2)} = \frac{q\dot{A}}{4\omega^2}. \tag{C3}$$

Order 3

$$\omega^{(3)} = -\frac{q^3A^3\kappa^2k_3}{2\omega^5} - \frac{q\ddot{A}k_3}{4\omega^3}, \tag{C4}$$

$$F_x^{(3)} = \frac{11q^3A^3\kappa^2}{16\omega^5} - \frac{q^3A^3}{2\omega^3} + \frac{15q^3A^3\kappa^2k_3}{16\omega^6} - \frac{q^3A^3k_3}{2\omega^4} + \frac{q\ddot{A}(\omega+k_3)}{8\omega^4}, \tag{C5}$$

$$F_y^{(3)} = -G_y^{(3)} = -\frac{5q^2A\dot{A}k_3}{8\omega^4}. \tag{C6}$$

Order 4

$$\omega^{(4)} = -\frac{5q^4A^4\kappa^4}{8\omega^7} + \frac{q^4A^4\kappa^2}{2\omega^5} - \frac{3\kappa^2q^2A\ddot{A}}{4\omega^5} + \frac{5q^2A\ddot{A}}{8\omega^3} + \frac{5k_3^2q^2\dot{A}^2}{8\omega^5}, \tag{C7}$$

$$F_x^{(4)} = -\frac{17A^4\kappa^2k_3q^4}{16\omega^7} + \frac{A^4k_3q^4}{2\omega^5} + \frac{195A^4\kappa^4q^4}{128\omega^8} - \frac{31A^4\kappa^2q^4}{16\omega^6} + \frac{A^4q^4}{2\omega^4} - \frac{Ak_3q^2\ddot{A}}{2\omega^5} - \frac{5\dot{A}^2k_3q^2}{16\omega^5} + \frac{9A\kappa^2q^2\ddot{A}}{16\omega^6} + \frac{5\dot{A}^2\kappa^2q^2}{16\omega^6} - \frac{Aq^2\ddot{A}}{2\omega^4} - \frac{11\dot{A}^2q^2}{32\omega^4}, \tag{C8}$$

$$F_y^{(4)} = -G_y^{(4)} = \frac{q^3A^2\dot{A}(34\omega^2 - 45\kappa^2)}{32\omega^6} - \frac{qA^{(3)}}{16\omega^4}. \tag{C9}$$

APPENDIX D: μ-PARAMETER ADIABATIC EXPANSION

The general solution for $F_\mu^{(n)}$, $G_\mu^{(n)}$ and $\omega_\mu^{(n)}$ is given by

$$\omega_\mu^{(n)} = \omega^{(n)}(\omega_\mu, \kappa_\mu, F_\mu, G_\mu) + \frac{\sigma\kappa_\mu}{\omega_\mu}[(G_\mu)_x^{(n-1)} + (F_\mu)_x^{(n-1)}] + \frac{\sigma^2}{2\omega_\mu}[(G_\mu)_x^{(n-2)} + (F_\mu)_x^{(n-2)}], \tag{D1}$$

$$(F_\mu)_x^{(n)} = F_x^{(n)}(\omega_\mu, \kappa_\mu, F_\mu, G_\mu) + \frac{1}{4\omega_\mu^2}\left\{\frac{\omega_\mu + k_3}{\omega_\mu - k_3}[2\sigma\kappa_\mu(G_\mu)_x^{(n-1)} + \sigma^2(G_\mu)_x^{(n-2)}] - 2\sigma\kappa_\mu(F_\mu)_x^{(n-1)} - \sigma^2(F_\mu)_x^{(n-2)}\right\}, \tag{D2}$$

$$(F_\mu)_y^{(n)} = F_y^{(n)}(\omega_\mu, \kappa_\mu, F_\mu, G_\mu) + \frac{1}{\kappa_\mu^2}[2\sigma\kappa_\mu(G_\mu)_y^{(n-1)} + \sigma^2(G_\mu)_y^{(n-2)}], \tag{D3}$$

where $\omega^{(n)}/F^{(n)}/G^{(n)}(\omega_\mu, \kappa_\mu, F_\mu, G_\mu)$ are given by the expressions (35), (36) and (37) with the changes $(\omega,\kappa,F,G) \to (\omega_\mu,\kappa_\mu,F_\mu,G_\mu)$. Note again that $G_\mu(k_3, qA)$ satisfies the same equations than $F_\mu(-k_3, -qA)$, and hence $G_\mu^{(n)}(k_3, qA) = F_\mu^{(n)}(-k_3, -qA)$. We also find an ambiguity in the imaginary part (D1). For simplicity we choose

$$(F_\mu)_y^{(n)} = -(G_\mu)_y^{(n)} = -\frac{(\omega_\mu - k_3)}{2\kappa_\mu^2}\left[(\dot{F}_\mu)_x^{(n-1)} + \sum_{i=1}^{n-1}\omega_\mu^{(n-i)}(F_\mu)_y^{(i)} + qA(F_\mu)_y^{(n-1)} - \frac{1}{\omega_\mu - k_3}(2\sigma\kappa_\mu(G_\mu)_y^{(n-1)} + \sigma^2(G_\mu)_y^{(n-2)})\right] \tag{D4}$$

With the initial conditions $(F_\mu)_x^{(0)} = (G_\mu)_x^{(0)} = 1$, $(F_\mu)_y^{(0)} = (G_\mu)_y^{(0)} = 0$ and $\omega_\mu^{(0)} = \omega_\mu$ and by fixing the ambiguity (D4), the solutions for the adiabatic functions $F_\mu^{(n)}$, $G_\mu^{(n)}$ and $\omega_\mu^{(n)}$ are univocally determined.

The renormalized electric current for an arbitrary mass scale is given by

$$\langle j^3\rangle_{\rm ren} = \frac{q}{2\pi^2}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3[(|h_{\vec{k}}^{II}|^2 - |h_{\vec{k}}^{I}|^2) - \langle j^3\rangle_{\vec{k}}^{(0-3)}(\mu)], \tag{D5}$$

with

$$\langle j^3\rangle_{\vec{k}}^{(0)}(\mu) = \frac{k_3}{\omega_\mu}, \tag{D6}$$

$$\langle j^3\rangle_{\vec{k}}^{(1)}(\mu) = \frac{\kappa_\mu^2 qA}{\omega_\mu^3} - \frac{2k_3\kappa_\mu\sigma}{\omega_\mu^3}, \tag{D7}$$

$$\langle j^3\rangle_{\vec{k}}^{(2)}(\mu) = -\frac{3\kappa^2k_3q^2A^2}{2\omega_\mu^5} - \frac{2qA\kappa_\mu\sigma(3\kappa_\mu^2 - 2\omega_\mu^2)}{\omega_\mu^5} + \frac{3k_3\sigma^2(2\kappa_\mu^2 - \omega_\mu^2)}{\omega_\mu^5}, \tag{D8}$$

$$\langle j^3\rangle_{\vec{k}}^{(3)}(\mu) = +\frac{\kappa_\mu^2q^3A^3(4\omega_\mu^2 - 5\kappa_\mu^2)}{2\omega_\mu^7} - \frac{2k_3\sigma^3(10\kappa_\mu^4 - 9\kappa_\mu^2\omega_\mu^2 + \omega_\mu^4)}{\kappa\omega_\mu^7} + \frac{3k_3q^2A^2\sigma\kappa_\mu(5\kappa_\mu^2 - 2\omega_\mu^2)}{\omega_\mu^7} + \frac{3qA\sigma^2(10\kappa_\mu^4 - 11\kappa_\mu^2\omega_\mu^2 + 2\omega_\mu^4)}{\omega_\mu^7} - \frac{\kappa_\mu^2q\ddot{A}}{4\omega_\mu^5}. \tag{D9}$$

APPENDIX E: SIMPLIFICATION OF THE EXPRESSION OF THE CURRENT AT LATE TIMES

In this appendix we prove that the second integral in the expression of the current at late times [see Eq. (56)],

$$I = \int_0^\infty k_\perp dk_\perp \int_{-\infty}^\infty dk_3\left[\frac{k_3 + qA_0}{\omega_{\rm out}} - \frac{k_3}{\omega} - \frac{\kappa^2qA_0}{\omega^3} + \frac{3\kappa^2k_3q^2A_0^2}{2\omega^5} + \frac{(\kappa^2 - 4k_3^2)\kappa^2q^3A_0^3}{2\omega^7}\right], \tag{E1}$$

vanishes. Taking into account the property $(1 + 2xy + y^2)^{-1/2} = \sum_{n=0}^\infty P_n(-x)y^n$, where $P_n(x)$ are the Legendre polynomials, we can expand the first term of the integral around $A_0 = 0$ as follows

$$\frac{k_3 + qA_0}{\omega_{\rm out}} = \sum_{n=0}^{\infty} c_n(\vec{k})(qA_0)^n,$$

where

$$c_0(\vec{k}) = \frac{k_3}{\omega},$$
$$c_n(\vec{k}) = \frac{1}{\omega^n}\left[P_{n-1}\left(-\frac{k_3}{\omega}\right) + \frac{k_3}{\omega}P_n\left(-\frac{k_3}{\omega}\right)\right] \quad \text{for } n > 0. \tag{E2}$$

One can see that the first four terms of this expansion give exactly the rest of the terms of the integral (E1) (the subtraction terms) with a global change of sign. Therefore they are cancelled and the integral can be written as

$$I = \int_0^\infty k_\perp dk_\perp \sum_{n=4}^{\infty}\left[(qA_0)^n \int_{-\infty}^{\infty} dk_3 c_n(\vec{k})\right]. \tag{E3}$$

Under the change of variable $x = -k_3/\omega$, the integral in k_3 can be rewritten as

$$\int_{-\infty}^{\infty} dk_3 c_n(\vec{k}) = \frac{1}{\kappa^{n-1}}\left(\int_{-1}^{1} dx (1-x^2)^{\frac{n-3}{2}} P_{n-1}(x) - \int_{-1}^{1} dx\, x (1-x^2)^{\frac{n-3}{2}} P_n(x)\right). \tag{E4}$$

The Legendre polynomials satisfy the property $P_n(-x) = (-1)^n P_n(x)$, so it is trivial to see that for any even n these integrals vanish. For odd values of n and $n \geq 3$ the function $(1-x^2)^{\frac{n-3}{2}}$ is a polynomial of order $n-3$. Using the property $\int_{-1}^{1} dx \mathrm{Pol}_a(x) P_b(x) = 0$ for $a < b$, where $\mathrm{Pol}_a(x)$ is a polynomial of order a, we get that the integrals in (E4) vanish for $n \geq 3$. This last property can be easily proven taking into account that $P_n(x)$ form a basis, and any function can be expanded as $f(x) = \sum_{b=0}^{\infty} c_b P_b(x)$ where $c_b = (b+1/2)\int_{-1}^{1} dx f(x) P_b(x)$, and if the function is a polynomial $f(x) = \mathrm{Pol}_a(x)$, for consistency $c_b = 0$ for any $b > a$. Therefore, for all values of n involved in (E3) the integral vanishes, and then $I = 0$, as we wanted to prove.

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Article 4

Note on the pragmatic mode-sum regularization method: Translational-splitting in a cosmological background

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The point-splitting renormalization method offers a prescription to calculate finite expectation values of quadratic operators constructed from quantum fields in a general curved spacetime. It has been recently shown by Levi and Ori that when the background metric possesses an isometry, like stationary or spherically symmetric black holes, the method can be upgraded into a pragmatic procedure of renormalization that produces efficient numerical calculations. In this paper we show that when the background enjoys three-dimensional spatial symmetries, like homogeneous expanding universes, the above pragmatic regularization technique reduces to the well-established adiabatic regularization method.

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I. INTRODUCTION

Obtaining accurate theoretical predictions from quantum field theory has become a topic of great interest nowadays for studies of the early universe and black holes. The naive calculation of physical observables associated with a quantum field ϕ, such as $\langle\phi^2(x)\rangle$ or $\langle T_{\mu\nu}(x)\rangle$, typically leads to divergent sums or integrals of field modes, thereby requiring the study of renormalization. While the systematics of renormalization in a general curved spacetime has been known for several decades now [1–5], the implementation of the standard prescription to get specific results is still difficult to put in practice even for the most simple spacetime backgrounds. This is because the regularization of ultraviolet divergences in a covariant way, and the construction of the subtraction terms, are based on the point-splitting technique [6–8], a purely analytical procedure that involves taking limits of points along geodesics. However, getting the field modes in a given spacetime background requires solving complicated differential equations, which can only be addressed numerically but in exceptional cases. A procedure to transform the covariant point-splitting technique into a numerically implementable method is thus almost mandatory if quantum field theory aims to produce results of practical interest for most gravitational scenarios.

The numerical implementation of the point-splitting regularization method is however a nontrivial task, specially for black hole backgrounds. In a Schwarzschild metric, the first important insight was introduced by Candelas in [9] by proposing an integral representation of the subtraction terms of point-splitting, allowing the possibility of subtracting the ultraviolet divergences within the integral of field modes, thereby yielding a formally finite result upon which the limit of points could be taken in advance. However, the numerical implementation of these integrals was still a difficult task and this idea was not pursued further. An alternative way to address the problem was proposed in [10], which did not involve the numerical evaluation of integrals, but which required an analytic WKB-type approximation of the field and a Wick rotation to analytically extend the metric to the Euclidean space. This method was successful in the Schwarzschild background and it was later extended for a general static and spherically symmetric metric in [11].

Unfortunately, these analytical techniques are not available for time-dependent backgrounds, as for instance in gravitational collapse, and thus this approach could not be extended to dynamical settings, that are of great interest in astrophysics. This problem recently motivated Levi and Ori [12,13] to develop what they called the pragmatic mode-sum method of regularization, which bypasses any analytic approximation, and that can be applied to any background metric as long as it displays an isometry. Their approach recovers the first insight proposed by Candelas of finding integral representations of the point-splitting subtraction terms, and proposes a successful method to implement numerically the integration over the field modes, based on the concept of generalized integrals. The technique has been proven to be useful in computing numerically $\langle\phi^2(x)\rangle$

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and $\langle T_{\mu\nu}(x)\rangle$ in black hole backgrounds in different complementary ways [14–20].

The importance of this new approach is that it can be applied to any metric provided that it has some symmetry. For instance, in order to calculate $\langle\phi^2(x)\rangle$ one splits the points as $\langle\phi(x)\phi(x')\rangle$, which is well defined, and then, after subtracting the necessary DeWitt-Schwinger term $G_{\rm DS}(x,x')$, one takes the limit in which the two points x, x' merge: $\langle\phi^2(x)\rangle = \lim_{x'\to x}(\langle\phi(x)\phi(x')\rangle - G_{\rm DS}(x,x'))$. Following the proposal in [12], in a stationary background there is a preferred direction for which this splitting could be taken, which is the direction of the time-translational Killing vector field, i.e., $x=(t,r,\theta,\varphi)$ and $x'=(t+\epsilon,r,\theta,\varphi)$ in the usual Boyer-Lindquist coordinates. Furthermore, the field $\phi(x)$ can be expanded in modes of well-defined frequency. Then, the point-splitting in the symmetric direction allow us to write the subtraction term $G_{\rm DS}(x,x')$ as integrals in the field-mode frequencies, by Fourier-transforming each term with respect to the splitting parameter ϵ. The resulting expression for the difference $\langle\phi(x)\phi(x')\rangle - G_{\rm DS}(x,x')$ is an integral in frequencies which is formally finite in the limit $x'\to x$, so this limit can be safely taken inside the integral. This is specially appropriate for numerical implementation, since only an integration is required to get the desired final result. It is important to stress that for the whole procedure to be well defined the time-translational symmetry is fundamental. Similar reasonings can be applied with spherical symmetry (implemented via angular splitting), or only axial symmetry (implemented via azimuthal point-splitting) [12–16].

So far the pragmatic mode-sum regularization method has been only applied for stationary black hole spacetimes. Can one implement this procedure for other, possibly dynamical, symmetric spacetimes? Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetimes, which are of interest in studies of cosmology, have three spatial Killing vector fields associated with spatial translations. Consequently, it is natural to use those symmetries to upgrade the point-splitting method and rewrite the subtraction term $G_{\rm DS}(x,x')$ as an integral in modes of momentum $\vec{k}$ (i.e., the constants of motion associated with the spatial translation symmetries). The goal of this work is to carry out this simple idea. As a result, we shall find that the subtraction integrals match the expressions offered by the method of adiabatic regularization [21].

The paper is organized in the following way. In Sec. II we outline the main idea of the pragmatic mode-sum regularization method. To this end we restrict the presentation to a stationary background and evaluate the renormalized two-point function $\langle\phi^2(x)\rangle$ by splitting the points in the associated timelike direction. In Sec. III we extend the method to provide a numerically implementable formula for $\langle\phi^2(x)\rangle$ in a spatially flat FLRW spacetime using the spatial translational symmetry of the metric. We end the section generalizing the method by considering an arbitrary renormalization point μ. In this work we follow the conventions in [4]; in particular we use the metric signature $(+,-,-,-)$.

II. PRAGMATIC MODE-SUM REGULARIZATION METHOD IN A STATIONARY BACKGROUND: *t*-SPLITTING

In this section we outline the idea underlying the pragmatic mode-sum regularization method introduced in [12], emphasizing those aspects that are relevant for our purposes. The method takes advantage of the symmetries of the spacetime metric to rewrite the renormalization subtractions in the point-splitting method into a numerically efficient way. In order to illustrate the procedure, let us focus on the computation of the two-point function of a scalar field by exploiting the stationary symmetry.

Let $\phi(x)$ be a scalar field of mass m living in a stationary spacetime of metric $g_{\mu\nu}$ that obeys the field equation $(\Box + m^2 + \xi R)\phi = 0$, where ξ is the coupling constant to the scalar curvature R. To formulate a quantum description of this field, one must construct a Hilbert space of states. As is well known, in a general curved spacetime there is no preferred prescription to do this. However, if the spacetime is stationary one can define creation and annihilation operators, $A^\dagger_\omega$ and A_ω, by decomposing the field operator into positive and negative frequency parts,

$$\phi(x) = \int_m^\infty d\omega[A_\omega f_\omega(x) + A^\dagger_\omega f^*_\omega(x)], \tag{1}$$

and define the vacuum state using the annihilation operators. The notion of field modes f_ω, f^*_ω of positive and negative frequency ω can be introduced in a natural way by the conditions $\mathcal{L}_K f_\omega = -i\omega f_\omega$, $\mathcal{L}_K f^*_\omega = i\omega f^*_\omega$, where K is the infinitesimal generator of the isometry (i.e., the Killing vector field) [5]. Then, using this set of field modes there is a preferred prescription to decompose the field operator as above [22]. We have omitted the additional quantum numbers required to specify a basis of modes, since they do not play any fundamental role in the following discussion. Choosing now a natural coordinate system $\{t, x^k\}$ such that $K = \partial/\partial t$, the above conditions imply

$$f_\omega(x) = e^{-i\omega t}\psi_\omega(\vec{x}), \tag{2}$$

where $\vec{x}$ is a shorthand for the three spatial coordinates x^k. The determination of the spatial functions $\psi_\omega(\vec{x})$ is achieved by solving numerically the Klein-Gordon equation.

The naive calculation of $\langle\phi^2(x)\rangle$ will produce a divergent expression, as expected, so a renormalization method is needed at this point. The DeWitt-Schwinger point-splitting method consists in taking the product of the field operator at two separated points x, x'; subtracting to this two-point function the corresponding asymptotic DeWitt-Schwinger proper-time expansion up to the last divergent term; and finally taking the coincident limit $x \to x'$ along a geodesic that connects the two points. The result of this procedure is what defines the renormalized two-point function:

$$\langle \phi^2(x) \rangle_{\rm ren} = \lim_{x' \to x} [\langle \{\phi(x), \phi(x')\} \rangle - G^{(1)}_{\rm DS}(x, x')], \quad (3)$$

where $\{\phi(x), \phi(x')\} \equiv \frac{1}{2}[\phi(x)\phi(x') + \phi(x')\phi(x)]$, and $G^{(1)}_{\rm DS}(x, x')$ is the symmetric part of the DeWitt-Schwinger subtraction term (Hadamard function). Following [7], it is possible to obtain an expansion for the symmetric two-point function in terms of covariant quantities evaluated at x and the geodesic distance between x and x'. Including only the relevant terms needed in the calculation of the renormalized two-point function (i.e., up to second-order derivatives of the metric), the subtraction term yields

$$G^{(1)}_{\rm DS}(x, x') = \frac{1}{8\pi^2}\left[-\frac{1}{\sigma} + (m^2 + (\xi - 1/6)R)\left(\gamma + \frac{1}{2}\log\left(\frac{m^2|\sigma|}{2}\right)\right) - \frac{m^2}{2} + \frac{1}{12}R_{\alpha\beta}\frac{\sigma^{;\alpha}\sigma^{;\beta}}{\sigma}\right], \quad (4)$$

where R is the scalar curvature and $R_{\alpha\beta}$ is the Ricci tensor. γ is the Euler constant and $\sigma(x, x') = \frac{1}{2}\tau(x, x')^2$, $\tau(x, x')$ being the proper distance along the geodesic connecting x to x' (for sufficiently close points this geodesic is unique [23]). The expansion (4) contains all the divergences of the two-point function. As a remark, expression (4) finds its origin in the integral expression for the Feynman Green function

$$G_{\rm DS}(x, x') = \frac{\Delta^{1/2}(x, x')}{(4\pi)^2}\int_0^\infty \frac{ds}{(is)^2} e^{-i(m^2 s + \frac{\sigma}{2s})} \times \sum_{n=0}^{\infty} a_n(x, x')(is)^n, \quad (5)$$

where $a_n(x, x')$ are the DeWitt coefficients, which can be solved recursively from the field equations using the input $a_0(x, x') = 1$ [5], and $\Delta(x, x')$ is the Van Vleck-Morette determinant defined as

$$\Delta(x, x') = -|g(x)|^{-1/2}\det[-\partial_\mu \partial_{\nu'}\sigma(x, x')]|g(x')|^{-1/2}. \quad (6)$$

The above expression can be written in terms of Hankel functions which, after expanding in an asymptotic series, give rise to (4). As we will see in Sec. III A, Eq. (5) will be the key starting point to generalize the subtraction terms in order to deal with the infrared divergence when $m \to 0$ (by means of the introduction of an arbitrary renormalization point μ). To implement the renormalization prescription with the pragmatic mode-sum regularization we just need (4), so we will forget about (5) for the moment.

At this point it becomes evident that, if the mode functions in (2) are to be solved numerically, the explicit calculation of (3) with (4) using numerical methods is far from obvious. Here is where the pragmatic mode-sum method comes into play. Following [12], we have to split the points x and x' in the direction associated with the symmetry, i.e., such that the metric has the same value in both points. Choosing $x = (t, \vec{x})$ and $x' = (t + \epsilon, \vec{x})$ with $\epsilon > 0$ an infinitesimal parameter, the mode expansion of the symmetric two-point function formally reads

$$\langle \{\phi(x), \phi(x')\} \rangle = \int_m^\infty d\omega \cos(\omega\epsilon)|\psi_\omega(\vec{x})|^2. \quad (7)$$

On the other hand, expanding σ in a Taylor series around $\epsilon = 0$ one finds that the general form of $G^{(1)}_{\rm DS}(x, x')$ has the form

$$G^{(1)}_{\rm DS}(x, x') = a(\vec{x})\frac{1}{\epsilon^2} + c(\vec{x})(\log(m\epsilon) + \gamma) + d(\vec{x}) + \mathcal{O}(\epsilon), \quad (8)$$

where $a(\vec{x})$, $c(\vec{x})$, and $d(\vec{x})$ are real functionals of the metric. The key point now is to express the ϵ-dependent terms as integrals in ω by using the following integral transforms:

$$\int_m^\infty d\omega\omega\cos(\omega\epsilon) = -\frac{1}{\epsilon^2} - \frac{m^2}{2} + \mathcal{O}(\epsilon), \quad (9)$$

$$\int_m^\infty \frac{d\omega}{\omega + m}\cos(\omega\epsilon) = -(\log(m\epsilon) + \gamma) - \log 2 + \mathcal{O}(\epsilon). \quad (10)$$

These integrals have to be understood as generalized integrals in the distributional sense. Inserting all these expressions in (3) one finds the following expression for the renormalized two-point function:

$$\langle \phi^2(x) \rangle_{\rm ren} = \lim_{\epsilon \to 0}\int_m^\infty d\omega\left(|\psi_\omega(\vec{x})|^2 + a(\vec{x})\omega + c(\vec{x})\frac{1}{\omega + m}\right)\cos(\omega\epsilon) - \bar{d}(\vec{x}), \quad (11)$$

where $\bar{d}(\vec{x}) = d(\vec{x}) - a(\vec{x})\frac{m^2}{2} - c(\vec{x})\log 2$. The integral is now expected to be convergent since the original point-splitting subtraction terms have been designed to cancel the divergences of the two-point function. Then, the limit and the integration can be interchanged and we finally obtain the following result for the renormalized two-point function:

$$\langle\phi^2(x)\rangle_{\rm ren} = \int_m^\infty d\omega\left(|\psi_\omega(\vec{x})|^2 + a(\vec{x})\omega + c(\vec{x})\frac{1}{\omega+m}\right) - \bar{d}(\vec{x}). \qquad (12)$$

Thus, with these manipulations the quantity $\langle\phi^2(x)\rangle_{\rm ren}$ can, at least in principle, be computed using ordinary numerical techniques.[1] In practice, however, there is one last issue that must be addressed, at least in some cases. To give an example, in a Schwarzschild background and for a massless field $m = 0$ one gets [12] $a = -(4\pi^2(1-2M/r))^{-1}$, $c = 0$, $d = M^2 \times (48\pi^2 r^4(1-2M/r))^{-1}$, which agrees with the result originally introduced in [9]. The point is that, when trying to implement the above integration numerically, one finds that it fails to converge. This is because when performing the integration between 0 and ω, increasing oscillations in ω appear. As pointed out in [12], the origin of these oscillations comes from the fact that black holes admit null geodesics connecting x and x', i.e., geodesics that start at some spatial point and after making one or several round trips around the black hole return to the same point with a delay time given by ϵ. At the values of ϵ corresponding to these geodesics the term $\langle\phi(x)\phi(x')\rangle$ presents singularities, which in Fourier domain is equivalent to oscillations in $|\psi_\omega(\vec{x})|^2$ [see Eq. (7)]. The wavelengths of the oscillations are related to the values of ϵ of these geodesics, that can be obtained through a straightforward analysis of the geodesic equation in Schwarzschild spacetime. To solve the issue of the divergent integration, one can apply a "self-cancellation" numerical method, explained in detail in [12], in order to cancel the oscillations and obtain the physical finite value of the renormalized two-point function. Fortunately, in the case we will study in this work these kinds of geodesics do not exist, so there will not be any convergence problem, and then the self-cancellation method will not be necessary.

As a final remark, for massless fields the term $\log(m^2|\sigma|/2)$ in (4) is ill defined. This infrared problem is usually bypassed by replacing the mass by a new arbitrary parameter μ in the logarithm. Therefore, in the massless case (12) actually reads

$$\langle\phi^2(x)\rangle_{\rm ren} = \int_0^\infty d\omega\left(|\psi_\omega(\vec{x})|^2 + a(\vec{x})\omega + c(\vec{x})\frac{1}{\omega+\mu}\right) - d(\vec{x}). \qquad (13)$$

We will reconsider this point later on, specially in the quantization of the field in the FLRW background.

[1] Notice though that the existence of an isometry was fundamental. Had the coordinate t failed to be associated with a Killing vector field, the above procedure could not have been carried out.

III. PRAGMATIC MODE-SUM REGULARIZATION METHOD IN A FLRW BACKGROUND: TRANSLATIONAL-SPLITTING

As pointed out in the Introduction, our aim is to extend the pragmatic mode-sum method to a cosmological setting. We shall work out the case of a scalar field in a FLRW spacetime, and consider a spatially flat universe with metric $ds^2 = dt^2 - a^2(t)d\vec{x}^2$, for simplicity. This spacetime is dynamical and t-splitting is no longer useful. On the contrary, since the background we are considering now is spatially homogeneous, it is natural to use the translational symmetry when applying the point-splitting prescription.

Given the spatial homogeneity of the spacetime background, the field operator ϕ can now be naturally expanded in the form

$$\phi(x) = \int d^3k[A_{\vec{k}}f_{\vec{k}}(x) + A^\dagger_{\vec{k}}f^*_{\vec{k}}(x)], \qquad (14)$$

where, again, $A_{\vec{k}}$ and $A^\dagger_{\vec{k}}$ are annihilation and creation operators satisfying canonical commutation relations, and $f_{\vec{k}}(x)$ denote a complete orthonormal family of solutions to the field equation satisfying $\mathcal{L}_{K^j}f_{\vec{k}} = ik^jf_{\vec{k}}$, where $\{K^j\}_{j=1,2,3}$ denote the three Killing vector fields associated with spatial translations. Thus, in a canonical coordinate chart $\{t,\vec{x}\}$ where $K^j = \partial/\partial x^j$ the field modes take the general form

$$f_{\vec{k}}(x) = \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{2(2\pi)^3a(t)^3}}h_k(t). \qquad (15)$$

These modes are assumed to obey the normalization condition with respect to the conserved Klein-Gordon product $(f_{\vec{k}}, f_{\vec{k}'}) = \delta^3(\vec{k}-\vec{k}')$, $(f_{\vec{k}}, f^*_{\vec{k}'}) = 0$. This condition translates into a Wronskian-type condition for the modes: $h^*_k\dot{h}_k - \dot{h}^*_kh_k = -2i$, where the dot means derivative with respect to time t. The complete specification of the modes usually requires assuming boundary conditions at early times. This is however not relevant for renormalization.

Let us now proceed to the regularization of the two-point function via the point-splitting method. As explained in the previous section the renormalized two-point function is defined as

$$\langle\phi^2(x)\rangle_{\rm ren} = \lim_{x'\to x}[\langle\{\phi(x),\phi(x')\}\rangle - G^{(1)}_{\rm DS}(x,x')], \qquad (16)$$

where the DeWitt-Schwinger subtraction term is given by (4), now with $R = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})$, $R_{00} = 3\frac{\ddot{a}}{a}$, $R_{ii} = -a^2(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2})$ for a FLRW metric.

As illustrated in the previous section the point-splitting regularization scheme becomes particularly useful when we evaluate the two-point function in two points where the metric has the same value. Therefore, taking advantage of the translational symmetry of the FLRW spacetime, we consider equal-time points $x \equiv (t, \vec{x})$ and $x' \equiv (t, \vec{x} + \vec{\epsilon})$ to do the splitting. Then, the equal-time two-point function reads

$$\langle\{\phi(x), \phi(x')\}\rangle = \frac{1}{2(2\pi a(t))^3} \int d^3k |h_k(t)|^2 \cos(\vec{k}\cdot\vec{\epsilon}). \tag{17}$$

One can now express the cosine in terms of exponentials and easily perform angular integration to reduce it to

$$\langle\{\phi(x), \phi(x')\}\rangle = \frac{1}{4\pi^2 a(t)^3} \int_0^\infty dk k^2 |h_k(t)|^2 \frac{\sin k\epsilon}{k\epsilon}, \tag{18}$$

where $k = |\vec{k}|$ and $\epsilon = |\vec{\epsilon}|$. To achieve an efficient numerical method of renormalization, the goal now is to rewrite the DeWitt-Schwinger term $G^{(1)}_{\rm DS}(x, x')$ in (4) as an integral in momentum space so that it can be fitted with the previous expression for the field modes. As in the previous section, we have to evaluate σ. To this end, it is useful to use the Riemann normal coordinates y^μ with origin at x. In these coordinates we have $\sigma(x, x') = \frac{1}{2} y_\mu y^\mu$. Following [24], y^μ can be expanded in terms of $\Delta x^\mu = x'^\mu - x^\mu$, and we obtain (for the first orders)

$$\sigma = \frac{1}{2}\Delta t^2 - \frac{a^2}{2}\Delta\vec{x}^2 - \frac{a\dot{a}}{2}\Delta\vec{x}^2\Delta t - \frac{a\ddot{a}}{6}\Delta\vec{x}^2\Delta t^2 - \frac{a^2\dot{a}^2}{24}\Delta\vec{x}^4 + \cdots. \tag{19}$$

Therefore in our case we can write

$$\sigma = -\frac{a^2}{2}\epsilon^2 - \frac{a^2\dot{a}^2}{24}\epsilon^4 - \frac{(a^2\dot{a}^4 + 3a^3\dot{a}^2\ddot{a})}{720}\epsilon^6 + \mathcal{O}(\epsilon^8). \tag{20}$$

The terms involving σ in (4) can be now expanded as follows [note that $\sigma^{;\alpha}$ must be calculated from (19)]

$$\frac{1}{\sigma} = -\frac{2}{a^2\epsilon^2} + \frac{\dot{a}^2}{6a^2} + \mathcal{O}(\epsilon^2), \tag{21}$$

$$R_{\alpha\beta}\frac{\sigma^{;\alpha}\sigma^{;\beta}}{\sigma} = 2\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} + \mathcal{O}(\epsilon^2). \tag{22}$$

Introducing these results in (4) we get

$$G^{(1)}_{\rm DS}(x, x') = \frac{1}{4\pi^2}\left(\frac{1}{a^2\epsilon^2} + \frac{1}{2}(m^2 + (\xi - 1/6)R)\left(\gamma + \log\left(\frac{ma}{2}\epsilon\right)\right) - \frac{m^2}{4} + \frac{R}{72}\right) + \mathcal{O}(\epsilon), \tag{23}$$

which turns out to be a function depending on $\vec{\epsilon}$ only through its modulus ϵ (this is due to the underlying isotropy of the FLRW metric). Now we have to rewrite the potential divergences of this expression as $\epsilon \to 0$ in terms of one-dimensional integrals in momentum space involving $\frac{\sin k\epsilon}{k\epsilon}$. To this end we consider the following integral transforms, which have to be understood as generalized integrals:

$$\int_0^\infty dk k \frac{\sin k\epsilon}{k\epsilon} = \frac{1}{\epsilon^2}, \tag{24}$$

$$\int_0^\infty dk \frac{k^2}{\omega^3}\frac{\sin k\epsilon}{k\epsilon} = -a^3\left(\gamma + \log\left(\frac{ma}{2}\epsilon\right)\right) + \mathcal{O}(\epsilon), \tag{25}$$

where $\omega(t) = \sqrt{k^2/a^2(t) + m^2}$. Substituting in (23) the terms depending on ϵ by these integrals we get

$$G^{(1)}_{\rm DS}(x, x') = \frac{1}{4\pi^2 a^3}\int_0^\infty dk \frac{\sin(k\epsilon)}{k\epsilon} \times \left[ka - \frac{k^2 m^2}{2\omega^3} + \frac{k^2(\frac{1}{6} - \xi)R}{2\omega^3}\right] - \frac{m^2}{16\pi^2} + \frac{R}{288\pi^2} + \mathcal{O}(\epsilon). \tag{26}$$

Using now the identity

$$\frac{1}{4\pi^2 a^3}\int_0^\infty dk \frac{\sin(k\epsilon)}{k\epsilon}\left[ka - \frac{k^2 m^2}{2\omega^3} - \frac{k^2}{\omega}\right] = \frac{m^2}{16\pi^2} + \mathcal{O}(\epsilon), \tag{27}$$

we can simplify the expression of $G^{(1)}_{\rm DS}$. Introducing it into (16) we can finally write

$$\langle\phi^2\rangle_{\rm ren} = \lim_{\epsilon\to 0}\frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \frac{\sin k\epsilon}{k\epsilon}\left[|h_k|^2 - \frac{1}{\omega} - \frac{(\frac{1}{6} - \xi)R}{2\omega^3}\right] - \frac{R}{288\pi^2}. \tag{28}$$

The sum of terms inside the parentheses has no ultraviolet divergences even for $\epsilon = 0$, so we can interchange the integral and the limit $\epsilon \to 0$ to find

$$\langle\phi^2\rangle_{\rm ren} = \frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \left[|h_k|^2 - \frac{1}{\omega} - \frac{(\frac{1}{6} - \xi)R}{2\omega^3}\right] - \frac{R}{288\pi^2}. \tag{29}$$

Note that the above expression can be naturally expressed in terms of a three-dimensional integral in the $\vec{k}$ modes associated with the three-dimensional translation symmetry

$$\langle\phi^2\rangle_{\rm ren} = \frac{1}{2(2\pi a)^3}\int d^3k \left[|h_k|^2 - \frac{1}{\omega} - \frac{(\frac{1}{6} - \xi)R}{2\omega^3}\right] - \frac{R}{288\pi^2}. \tag{30}$$

This result agrees exactly with the renormalized two-point function obtained by using the so-called adiabatic regularization method developed by Parker and Fulling in the early 1970s for cosmological backgrounds and scalar fields [21] (see [25] for spin-1/2 fields). One can check the result with Eq. (A9) in Appendix A. As explained before, the problem of frequency oscillations in the Schwarzschild black hole context, pointed out in [12], does not emerge for FLRW metrics. Therefore, after solving numerically the Klein-Gordon equation for the modes $h_k(t)$, ordinary numerical integration techniques can be applied directly to calculate the final expression (29).

It is also straightforward to see that the result can be extended to the renormalized stress-energy tensor in a FLRW background. To this end one needs a more complete form of the DeWitt-Schwinger expansion of the two-point function [7,26]

$$G^{(1)}_{\rm DS}(x,x') = \frac{\Delta^{1/2}}{8\pi^2}\left\{-\frac{1}{\sigma} + m^2\left(\gamma + \frac{1}{2}\ln\left|\frac{1}{2}m^2\sigma\right|\right)\left(1-\frac{1}{4}m^2\sigma\right) - \frac{1}{2}m^2 + \frac{5}{16}m^4\sigma \right.$$
$$\left. - a_1\left[\left(\gamma + \frac{1}{2}\ln\left|\frac{1}{2}m^2\sigma\right|\right)\left(1-\frac{1}{2}m^2\sigma\right) + \frac{1}{2}m^2\sigma\right] - \frac{1}{2}a_2\sigma\left[\gamma + \frac{1}{2}\ln\left|\frac{1}{2}m^2\sigma\right| - \frac{1}{2}\right] + \frac{a_2}{2m^2}\right\}, \tag{31}$$

where a_1 and a_2 are the first DeWitt coefficients. The renormalized vacuum expectation value of the stress-energy tensor can be obtained by acting with a nonlocal operator to the renormalized symmetric part of the two-point function

$$\langle T_{\mu\nu}\rangle = \lim_{x\to x'}\mathcal{D}_{\mu\nu}(x,x')[\langle\{\phi(x),\phi(x')\}\rangle - G^{(1)}_{\rm DS}(x,x')]. \tag{32}$$

This differential operator $\mathcal{D}_{\mu\nu}(x,x')$ contains different quadratic terms of covariant derivatives [7,26]; therefore, we need to expand (31) up to and including the order $\mathcal{O}(\epsilon^2)$ because terms proportional to ϵ^2 can give rise to finite terms in $\langle T_{\mu\nu}\rangle$. Proceeding as before and expanding (31) to order $\mathcal{O}(\epsilon^2)$ we arrive at the expression (B4) (see Appendix B for details) which contains terms with four derivatives of the metric. This expression agrees with the subtraction terms of the two-point function obtained by adiabatic regularization at fourth adiabatic order (B5). Therefore, the pragmatic form of the subtraction terms for the stress-energy tensor, when the translational symmetry is considered, reduces to the renormalization terms of adiabatic regularization. The explicit formulas of interest required to do the direct numerical implementation can be seen, for instance, in [27]. These results explain the great versatility of the adiabatic method with numerical calculations [28–31].

A. Massless case and the renormalization scale μ

For massless fields expression (4) is ill defined due to a logarithmic divergence. The usual approach to bypass this infrared divergence is to introduce an upper cutoff in the proper-time integral (5) [5], or to replace m^2 by an arbitrary mass scale μ^2 in the problematic logarithmic term. Here we will follow an alternative strategy based on [32] that consists in replacing m^2 by $m^2+\mu^2$ in the exponent of the DeWitt-Schwinger integral form (5). The advantage of this approach is that it leads to a natural decoupling mechanism of heavy massive fields. Following this idea, we have

$$G_{\rm DS}(x,x') = \frac{\Delta^{1/2}}{(4\pi)^2}\int_0^\infty \frac{ds}{(is)^2}e^{-i((m^2+\mu^2)s+\frac{\sigma}{2s})}\sum_{n=0}^{\infty}\bar{a}_n(is)^n. \tag{33}$$

In order to be consistent with (5), the first DeWitt coefficients need to be modified in the following way: $\bar{a}_0(x,x')=1, \bar{a}_1(x,x')=a_1(x,x')+\mu^2, \bar{a}_2(x,x')=a_2(x,x')+a_1(x,x')\mu^2+\frac{1}{2}\mu^4$. Now one can proceed as in the case in which $\mu=0$. We can write (33) in terms of Hankel functions and later expand them in asymptotic series to finally get the following expression for the subtraction term of the two-point function in the point-splitting renormalization method:

$$G^{(1)}_{\rm DS}(x,x') = \frac{1}{8\pi^2}\left[-\frac{1}{\sigma} + (m^2+(\xi-1/6)R)\right.$$
$$\times\left(\gamma + \frac{1}{2}\log\left(\frac{m^2+\mu^2}{2}|\sigma|\right)\right)$$
$$\left. - \frac{m^2+\mu^2}{2} + \frac{1}{12}R_{\alpha\beta}\frac{\sigma^{;\alpha}\sigma^{;\beta}}{\sigma}\right]. \tag{34}$$

Note that the parameter μ^2 appears nontrivially in this expression. Not only does it appear in the logarithmic term but it also emerges in the constant term, and not in the usual combination $m^2+(\xi-1/6)R$ multiplying the logarithm. This effect is responsible of the decoupling of heavy particles in the computations of the renormalized energy-momentum tensor [32].

Considering a FLRW spatially flat spacetime we can write the generalized subtraction term with integrals in modes of k by using again the translational symmetry. To do so, we just have to replace m^2 by $m_{\rm eff}^2 = m^2 + \mu^2$ in (25) and using the integral representations of the divergent terms [Eqs. (24) and (25)] in (34) we get

$$G_{\rm DS}^{(1)}(x,x') = \frac{1}{4\pi^2 a^3}\int_0^\infty dk \frac{\sin(k\epsilon)}{k\epsilon} \times \left[ka - \frac{k^2 m^2}{2\omega_{\rm eff}^3} + \frac{k^2(\frac{1}{6}-\xi)R}{2\omega_{\rm eff}^3}\right] - \frac{m_{\rm eff}^2}{16\pi^2} + \frac{R}{288\pi^2} + \mathcal{O}(\epsilon), \tag{35}$$

where $\omega_{\rm eff}^2 = \frac{k^2}{a^2} + m^2 + \mu^2$. If we consider the identity (27) with m^2 replaced by $m_{\rm eff}^2$ we can rewrite the expression above as follows:

$$G_{\rm DS}^{(1)}(x,x') = \frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \frac{\sin(k\epsilon)}{k\epsilon} \times \left[\frac{1}{\omega_{\rm eff}} + \frac{(\frac{1}{6}-\xi)R}{2\omega_{\rm eff}^3} + \frac{\mu^2}{2\omega_{\rm eff}^3}\right] + \frac{R}{288\pi^2} + \mathcal{O}(\epsilon). \tag{36}$$

This is the generalized subtraction term for the two-point function written as an integral in modes of the momentum k. Note that a new term proportional to μ^2 appears. This result agrees with adiabatic regularization when we introduce the arbitrary parameter μ requiring the same conditions (see Appendix A for details).

IV. CONCLUSIONS

In this work we have applied the pragmatic mode-sum regularization method proposed by Levi and Ori to study the numerical implementability of renormalization for quantum fields in FLRW spacetime backgrounds. This was possible thanks to the isometry under spatial translations of the underlying metric. The results obtained are in agreement with the well-known prescription of adiabatic regularization, developed by Parker and Fulling in the early 1970s. Adiabatic regularization can now be understood as the natural renormalization procedure that emerges when the point-splitting technique is applied using the spatial isometries of the FLRW metric.

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APPENDIX A: ADIABATIC REGULARIZATION WITH AN ARBITRARY μ

In this Appendix we provide a very concise presentation of the adiabatic regularization method. Furthermore we also introduce the adiabatic procedure in a generalized way so as to account for the introduction of a renormalization scale μ. It was first sketched in [33] by replacing m^2 by μ^2 in the zeroth adiabatic order. However, following [32] a better, and physically motivated procedure, is to replace $m^2 \to m^2 + \mu^2$. The underlying reason is to guarantee the decoupling of heavy massive fields.

Adiabatic renormalization is based on a generalized WKB-type asymptotic expansion of the modes (15) according to the ansatz

$$h_k(t) \sim \frac{1}{\sqrt{W_k(t)}} e^{-i\int^t W_k(t')dt'}, \tag{A1}$$

which guarantees the Wronskian condition $h_k \dot{h}_k^* - h_k^* \dot{h}_k = -2i$. One then expands W_k in an adiabatic series, in which each contribution is determined by the number of time derivatives of the expansion factor $a(t)$

$$W_k(t) = \omega^{(0)}(t) + \omega^{(2)}(t) + \omega^{(4)}(t) + \cdots, \tag{A2}$$

where the leading term $\omega^{(0)}(t) \equiv \omega(t) = \sqrt{k^2/a^2(t) + m^2}$ is the usual physical frequency. Higher order contributions can be univocally obtained by iteration (for details, see [4]), which come from introducing (A1) into the equation of motion for the modes. The adiabatic expansion of the modes can be easily translated to an expansion of the two-point function $\langle\phi(x)\phi(x')\rangle \equiv G(x,x')$ at coincidence $x = x'$:

$$G_{\rm Ad}(x,x) = \frac{1}{2(2\pi)^3 a^3} \times \int d^3k[\omega^{-1} + (W^{-1})^{(2)} + (W^{-1})^{(4)} + \cdots]. \tag{A3}$$

As remarked above, the expansion must be truncated to the minimal adiabatic order necessary to cancel all ultraviolet divergences that appear in the formal expression of the vacuum expectation value that one wishes to compute. The calculation of the renormalized variance $\langle\phi^2\rangle$ requires only second adiabatic order.

The above process can be repeated now by replacing m^2 by $m^2 + \mu^2$ in the zeroth adiabatic order ω. Therefore the expansion for W_k depends now on μ

$$W_k(t) = \omega_{\rm eff}^{(0)}(t,\mu) + \omega_{\rm eff}^{(2)}(t,\mu) + \omega_{\rm eff}^{(4)}(t,\mu) + \cdots, \tag{A4}$$

where the leading term is $\omega_{\rm eff}^{(0)}(t) \equiv \omega_{\rm eff}(t) = \sqrt{k^2/a^2(t) + m^2 + \mu^2}$. The higher orders are univocally recalculated. For the second order $\omega_{\rm eff}^{(2)}(t,\mu)$ which is enough to renormalize the two-point function, we have

$$\omega_{\rm eff}^{(2)} = \frac{5k^4\dot{a}^2}{8\omega_{\rm eff}^5 a^6} - \frac{3k^2\dot{a}^2}{4\omega_{\rm eff}^3 a^4} + \frac{k^2 a}{4\omega_{\rm eff}^3 a^3} + \frac{3\xi\dot{a}^2}{\omega_{\rm eff} a^2} + \frac{3\xi\ddot{a}}{\omega_{\rm eff} a} - \frac{3\dot{a}^2}{8\omega_{\rm eff} a^2} - \frac{3\ddot{a}}{4\omega_{\rm eff} a} - \frac{\mu^2}{2\omega_{\rm eff}}. \tag{A5}$$

The new terms proportional to μ^2 serve to remove the divergences, in accordance with the new definition of $\omega_{\rm eff}^{(0)}(t,\mu)$ while maintaining locality and general covariance. Note that μ^2 should be regarded as a parameter of adiabatic order 2.

Therefore, the subtraction term for the two-point function is given by

$$^{(2)}G_{\rm Ad}(x,x) = \frac{1}{2(2\pi)^3 a^3}\int d^3k \left[\frac{1}{\omega_{\rm eff}} - \frac{\omega_{\rm eff}^{(2)}}{\omega_{\rm eff}^2}\right]. \tag{A6}$$

After a little bit of algebra the terms in the integral can be written like

$$^{(2)}G_{\rm Ad}(x,x) = \frac{1}{4\pi^2 a^3}\int k^2 dk \left[\frac{1}{\omega_{\rm eff}} + \frac{\mu^2}{2\omega_{\rm eff}^3} + \frac{(\frac{1}{6}-\xi)R}{2\omega_{\rm eff}^3} + \left(\frac{m_{\rm eff}^2\dot{a}^2}{2a^2\omega_{\rm eff}^5} + \frac{m_{\rm eff}^2\ddot{a}}{4a\omega_{\rm eff}^5} - \frac{5m_{\rm eff}^4\dot{a}^2}{8a^2\omega_{\rm eff}^7}\right)\right]. \tag{A7}$$

The last terms in the parentheses are finite and can be integrated to give

$$\frac{1}{4\pi^2 a^3}\int k^2 dk \left[\frac{m_{\rm eff}^2\dot{a}^2}{2a^2\omega_{\rm eff}^5} + \frac{m_{\rm eff}^2\ddot{a}}{4a\omega_{\rm eff}^5} - \frac{5m_{\rm eff}^4\dot{a}^2}{8a^2\omega_{\rm eff}^7}\right] = \frac{R}{288\pi^2}. \tag{A8}$$

Finally we obtain the same subtraction term as in the pragmatic mode-sum regularization method

$$^{(2)}G_{\rm Ad}(x,x) = \frac{1}{4\pi^2 a^3}\int_0^\infty dk k^2 \left[\frac{1}{\omega_{\rm eff}} + \frac{(\frac{1}{6}-\xi)R}{2\omega_{\rm eff}^3} + \frac{\mu^2}{2\omega_{\rm eff}^3}\right] + \frac{R}{288\pi^2}. \tag{A9}$$

Notice that this result agrees with (36) in the coincidence limit, $\epsilon \to 0$, and both agree with (29) when $\mu = 0$.

APPENDIX B: HIGHER ORDER EXPANSION

In this Appendix we expand the two-point function $G_{\rm DS}(x,x')$ to order ϵ^2. This expansion is enough to compute the vacuum expectation value of the stress-energy tensor $\langle T^{\mu\nu}\rangle$ by acting with a nonlocal operator to the symmetric part of the renormalized two-point function, $\langle T_{\mu\nu}\rangle = \lim_{x\to x'}\mathcal{D}_{\mu\nu}(x,x')[\langle\{\phi(x),\phi(x')\}\rangle - G_{\rm DS}^{(1)}(x,x')]$ [7,26].

We begin by expanding (5) to linear order in σ and up to and including four derivatives of the metric, which leads us to (31). For simplicity we will deal with the case $\xi = \frac{1}{6}$. The following expansions are enough to build the renormalized stress-energy tensor ($\sigma^\alpha \equiv \sigma^{;\alpha}$):

$$\Delta^{1/2} = 1 - \frac{1}{12}R_{\alpha\beta}\sigma^\alpha\sigma^\beta - \frac{1}{24}R_{\alpha\beta;\gamma}\sigma^\alpha\sigma^\beta\sigma^\gamma + \left(\frac{1}{288}R_{\alpha\beta}R_{\gamma\delta} + \frac{1}{360}R^\rho{}_\alpha{}^\tau{}_\beta R_{\rho\gamma\tau\delta} - \frac{1}{80}R_{\alpha\beta;\gamma\delta}\right)\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta + \cdots, \tag{B1}$$

$$a_1 = \left[\frac{1}{90}R_{\alpha\rho}R^\rho{}_\beta - \frac{1}{180}R^{\rho\tau}R_{\rho\alpha\tau\beta} - \frac{1}{180}R_{\rho\tau\kappa\alpha}R^{\rho\tau\kappa}{}_\beta + \frac{1}{120}R_{\alpha\beta;\rho}{}^\rho - \frac{1}{360}R_{;\alpha\beta}\right]\sigma^\alpha\sigma^\beta + \cdots, \tag{B2}$$

$$a_2 = -\frac{1}{180}R^{\rho\tau}R_{\rho\tau} + \frac{1}{180}R^{\rho\tau\kappa\iota}R_{\rho\tau\kappa\iota} - \frac{1}{180}R_{;\rho}{}^\rho + \cdots, \tag{B3}$$

where σ^α is computed up to order ϵ^5 using the expansion (19). Expanding (31) with (B1), (B2), and (B3) we arrive at the following expansion for the two-point function up to and including the order $\mathcal{O}(\epsilon^2)$:

$$G^{(1)}_{\rm DS}(x,x') = \frac{1}{4\pi^2\epsilon^2 a^2} + \frac{1}{480\pi^2 m^2}\left[10m^2\left(\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right) - \left(\frac{a^{(4)}}{a}+\frac{\ddot a^2}{a^2}\right) + 3\left(\frac{\dot a^2\ddot a}{a^3}-\frac{a^{(3)}}{a^2}\dot a\right) + 60m^4\left(\gamma-\frac{1}{2}+\log\left(\frac{m\epsilon}{2}a\right)\right)\right]$$
$$+\frac{\epsilon^2}{2880\pi^2}\left[\left(\frac{3}{2}a^{(4)}a+6\ddot a^2-\frac{\dot a^4}{a^2}+\frac{21}{2}a^{(3)}\dot a+\frac{23}{2}\frac{\dot a^2\ddot a}{a}\right)+30m^2(a\ddot a+2\dot a^2)\left(\gamma-\frac{1}{2}+\log\left(\frac{m\epsilon}{2}a\right)\right)\right.$$
$$\left.+45m^4a^2\left(\gamma-\frac{5}{4}+\log\left(\frac{m\epsilon}{2}a\right)\right)\right]+\mathcal{O}(\epsilon^3), \tag{B4}$$

where $a^{(4)} \equiv \ddddot{a}$ and $a^{(3)} \equiv \dddot{a}$. This expression contains terms with four derivatives of the metric $(a^{(4)}, \dot a^4, \dot a a^{(3)}, \cdots)$.

On the other hand, (B4) agrees with

$$^{(4)}G^{(1)}_{\rm Ad}(x,x') = \frac{1}{4\pi^2a^3}\int_0^\infty k^2dk\frac{\sin k\epsilon}{k\epsilon}\left[\frac{1}{\omega}+\frac{(\frac{1}{6}-\xi)R}{2\omega^3}+\frac{m^2\dot a^2}{2a^2\omega^5}+\frac{m^2\ddot a}{4a\omega^5}-\frac{5m^4\dot a^2}{8a^2\omega^7}+(W^{-1})^{(4)}\right], \tag{B5}$$

when it is expanded at order $\mathcal{O}(\epsilon^2)$. Equation (B5) is the expansion of adiabatic regularization at fourth adiabatic order [4]. The integral on $(W^{-1})^{(4)}$ is finite and contains terms with four derivatives of the metric.

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Article 5

Quantum corrections to the Schwarzschild metric from vacuum polarization

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We explore static and spherically symmetric solutions of the 4-dimensional semiclassical Einstein's equations using the quantum vacuum polarization of a conformal field as a source. These solutions may be of interest for the study of exotic compact objects (ECOs). The full backreaction problem is addressed by solving the semiclassical Tolman-Oppenheimer-Volkoff (TOV) equations making use of effective equations of state inspired by the trace anomaly and an extra simplifying and reasonable assumption. We combine analytical and numerical techniques to solve the resulting differential equations, both perturbatively and nonperturbatively in $\hbar$. In all cases the solution is similar to the Schwarzschild metric up to the vicinity of the classical horizon $r = 2M$. However, at $r = 2M + \varepsilon$, with $\varepsilon \sim O(\sqrt{\hbar})$, we find a coordinate singularity. In the case of matching with a static star, this leads to an upper bound in the compactness, and sets a constraint on the family of stable ECOs. We also study the corrections that the quantum-vacuum polarization induces on the propagation of waves, and discuss the implications. For the pure vacuum case, we can further extend the solution by using appropriate coordinates until we reach another singular point, where this time a null curvature singularity arises and prevents extending beyond. This picture qualitatively agrees with the results obtained in the effective two-dimensional approach, and reinforces the latter as a reasonable method.

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I. INTRODUCTION

Advances in gravitational wave (GW) astronomy to detect and analyze GWs in the last years [1], as well as the recent progress in very long baseline interferometry [2], are opening new avenues to study strong-field gravity and the physics of black holes. In particular, with the advent of large amounts of data from GW and electromagnetic observations in the future, it will become possible to test and to quantify in precise terms the existence of horizons. As a result, there is a growing interest in studying models of dark, compact horizonless astrophysical objects that may mimic very closely the behavior of black holes in the GW data, and in examining different physical mechanisms that could be used to uncover these exotic compact objects (ECOs) with observations [3].

While there exists a large class of different models that manage to simulate black holes, most of them require going beyond the Standard Model of particles and/or general relativity (GR) [4–10]. This is because similar values of BH compactness are required to mimic GW observations, but stable astrophysical objects with such compactness are forbidden within GR by Buchdahl's theorem and the classical energy conditions. An appealing possibility is to consider quantum effects (while preserving classical gravity as described by conventional GR), as they can potentially avoid the assumptions of this theorem without requiring exotic assumptions. This involves facing the difficulties of the renormalized stress-energy tensor $\langle T_{ab}\rangle$, describing the gravitational vacuum polarization of quantum fields, and also solving the corresponding semiclassical backreaction equations. So far all methods developed to compute $\langle T_{ab}\rangle$ in quantum field theory in curved spacetime, either analytical or numerical, assume a fixed background metric. Even fixing the background, the explicit computation of $\langle T_{ab}\rangle$ is complicated and only a few examples are known, mainly in cosmology [11–13] and for stationary configurations [14–16]. As a consequence, the problem of solving the full semiclassical Einstein's equations is terribly complicated, even approximately. Since the nontrivial $(t-r)$ part of a spherically symmetric metric is two-dimensional, a popular approach in the past has been to consider the analogous problem in effective $1+1$ dimensions. A first attempt in this direction is to truncate the theory to the s-wave sector of the matter field and implement dimensional reduction by integrating the angular

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degrees of freedom. One ends up with an effective two-dimensional theory (i.e., a particular dilaton-gravity theory [17]), which, after further simplifying assumptions (near-horizon approximation), has a semiclassical description univocally determined by the two-dimensional trace anomaly $\langle T\rangle = \frac{\hbar}{24\pi}R^{(2)}$ (this is usually referred to as the Polyakov theory approximation [17]). In two-dimensions the trace anomaly is sufficient to fix the quantum stress-energy tensor, which in turn can be used to produce a reasonable approximation for evaluating static quantum corrections to the Schwarzschild geometry in vacuum. The semiclassical solution is similar to the classical Schwarzschild solution until very close to the event horizon, but the near-horizon geometry is replaced by a bouncing surface for the radial coordinate, mimicking the throat of a nonsymmetric wormhole. A curvature singularity is found beyond the throat [18]. This picture has been confirmed with more analytical details in [19] and also in [20] (using a natural deformation of the Polyakov theory approximation), and very interesting extensions for stellar configurations have been analyzed in [21–23].

The above two-dimensional effective method is expected to provide important insights, but since the problem is very relevant and it is not entirely clear to what extent the two-dimensional approach is really a good approximation, an intrinsic four-dimensional approach is demanded. This is one of the aims of this work. Our strategy here will be to solve the full semiclassical Einstein's equations but without explicitly calculating $\langle T_{ab}\rangle$. Instead, we shall approach the problem as in classical general relativity, by simply giving equations of state and some appropriate boundary conditions. One of the equations of state will be determined by the four-dimensional trace anomaly, which is independent of the choice of quantum state. More specifically, we will consider a conformal quantum field, in which the trace of $\langle T_{ab}\rangle$ is entirely determined by the anomaly. Then, we will assume a natural condition on the tangential pressure which we expect to capture the main qualitative aspects of the actual solution (we differ here from the assumptions given in [24]). This will make the problem manageable and will allow us to approach the problem directly in four dimensions.

In this new framework we will also be interested in investigating whether there exists physically reasonable, horizonless "vacuum" geometries which may mimic black holes (e.g., wormholes), as well as analyzing what implications the quantum vacuum-polarization from the exterior geometry may have on static ECOs. Uniqueness theorems in classical GR tell us that the exterior vacuum solution of any ECO must be described by the Schwarzschild metric, and this is widely taken for granted in the literature. However, quantum fields exist all around, and their presence may break this degeneracy with respect to black holes.

Even though semiclassical gravity may provide a conservative framework for studying the formation and/or exterior geometry of exotic astrophysical objects, for solar-mass scales it is often expected that quantum effects should only lead to extremely low corrections of the classical solutions, in such a way that from an observational point of view the difference is totally negligible. Remarkably, recent works developed by different independent groups have shown that even tiny corrections to the metric may significantly alter the quasinormal mode (QNM) frequency spectrum of black holes [25–28], opening the possibility of constraining these quantum corrections with GW spectroscopy. Incidentally, this provides a fantastic opportunity to test quantum field theory in astrophysics and adds further motivation to address the historical difficulties encountered when solving the semiclassical Einstein's equations.

The paper is organized as follows. In Sec. II we provide the setup of the calculation by writing down the central equations, as well as by specifying and motivating the assumptions in our problem. Then in Sec. III we solve the differential equations, combining both analytical and numerical techniques, and highlight the main features of the solution obtained, as well as the implications for ECOs. In Sec. IV we obtain the maximal extension and describe the curvature singularity that arises. Section V is devoted to physical applications of the obtained semiclassical metric. In particular we derive the dynamical equations governing scalar and electromagnetic waves, estimate the associated light-ring frequencies using the WKB approximation, and compare them with the Schwarzschild case. Finally, in Sec. VI we present our conclusions.

Our conventions are as follows. We work in geometrized units $G = c = 1$ and keep $\hbar$ explicit throughout. The metric signature has signature $(-,+,+,+)$, ∇_a will denote the associated Levi-Civita connection, the Riemann tensor is defined by $2\nabla_{[a}\nabla_{b]}v_c =: R_{abc}{}^{d}v_d$ for any 1-form v_d; the Ricci tensor is defined by $R_{ab} \coloneqq R_{acb}{}^{c}$; and the scalar curvature is $R \coloneqq g^{ab}R_{ab}$. All tensors and functions are assumed to be smooth, unless otherwise stated.

II. SEMICLASSICAL TOV EQUATIONS IN QUANTUM VACUUM

Our aim in this work is to study solutions of the semiclassical Einstein's equations

$$G_{ab} = 8\pi(\langle T_{ab}\rangle + T_{ab}^{\mathrm{classical}}), \tag{1}$$

in order to find an effective metric that may describe quantum corrections to classical black hole spacetimes induced by the quantum vacuum, or even a new family of solutions. Here $T_{ab}^{\mathrm{classical}}$ represents some classical gravitational source, while $\langle T_{ab}\rangle$ denotes the expectation value of the stress-energy tensor, evaluated for some vacuum state $|0\rangle$ of some given quantum field living on the background metric g_{ab} that solves the above equations. For $T_{ab}^{\mathrm{classical}} = 0$ and in the absence of quantum fields the spherically symmetric solution is a Schwarzschild black hole due to Birkhoff's theorem. But if a quantum field is included, $\langle T_{ab}\rangle \neq 0$, and we expect to get a Schwarzschild-type deformed metric due to quantum vacuum effects ascribed to that field. Solving this problem requires finding a vacuum state $|0\rangle$ and a metric g_{ab} that together solve (1). For reasons that we will discuss in more detail below, this is an extraordinary problem and there are currently no systematic techniques available to address the full question. Our strategy will consist in fixing some desirable properties for the vacuum state and solving the resulting PDE for g_{ab}. More precisely, we will demand the vacuum state to be static and invariant under the group of rotations. This may be thought of as the most immediate quantum generalization of the classical Schwarzschild vacuum. The solution to (1) will then correspond to a spherically symmetric and static metric, which in global coordinates $\{t, r, \theta, \phi\}$ can be written as [29]

$$ds^2 = -e^{-2\phi(r)}dt^2 + \frac{dr^2}{1-\frac{2m(r)}{r}} + r^2 d\Omega^2. \tag{2}$$

Physically, the assumption of staticity is fundamental for studying the exterior vacuum region of exotic compact objects (ECOs) that are stable. For black holes, on the other hand, it is well-known that the assumption of staticity leads to the Boulware state, which gives rise to divergences in the stress-energy tensor at the classical horizon [30]. However, this conclusion holds only when the renormalized stress-energy tensor is computed for a test quantum field on a fixed Schwarzschild background. In this work we will evaluate the implications of staticity when considering the whole problem, including the backreaction effect that the quantum vacuum may produce in the metric.

To get the specific values of the metric components in (2) we have to solve (1) for $T^{\text{classical}}_{ab} = 0$. For a static and spherically symmetric vacuum state the most general expression for the renormalized stress-energy tensor $\langle T_{ab}\rangle$ is

$$\langle T_{ab}\rangle = -\langle\rho(r)\rangle u_a u_b + \langle p_r(r)\rangle r_a r_b + \langle p_t(r)\rangle q_{ab}, \tag{3}$$

where $u_a = e^{-\phi}\nabla_a t$ is a timelike vector normalized as $u^2 = -1$, $r_a = (1-\frac{2m(r)}{r})^{-1/2}\nabla_a r$ is a unit spacelike vector, and q_{ab} is the metric on the unit 2-sphere. The metric can be written covariantly as $g_{ab} = -u_a u_b + r_a r_b + q_{ab}$. There are only three independent equations from the semiclassical Einstein equations. On the other hand, there is one nontrivial Bianchi identity. Collecting the tt and rr Einstein's equations and this Bianchi identity we get the following equations

$$\frac{dm(r)}{dr} = 4\pi r^2\langle\rho(r)\rangle, \tag{4}$$

$$\frac{d\phi(r)}{dr} = -\frac{m(r) + 4\pi r^3\langle p_r(r)\rangle}{r^2(1-\frac{2m(r)}{r})}, \tag{5}$$

$$\frac{d\langle p_r(r)\rangle}{dr} = -\frac{m(r) + 4\pi r^3\langle p_r(r)\rangle}{r^2(1-\frac{2m(r)}{r})}(\langle\rho(r)\rangle + \langle p_r(r)\rangle) - \frac{2}{r}(\langle p_r(r)\rangle - \langle p_t(r)\rangle). \tag{6}$$

When $\langle p_r\rangle \neq \langle p_t\rangle$, there are anisotropic pressures. In the isotropic case this system of equations reduces to the usual Tolman-Oppenheimer-Volkoff (TOV) equations. In the rest of the work we will refer to this system of equations as the semiclassical TOV equations.

In this system there are 5 unknowns (3 from the stress-energy tensor and 2 from the metric) for 3 equations. Normally one would compute $\langle T_{ab}\rangle$ and express the result in terms of $\phi(r)$ and $m(r)$ in order to get the system above solved. Instead, we will impose two functional relations between the components of the stress-energy tensor, in order to avoid such a difficult (or unattainable) calculation. First, we will consider the case of a massless quantum field conformally coupled to the spacetime. The advantage of doing this is that the relation between the three independent components of the stress energy tensor is univocally fixed by the trace anomaly $\langle T^a_a\rangle$ as

$$-\langle\rho\rangle + \langle p_r\rangle + 2\langle p_t\rangle = \langle T^a_a\rangle, \tag{7}$$

and the trace anomaly is uniquely determined by the geometry of the spacetime

$$\langle T^a_a\rangle = \frac{\hbar}{2880\pi^2}(\alpha C^{abcd}C_{abcd} + \beta R^{ab}R_{ab} + \gamma R^2 + \delta\Box R). \tag{8}$$

In this expression C_{abcd} is the Weyl tensor, R_{ab} the Ricci tensor, R the Ricci scalar and α, β, γ, δ are real numbers. Most importantly, this result is independent of the choice of the quantum state. The idea of exploiting the trace anomaly goes back to [31]. The constant coefficients depend on the particular field under consideration. It should be noted though that there exits an intrinsic ambiguity in the trace anomaly for the coefficient δ [32]. This ambiguity is related to the choice of the renormalization scheme. The term with $\Box R$ can always be removed by adding a local counterterm in the Lagrangian so, from now on we set $\delta = 0$. This simplifies the problem considerably, since it will avoid derivatives of second and third order of the metric in the field equations.

By evaluating (8) with our metric and using the semiclassical TOV equations written above one can obtain a simplified expression for the trace anomaly in terms of $\langle\rho\rangle$, $\langle p_r\rangle$ and $\langle p_t\rangle$. This leads to the following equation of state

$$-\langle\rho\rangle+\langle p_r\rangle+2\langle p_t\rangle=\frac{\hbar}{270}\left[\alpha\left(\frac{3m}{4\pi r^3}-\langle\rho\rangle+\langle p_r\rangle-\langle p_t\rangle\right)^2+6\beta(\langle\rho\rangle^2+\langle p_r\rangle^2+2\langle p_t\rangle^2)+6\gamma(-\langle\rho\rangle+\langle p_r\rangle+2\langle p_t\rangle)^2\right]. \tag{9}$$

For definiteness in this work we restrict to scalar fields, for which the coefficients are $\alpha=\beta=1$ and $\gamma=-1/3$. For these values the above expression can be further simplified to

$$-\langle\rho\rangle+\langle p_r\rangle+2\langle p_t\rangle=\frac{\hbar}{180}\left[\frac{m}{\pi r^3}\left(3\frac{m}{\pi r^3}+8(-\langle\rho\rangle+\langle p_r\rangle-\langle p_t\rangle)\right)+8\langle\rho\rangle(\langle\rho\rangle-\langle p_r\rangle+2\langle p_t\rangle)+8(\langle p_r\rangle-\langle p_t\rangle)^2\right]. \tag{10}$$

We need another restriction to make our system of equations solvable. Unfortunately there are no other universal geometric properties of the stress-energy tensor that may allow us to fix a similar relation between the different components of the stress-energy tensor. To proceed further we need to impose a condition on $\langle T_{ab}\rangle$ based on what we may expect from the quantum state. We will consider here that $\langle p_r\rangle=\langle p_t\rangle$. This simplifying assumption is inspired by the "zero-order" result that one gets when calculating $\langle T_{ab}\rangle$ in a fixed Schwarzschild background when $r\to 2M$, and we expect this near-horizon approximation to capture the qualitative behavior of the actual solution. Indeed, in a Schwarzschild spacetime background the vacuum expectation value $\langle T_{ab}\rangle$ of a conformal scalar field in the static spherically symmetric state behaves, in the vicinity of the horizon, as [30]

$$\langle T^\nu_\mu\rangle\sim-\frac{\hbar}{2\pi^2(1-2M/r)^2}\int_0^\infty\frac{d\omega\omega^3}{e^{8\pi M\omega}-1}\begin{bmatrix}-1&0&0&0\\0&\frac{1}{3}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{3}\end{bmatrix}. \tag{11}$$

Both $\langle T^\theta_\theta\rangle\equiv\langle p_t\rangle$ and $\langle p_r\rangle\equiv\langle T^r_r\rangle$ merge for $r\to 2M$, but as one moves away from the vicinity of the horizon, the tangential and radial pressures start to differ. In fact, for $r\to\infty$ one has $\langle p_r\rangle=-\frac{1}{3}\langle p_t\rangle\sim\mathcal{O}(r^{-5})$ [33]. Therefore, our assumption is expected to work only qualitatively as an approximation to the actual relationship, whose knowledge requires computing $\langle T_{ab}\rangle$ in detail. This simplification is expected to capture the main physical ingredients of our field theory (the results obtained will be exact at least in a neighborhood of the classical horizon).

Our approach can be easily compared with other works by fixing this free condition with different assumptions. For instance, the effective two-dimensional Polyakov approximation [18,19,21,22] can be regarded as fixing trivially the tangential pressure $\langle p_t\rangle=0$ (or with additional extra deformations [20,23]) and restricting the trace anomaly to its two-dimensional value. Instead, we are trying to solve the 4D problem directly without assuming *a priori* that it is similar to the 2-dimensional case. On the other hand, the approach of [24] also quantizes the matter field in four dimensions, but assumes that $\langle p_t\rangle$ is regular as $r\to 2M$, even in the Schwarzschild background. Instead, our assumption is compatible with Eq. (11).

III. SEMICLASSICAL METRIC SOLUTION

A. Perturbative analytical solution

The leading order contributions of the stress-energy tensor are expected to behave as $\langle\rho\rangle\sim O(\hbar^1)$, $\langle p\rangle\sim O(\hbar^1)$ [where $\langle p\rangle=\langle p_r\rangle=\langle p_t\rangle$]. We can thus look for perturbative solutions of the semiclassical TOV equations, solving the system order by order in powers of $\hbar$. In this subsection we will obtain the first order correction using analytical techniques, and in the next subsection we will analyze the validity of this approach by solving the system of equations numerically.

Solving the TOV equations at order $\hbar^0$ gives $m(r)=M+O(\hbar)$ and $\phi\sim-\frac{1}{2}\log(1-2M/r)+O(\hbar)$, where M is an arbitrary constant of integration, which can be identified with the ADM mass. This is the Schwarzschild metric, as expected at order $\hbar^0$. To get something interesting we have to solve the equations at first order in $\hbar$. Let us define $m=M+m_1\hbar+O(\hbar^2)$, $\phi\sim-\frac{1}{2}\log(1-2M/r)+\phi_1\hbar+O(\hbar^2)$, $\langle\rho\rangle=\rho_1\hbar+O(\hbar^2)$, $\langle p\rangle=p_1\hbar+O(\hbar^2)$. Then the system of equations at first order in $\hbar$ is given by

$$\frac{dm_1}{dr}=4\pi r^2\rho_1, \tag{12}$$

$$\frac{d\phi_1}{dr}=-\frac{m_1}{r^2f^2}-\frac{4\pi r p_1}{f}, \tag{13}$$

$$\frac{dp_1}{dr}=-\frac{M}{r^2f}(\rho_1+p_1), \tag{14}$$

$$-\rho_1+3p_1=\frac{M^2}{60\pi^2r^6}, \tag{15}$$

where $f = 1 - \frac{2M}{r}$. This system can be solved analytically, obtaining the following expressions for the pressure and density[1]

$$\langle p \rangle = -\frac{\hbar M^3}{480\pi^2 r^7 f^2}\left(\frac{1}{7} + f\right) + \mathcal{O}(\hbar^2), \tag{16}$$

$$\langle \rho \rangle = \hbar\left(-\frac{M^3}{160\pi^2 r^7 f^2}\left(\frac{1}{7}+f\right) - \frac{M^2}{60\pi^2 r^6}\right) + \mathcal{O}(\hbar^2), \tag{17}$$

and the following ones for the metric components

$$m = M + \frac{\hbar}{40320\pi M f}(9 - 36f\log(f) + 10f - 174f^2 + 246f^3 - 91f^4) + O(\hbar^2), \tag{18}$$

$$\phi = -\frac{1}{2}\log f + \frac{\hbar}{80640\pi M^2 f^2}(3 + 36(-1+3f)f\log(f) - 35f + 152f^2 - 132f^3 + 5f^4 + 7f^5) + O(\hbar^2). \tag{19}$$

To fix the constants of integration we have assumed the natural boundary conditions $\langle p \rangle(r \to \infty) = 0$, $\langle \rho \rangle(r \to \infty) = 0$ and the metric tending to the Schwarzschild one as $r \to \infty$. For pedagogical purposes, we display the asymptotic form of the metric around $r = 2M$

$$ds^2 = -\left(f(r) - \hbar\left(\frac{1}{13440\pi M^2 f(r)} + \mathcal{O}(\log f)\right) + \mathcal{O}(\hbar^2)\right)dt^2 + \frac{dr^2}{f(r) - \hbar(\frac{1}{4480\pi M^2 f(r)} + \mathcal{O}(\log f)) + \mathcal{O}(\hbar^2)} + r^2 d\Omega^2. \tag{20}$$

In the Appendix we prove that the curvature at this singular point is finite, so this is just a coordinate singularity. In fact, this is just the classical Schwarzschild coordinate singularity at $r = 2M$ shifted to the value r_0 defined by $g_{rr}^{-1}(r_0) = 0$. Using the expression (18) and imposing $2m(r_0) = r_0$, we easily obtain

$$r_0 = 2M + \frac{\sqrt{\hbar}}{4\sqrt{70\pi}} + \mathcal{O}(\hbar). \tag{21}$$

In geometrized units $\sqrt{\hbar} = l_p$ is the Planck length. This singular, limiting point defines the end of validity of our coordinate system, which would traditionally indicate the location of a "horizon" at $r = r_0$. However, note that, unlike the Schwarzschild case, in this point the component g_{tt} of the metric (the so called redshift function) does not vanish, but takes the value

$$g_{tt}(r_0) = -\frac{\sqrt{\hbar}}{12\sqrt{70\pi}M} + \mathcal{O}(\hbar). \tag{22}$$

This implies that the static spacetime that we have obtained does not contain a horizon, i.e. it is not defining a black hole [34]. We check this in the Appendix.

Note that, even though (16) and (17) are generally very small (because of the prefactor $\hbar$), they become relevant around $r \sim r_0$, since in this limit the factor $f(r)$ in the denominator can compensate $\hbar$. In other words, quantum effects are quite important near the location of what was classically the horizon. The Krechtmann scalar is also found to be significantly corrected at the singular point (see Appendix). These observations lead us to the following subsection.

B. Nonperturbative numerical solution

As we can see the results obtained above at first order in $\hbar$ also depend on $f(r)$, which takes values of order $\sqrt{\hbar}$ near the singular point $r = r_0$. This dependence compensates the small value of $\hbar$ in some expressions above. Because of this, a natural question is whether the perturbative method is a good approximation near to the singular point. To answer this we can solve the TOV equations at second order in $\hbar$ and analyze whether near the singular point the solution is consistent with the perturbative hypothesis (i.e. that the order $\hbar^1$ is larger than the order $\hbar^2$, etc.). The analysis is tedious and we avoid showing the details. What we obtain is that the $\hbar^2$ contribution to the pressure and the density is proportional to $\hbar^2/f(r)^4$. Near to the singular point $f(r)^4$ is of order $\hbar^2$, so this term competes with the first order contribution (16), which is proportional to $\hbar/f(r)^2$. Therefore we find that, near the singular point, the higher order contributions in $\hbar$ are not necessarily smaller than the first one and perturbation theory actually breaks down. Therefore, we cannot rely on the perturbative series in the vicinity of r_0 and we are forced to solve the differential TOV equations exactly, which can only be done numerically. Still, we shall find that the perturbative approach presented in the previous subsection is a good approximation to the problem, and it qualitatively predicts well the behavior of the nonperturbative solution.[2]

[1]The negative sign and the dependence on $1/f^2$ obtained in these expressions are in agreement with the exact results obtained on the fixed (Schwarzschild) background near the horizon for the Boulware vacuum state [see (11)].

[2]In this paper we work in the semiclassical regime in which fluctuations of the stress-energy tensor are negligible compared to its mean value. Going beyond this framework would require working with techniques in stochastic gravity [13], which is out of the scope of the present paper. By nonperturbative we mean the exact solution of the TOV equations within the semiclassical framework.

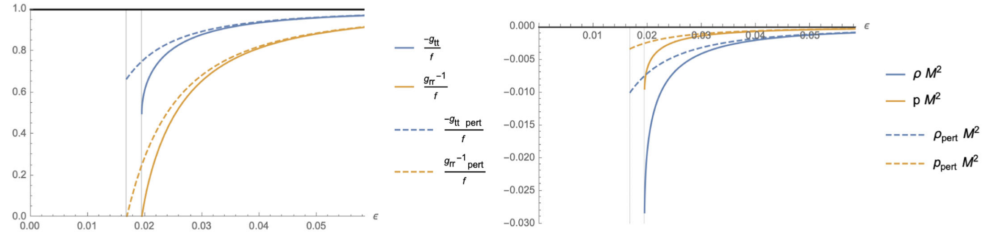

FIG. 1. Numerical results obtained for the metric components and the renormalized energy density and pressure near the singular point $\epsilon = 0.01949$ (where $r = 2M + \epsilon\sqrt{\hbar}$). We have chosen $\hbar/M^2 = 10^{-5}$, but the plots do not change significantly for other values. We compare them with the perturbative solution (dashed curves), for which the singular point is $\epsilon = 0.01686$.

We now turn to solve numerically the TOV equations (4)–(6) using the equation of state (10) and $\langle p_r \rangle = \langle p_t \rangle$. We place the boundary conditions at $r = 1000M$, and demand that at this location the solution is approximately the Schwarzschild metric.[3] An important issue that one faces when solving the equations numerically is that the value of $\hbar$ is much smaller than M. To be able to distinguish the implications of a nonzero but tiny value of $\hbar$ from the numerical error, one needs a huge computer accuracy. To avoid this issue, a useful strategy is to use first some artificial high values of $\hbar$ (between $10^{-5}M^2$ and $10^{-15}M^2$), study the dependence of the results on $\hbar$, and then extrapolate the relevant quantities to the actual value of $\hbar$. By solving numerically the equations for different values of $\hbar$ and calculating for each case the value of r_0 we obtain results that approximately fit the expression $r_0 \approx 2M + 0.01947\sqrt{\hbar}$. This shift differs from the one estimated by the perturbative method ($r_0 \approx 2M + 0.01686\sqrt{\hbar}$) but the functional dependence on $\hbar$ remains the same.

In Fig. 1 we plot the components of the metric obtained numerically, normalized by the factor $f(r) = 1–2M/r$, as well as the renormalized energy density and pressure. They are plotted as a function of $\epsilon = \frac{r-2M}{\sqrt{\hbar}}$. With this new radial variable the singular point r_0 does not depend on the specific value of $\hbar$. These plots are taken for $\hbar/M^2 = 10^{-5}$, but we have analyzed them for other values and have seen that they do not significantly depend on the chosen value of $\hbar$ near the singular point. From these plots one can see that, as in the perturbative solution, the component g_{rr}^{-1} tends to 0 at the singular point $r = r_0$, while g_{tt} tends to a nonzero value. The energy density and pressure differ from 0 as they approach the singular point, as expected. More precisely $g_{tt} \sim O(\sqrt{\hbar}/M)$, $g_{rr}^{-1} \sim (r - r_0)/M$, $\rho \sim O(\hbar^0)$, and $p \sim O(\hbar^0)$ as $r \to r_0$. This is the same dependence on $\sqrt{\hbar}/M$ as that obtained by the perturbative approach, although the numerical coefficients are different. This allows us to consider the perturbative solution as a qualitatively good approximation.

[3]To get more precision we can choose the corrected solution at first order in $\hbar$ obtained above, but the results near the singular point are numerically indistinguishable.

We can summarize the above numerical result in terms of the following generic expression for the metric

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2, \qquad (23)$$

where $g_{rr}^{-1} \to 0$, as $r \to r_0 > 2M$ and $g_{tt}(r_0) \neq 0$. Furthermore, $g_{rr}^{-1} \sim (r - r_0)/M$ and $g_{tt}(r) \sim \mathcal{O}(\sqrt{\hbar}/M)$ in a neighborhood of r_0.

IV. EXTENSION BEYOND THE COORDINATE SINGULARITY

The metric (23) [or (20)] is only meaningful when $r > r_0$ because of the coordinate singularity at $r = r_0$. We recall (see Appendix) that the curvature scalars are finite at $r = r_0$. Physically this effective metric can be used to describe the exterior spacetime of a static, spherically symmetric star, including the vacuum polarization effects of quantum fields around. But in close analogy to the classical Schwarzschild case when expressed in $\{t, r, \theta, \phi\}$ coordinates, one may attempt to extend the spacetime across the $r = r_0$ point and examine if there exists a purely (quantum) vacuum solution. As remarked at the end of Sec. III. A, the usual Eddington-Finkelstein coordinates fail to provide a regular metric, which prevents the usual analytical extension beyond $r = r_0$.

By looking at the specific form of the metrics (23) or (20) one realizes that they can be used to construct a portion of a static, traversable (and Lorentzian) wormhole [35,36]. By introducing the usual proper-length coordinate $l(r) \equiv \int_{r_0}^{r} 1/\sqrt{1 - 2m(r')/r'}dr' \geq 0$ the metric can be rewritten to fit the Morris-Thorne ansatz

$$ds^2 = -e^{-2\phi(l)}dt^2 + dl^2 + r(l)^2 d\Omega^2. \qquad (24)$$

Therefore, one can extend the spacetime beyond the critical point $r = r_0$ or $l = 0$ (which physically represents the throat of the wormhole) by analytically extending to negative values of l. The function $r = r(l)$ is determined by inverting the equation $l = l(r)$ given above, but only when $l > 0$. For $l < 0$ the function $r = r(l)$ must be determined by other means.

A. Setup

Instead of working with the metric ansatz (2) and then transforming to (24) by a change of variables, we can alternatively solve the problem from scratch using the latter metric directly and explore if there exist wormhole solutions. The equivalent system of TOV equations now reads (we find convenient to introduce the defining relation $g(l) \equiv \frac{dr}{dl}$)

$$\frac{dr}{dl} = g, \tag{25}$$

$$\frac{dg}{dl} = \frac{1 - 8\pi r^2 \langle \rho \rangle + g^2}{2r}, \tag{26}$$

$$\frac{d\phi}{dl} = \frac{-1 - 8\pi r^2 \langle p_r \rangle + g^2}{2rg}, \tag{27}$$

$$\frac{d\langle p_r \rangle}{dl} = \frac{(-1 - 8\pi r^2 \langle p_r \rangle + g^2)(\langle p_r \rangle + \langle \rho \rangle)}{2rg} + \frac{2g(\langle p_t \rangle - \langle p_r \rangle)}{r}. \tag{28}$$

There are six unknowns for four equations. Again, we can impose two equations of state to get a solvable model. As before, we shall take $\langle p_t \rangle = \langle p_r \rangle$ (notice that the contribution of $\langle p_t \rangle - \langle p_r \rangle$ is negligible near the throat, where as we will see $g(0) = 0$) and $\langle T^a_a \rangle$ given by the trace anomaly:

$$-\langle \rho \rangle + 3\langle p \rangle = \frac{\hbar}{180}\left[\frac{1-g^2}{2\pi r^2}\left(3\frac{1-g^2}{2\pi r^2} - 8\langle \rho \rangle\right) + 8\langle \rho \rangle(\langle p \rangle + \langle \rho \rangle)\right]. \tag{29}$$

To get wormhole solutions we must impose several conditions. Without loss of generality, we can locate the throat at $l = 0$. One of the sectors of the throat (that would represent the universe we live in) must be asymptotically flat, and inertial observers at infinity must measure time with t. We choose that sector corresponding to $l > 0$. Then the previous condition requires $\phi(\infty) = 0$, $\langle p_r \rangle(\infty) = \langle \rho \rangle(\infty) = 0$. On the other hand, the coordinate l should agree with the radial function $r(l)$ at infinity, i.e. $r(l) \to l$ as $l \to \infty$. Furthermore, for sufficiently large distances away from the throat, g must be given by the Morris-Thorne coordinate transformation (the solution should mimic a hole at large distances), i.e. $g(l) \sim \sqrt{1 - 2m(l)/l}$ and therefore $g(\infty) = 1$.

This set of boundary conditions, together with the two equations of state specified above, can be used to obtain a unique solution to the above system of differential equations, integrating all the way down from $l = +\infty$ until negative values of l. Notice that in general there will be no mirror-reflection symmetry at the throat. The results are shown in the next subsection. For the solution to represent a wormhole, note that (i) the throat must have a finite, nonvanishing radius, so $r(0) = r_0 > 0$, and (ii) the throat area must correspond to a minimum, therefore $g(0) = 0$.

Before discussing the results, we remark an important issue. It may seem that the above system of equations is not well defined at the throat $l = 0$ because of $g(0) = 0$ in the denominator of some equations. But notice that, according to Einstein's equation,

$$\langle p_r(r) \rangle = -\frac{1}{8\pi}\left[\frac{2m}{r^3} - 2\left(1 - \frac{2m}{r}\right)\frac{\partial_r \phi}{r}\right], \tag{30}$$

so at the throat (where $2m(r) = r$) we also have $p_r(0) = -1/(8\pi r_0^2)$, provided that $\partial_r \phi(0)$ is well-defined at the throat or that it does not blow up as quickly as $(1 - 2m/r)^{-1}$ (this is verified in this case, using the numerical solution obtained in the previous section one can see that $\partial_r \phi \sim (1 - 2m(r)/r)^{-1/2}$ when $r \to r_0$).

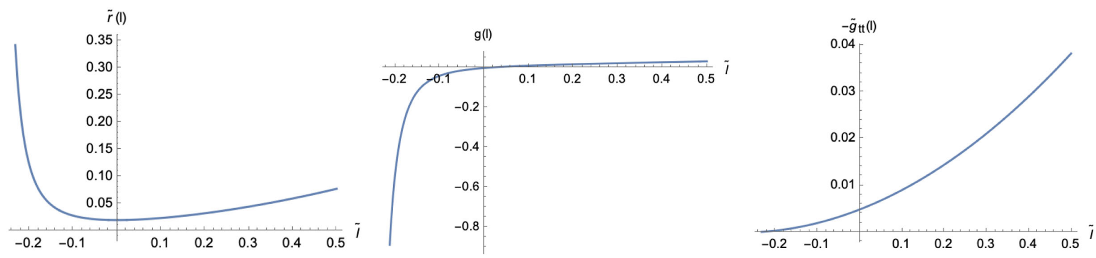

FIG. 2. Numerical results obtained for the components of the metric (24) and $g(l) = r'(l)$ in terms of $\tilde{l} = l\hbar^{-1/4}M^{-1/2}$. The represented interval of $\tilde{l}$ includes the throat ($\tilde{l} = 0$) and the curvature singularity ($\tilde{l} \approx -0.278$). We have defined the quantities $\tilde{r} = (r - 2M)\hbar^{-1/2}$ and $\tilde{g}_{tt} = g_{tt}\hbar^{-1/2}M$, in such a way that their values at the throat do not depend on the chosen value of $\hbar$. We have chosen $\hbar/M^2 = 10^{-3}$ for these plots, but they have a similar form for other values.

Therefore, the numerator of (27) and (28) also vanishes whenever the denominator does, and we have a $0/0$ ambiguity. To ensure that we can extend the metric across the throat one needs to check first that the limit $l \to 0$ tends to a finite value under the boundary conditions specified above. Numerically we find that near the throat $1 + 8\pi r^2 p \sim O(l)$ and $g \sim O(l)$, so we can conclude that the limit of the quotient will be finite.

B. Results

In Fig. 2 we show the result of solving numerically the system of equations (25), (26), (27), (28) and (29) under the conditions specified in the previous subsection. As in Sec. III. B, to capture the implications of a nonvanishing but tiny value of $\hbar$ on the equations, we do the calculation for several high values of $\hbar$ (so that their effect is numerically distinguishable), then we perform a fit of the results to be able to extrapolate the value of interest with the actual value of Planck's constant. In our calculation the throat is located at $l = 0$, note how at this point there is a bounce in the function $r(l)$ (its derivative $g(l)$ changes sign).

Furthermore, we find that in the interior region, $l < 0$, a new singular point appears at $l_s \sim -0.278\hbar^{1/4}\sqrt{M}$. It is a singular point because the redshift function vanishes there, $g_{tt}(l_s) = 0$. As we approach to l_s we find that the renormalized density, the pressure and the scalar of curvature $R = 8\pi(-\rho + 3p)$ all tend to diverge. This signals the existence of a curvature singularity. To confirm the existence of this singularity from an analytical viewpoint we can examine the expression of the scalar curvature in terms of the metric components:

$$R(l) = \frac{g'_{tt}(l)^2}{2g_{tt}(l)^2} - \frac{g''_{tt}(l)r(l) + 2g'_{tt}(l)r'(l)}{g_{tt}(l)r(l)} - \frac{2(2r(l)r''(l) + r'(l)^2 - 1)}{r(l)^2}. \tag{31}$$

Since at the singular point $g_{tt}(l_s) = 0$ (see Fig. 2) some terms of this expression diverge at this point. Although $g'_{tt}(l)$ also vanishes at $l = l_s$, numerical computations show that it decreases slower than $g_{tt}(l)$. To see the causal character of this curvature singularity, let us consider the induced metric on a $l =$ constant three-dimensional hypersurface: $d\bar{s}^2 = g_{tt}(l)dt^2 + r(l)^2 d\Omega^2$. At the singularity $l = l_s$ we have $g_{tt}(l_s) = 0$, so the metric becomes degenerate: $d\bar{s}^2 = 0 + r(l_s)^2 d\Omega^2$. Therefore, the surface $l = l_s$ becomes a null hypersurface [37], and this curvature singularity is null. Figure 3 provides a Penrose diagram that shows all these features.

An important question is how long it would take for an observer crossing the throat to reach this curvature singularity. To study this let us consider a radial and time-like geodesic starting at $l = 0$ (throat) and ending at the singular point $l = l_s$. The relevant geodesic equation for a static and spherically symmetric metric $ds^2 = g_{tt}(l)dt^2 + g_{ll}(l)dl^2 + r(l)^2 d\Omega^2$ is given by

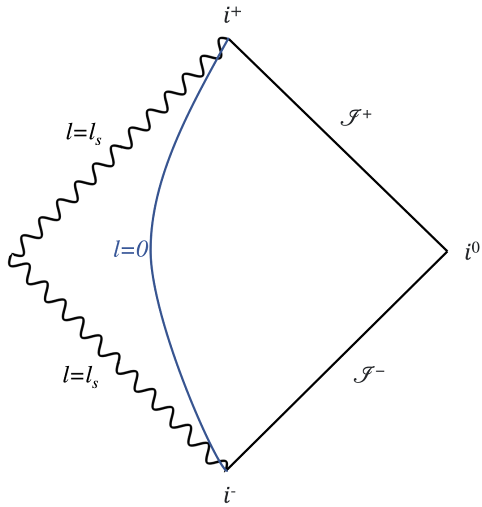

FIG. 3. Penrose diagram showing the wormhole throat ($l = 0$) and the null curvature singularity ($l = l_s$).

$$\frac{dl}{d\tau} = \pm\sqrt{-g_{ll}^{-1}\left(E^2 g_{tt}^{-1} + \frac{L^2}{r^2} + \mu\right)} \tag{32}$$

where τ is the proper time, $\mu = +1, 0, -1$ for timelike, null and spacelike geodesics respectively, and E and L are constants of motion given by $E = -g_{tt}\frac{dt}{d\tau}$ and $L = r^2\frac{d\phi}{d\tau}$. In our case $g_{ll} = 1$, $\mu = 1$, $L = 0$, and $dl/d\tau < 0$ (the geodesic is approaching the singularity), so

$$\frac{dl}{d\tau} = -\sqrt{E^2 g_{tt}(l)^{-1} - 1}. \tag{33}$$

The proper time needed to reach the curvature singularity from the throat is then given by

$$\Delta\tau = -\int_0^{l_s} \frac{dl}{\sqrt{E^2 g_{tt}(l)^{-1} - 1}}. \tag{34}$$

(the condition that the geodesic propagates into the future, $\partial_\tau t > 0$, implies $E^2 g_{tt}(l)^{-1} > 1$ and guarantees that the integral is real). The order of magnitude of this quantity can be estimated as follows. From (22) we know that $g_{tt}(0) \sim \sqrt{\hbar}/M$. Assuming $E \sim 1$, we have $E^2 g_{tt}^{-1}(l) \gg 1$ in the region of the integration. Since $|l_s| \sim \hbar^{1/4}\sqrt{M} \ll 1$ we can also Taylor expand the integral to finally get

$$\Delta\tau \sim -\int_0^{l_s} \sqrt{g_{tt}(l)}dl \sim \sqrt{g_{tt}(0)}|l_s| + O(l_s^2) \sim \sqrt{\hbar} \tag{35}$$

So an observer crossing the throat will almost immediately see the presence of the curvature singularity.

Finally, we want to stress that the occurrence of the curvature singularity has been obtained for a purely vacuum semiclassical solution. The presence of matter producing very compact stellar objects (ECOs) makes only the outer part of the solution physically relevant. Moreover, these results also suggest a maximum in the compactness of ECOs. This maximum would be given by the minimum of the radial function $r(l)$, i.e. the throat ($r = r_0$). Therefore this maximum of compactness [measured as $2M/r(l)$] is of order

$$\frac{2M}{r_0} \sim 1 - 0.01686\frac{\sqrt{\hbar}}{2M}. \tag{36}$$

We regard (36) as one of the main results of this work. Probing the exterior of the semiclassical metric via scalar and vector perturbations will be the topic of the next section.

Remark: Another way to extend the metric beyond the coordinate singularity $r = r_0$ consists in defining a coordinate $\bar{r}$ by $\frac{d\bar{r}}{dr} = e^{-\phi(r)}(1 - \frac{2m(r)}{r})^{-1/2}$. In this case the metric has the form

$$ds^2 = -G(\bar{r})dt^2 + \frac{dr^2}{G(\bar{r})} + R(\bar{r})^2 d\Omega^2. \tag{37}$$

Using this metric as an ansatz for solving the semiclassical TOV equations we found that the functions G and R can be analytically extended beyond the coordinate singularity $\bar{r} = \bar{r}_0$. In particular $R(\bar{r})$ reaches a minimum at $\bar{r}_0$ and starts increasing for lower values, as expected for a wormhole metric. On the other hand $G(\bar{r})$ continues to decrease until it reaches the value $\bar{r} = \bar{r}_s$, where $G(\bar{r}_s) = 0$. At this point we again find a curvature singularity, which is equivalent to the one found in the other extension explained above. Therefore, with this alternative extension, we obtain the same conclusions. However the approach described above allows a higher accuracy in the numerical calculations.

V. PROPAGATION OF WAVES IN THE SEMICLASSICAL METRIC

The propagation of waves on a given spacetime background provides a way to test some features of this metric by studying the scattering properties of the wave. Furthermore, they provide a means to test the stability of the metric under linear perturbations, which is a necessary condition for any semiclassical metric that aims to describe acceptable astrophysical systems. In this section we will study scalar and electromagnetic perturbations around the semiclassical metric constructed in Sec. III. In particular, we will compute the leading order corrections to the light-ring frequency modes. These frequencies depend only on the geometry around the light-ring of the classical black hole, and describe the early ringdown stage in gravitational wave observations of binary mergers. While the late ring-down stage is expected to be described by the proper QNM frequencies [38,39], the calculation of these is out of the scope of the present paper.

A. Scalar perturbations

Let us study the behavior of a massless scalar field coupled to a general static and spherically symmetric background, $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2$. The field satisfies the Klein-Gordon equation

$$(\Box + \xi R)\phi = 0. \tag{38}$$

Since the metric is static and spherically symmetric, we can look for solutions of the following form

$$\phi_{\omega lm} = \frac{1}{r} e^{-i\omega t} Y_{lm}(\theta, \psi)\Psi_{\omega l}(r). \tag{39}$$

For a curved spacetime the D'Alembert operator is given by $\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$. The Klein-Gordon equation decouples and, after some calculations one obtains the following equation for the radial function

$$F^2\Psi''_{\omega l} + FF'\Psi'_{\omega l} + (\omega^2 - V_l)\Psi_{\omega l} = 0, \tag{40}$$

where

$$F(r) = \sqrt{-\frac{g_{tt}}{g_{rr}}}, \tag{41}$$

$$V(r) = -g_{tt}(r)\left(\frac{l(l+1)}{r^2} - \xi R(r)\right) + \frac{F(r)F'(r)}{r}. \tag{42}$$

If we define a generalized tortoise coordinate as $\partial_{r^*} = F(r)\partial_r$, then Eq. (40) can be rewritten in the usual Regge-Wheeler form $\partial_{r_*}^2\Psi_{\ell m} + (\omega^2 - V)\Psi_{\ell m} = 0$. In particular, for a Schwarzschild background we recover the usual expression.

Now let us study this potential for our particular case, given by $g_{tt} = -e^{-2\phi(r)}$ and $g_{rr} = 1 - \frac{2m(r)}{r}$, where $\phi(r)$ and $m(r)$ have been obtained in Sec. III. Using the semiclassical TOV equations we can rewrite the potential as

$$V(r) = e^{-2\phi(r)}\left(\frac{l(l+1)}{r^2} + \frac{2m(r)}{r^3} + 4\pi((1-6\xi)\langle p(r)\rangle + (-1+2\xi)\langle\rho(r)\rangle)\right). \tag{43}$$

It is easy to see that in the Schwarzschild limit $\hbar \to 0$ this expression reduces to the usual effective potential for scalar fields. We can now use the perturbative solution of the corrected Schwarzschild metric to obtain the correction of the Regge-Wheeler potential at first order in $\hbar$. The resulting expression, at first order in $\hbar$, yields

$$V(r) = V_0(r) + \frac{\hbar}{20160\pi M^2 r^6 f}[-2688M^4\xi f^2 + Mr^3(-3\lambda + (53+32\lambda)f - 40(1+3\lambda)f^2 + 12(-36+\lambda)f^3 + (664+7\lambda)f^4 - 245f^5) + 18r^4((1+\lambda) - (5+3\lambda)f + 4f)f\log(f)] + O(\hbar^2), \tag{44}$$

where $f = 1 - \frac{2M}{r}$, $\lambda = l(l+1)$ and $V_0(r) = f(\frac{\lambda}{r^2} + \frac{2M}{r^3})$, which is the effective potential for scalar fields on a Schwarzschild background. Note that this expression does not depend on ξ. This is because the scalar curvature is given by $R = 8\pi(-\rho + 3p)$, and expanding around $r = 2M$ we have $\rho \approx 3p$ at leading order [see (17) and (16)], so the term ξR is subleading. On the other hand, near the throat $f(r_0) \sim O(\sqrt{\hbar}/M)$ so the quantum correction of the effective potential is of order $O(\sqrt{\hbar}/M^3)$ near the throat, while it is of order $O(\hbar/M^4)$ in general.

Using this expression for the corrected effective potential, we can now obtain the quantum corrections at first order in $\hbar$ to the light ring frequencies. The computation of these frequencies requires numerical methods. However, one can obtain a reasonable estimation by using the WKB approximation [40].

Let us briefly review the calculation for a Schwarzschild metric. In this framework the light ring frequencies at 0th adiabatic order are given by

$$\omega_n^2 = V(r_m^*) - i\sqrt{-2V''(r_m^*)}\left(n + \frac{1}{2}\right), \tag{45}$$

where r_m^* is the value of the tortoise coordinate at which the potential is maximum, and the primes mean derivatives with respect to r^*. In the case of a Schwarzschild background the maximum of the potential is located at

$$r_m = \frac{3(\lambda-1) + \sqrt{(9\lambda+14)\lambda+9}}{2\lambda}M, \tag{46}$$

which for large l tends to $r_m \sim 3M$. [The case $\lambda = 0$ ($l = 0$) has to be studied separately, we analyze it at the end of this section]. Using this expression, we can obtain the frequency of the light-ring modes for a scalar perturbation in a classical black hole

$$\omega_{\mathrm{Sch}}^2 = \frac{1}{M^2}\left(1 - \frac{2}{\tilde{r}_m}\right)\left(\frac{\lambda}{\tilde{r}_m^2} + \frac{2}{\tilde{r}_m^3}\right) - i\left(n + \frac{1}{2}\right)\frac{2}{M^2\tilde{r}_m^4}\sqrt{\left(1 - \frac{2}{\tilde{r}_m}\right)(-96\tilde{r}_m - 10(3\lambda-7)\tilde{r}_m^2 + 4(5\lambda-3)\tilde{r}_m^3 - 3\lambda\tilde{r}_m^4)}, \tag{47}$$

where $\tilde{r}_m = r_m/M$.

Now let us see how this expression changes if we add quantum corrections at first order in $\hbar$. The corrected effective potential (44) has its maximum at $r = r_m + \frac{\hbar}{M}\epsilon + O(\hbar^2)$, where

$$\epsilon = \frac{1}{5040\pi}\tilde{r}_m^{-4}(40 + 12(\lambda-1)\tilde{r}_m - 3\lambda\tilde{r}_m^2)^{-1}\left(1 - \frac{2}{\tilde{r}_m}\right)^{-2}\Big[-392(48\xi - 35) + 8(21\lambda + 3360\xi - 2663)\tilde{r}_m - 2(249\lambda + 6384\xi - 5057)\tilde{r}_m^2 + 12(6\lambda + 168\xi - 59)\tilde{r}_m^3 + 3(89\lambda - 177)\tilde{r}_m^4 + 27(3 - 5\lambda)\tilde{r}_m^5 + 18\lambda\tilde{r}_m^6 + \frac{9}{2}\tilde{r}_m^5(-32 - 9(\lambda-1)\tilde{r}_m + 2\lambda\tilde{r}_m^2)\left(1 - \frac{2}{\tilde{r}_m}\right)^2\log\left(1 - \frac{2}{\tilde{r}_m}\right)\Big]. \tag{48}$$

Using the equation (45) we obtain the following expression for the corrected frequencies at first order in $\hbar$

$$\mathrm{Re}[\omega^2] = \mathrm{Re}[\omega_{\mathrm{Sch}}^2] + \frac{\hbar}{630\pi M^4}\frac{336(\lambda+2)\xi - 201\lambda - 560 + \tilde{r}_m(-84(\lambda+3)\xi + 13\lambda^2 + 42\lambda + 210)}{\lambda\tilde{r}_m^8(1 - \frac{2}{\tilde{r}_m})} + O(\hbar^2). \tag{49}$$

$$\mathrm{Im}[\omega^2] = \mathrm{Im}[\omega^2_{\mathrm{Sch}}] - \frac{\hbar}{2520\pi M^4}\left(n+\frac{1}{2}\right)\tilde{r}_m^{-8}\left(1-\frac{2}{\tilde{r}_m}\right)^{-3/2}(-96\tilde{r}_m - 10(3\lambda-7)\tilde{r}_m^2 + 4(5\lambda-3)\tilde{r}_m^3 - 3\lambda\tilde{r}_m^4)^{-1/2}$$
$$\cdot\Big[1176(144\xi-125) + 12\tilde{r}_m(-749\lambda - 25760\xi + 645120\pi\tilde{r}_m\epsilon + 23011)$$
$$+ 4\tilde{r}_m^2(3977\lambda + 52584\xi + 25200\pi(21\lambda-121)\tilde{r}_m\epsilon - 45476)$$
$$- 2\tilde{r}_m^3(3468\lambda + 31584\xi + 25200\pi(63\lambda-139)\tilde{r}_m\epsilon - 20771)$$
$$+ 2\tilde{r}_m^4(-1069\lambda + 3528\xi + 5040\pi(170\lambda-171)\tilde{r}_m\epsilon + 1956) - 9\tilde{r}_m^5(-271\lambda + 560\pi(77\lambda-30)\tilde{r}_m\epsilon + 312)$$
$$+ 9\tilde{r}_m^6(-71\lambda + 3360\pi\lambda\tilde{r}_m\epsilon + 30) + 54\lambda\tilde{r}_m^7 + \frac{9}{2}\tilde{r}_m^4(-768 - 14(15\lambda-59)\tilde{r}_m - 6(47-35\lambda)\tilde{r}_m^2 - 5(13\lambda-6)\tilde{r}_m^3$$
$$+ 6\lambda\tilde{r}_m^4)\left(1-\frac{2}{\tilde{r}_m}\right)\log\left(1-\frac{2}{\tilde{r}_m}\right)\Big] + O(\hbar^2). \tag{50}$$

As mentioned above, the case $l=0$ requires special attention. In this case the effective potential has its maximum at

$$r = \frac{8M}{3} + \frac{\hbar}{430080\pi M}(1008\xi - 1767 + 2048\log(2)) + O(\hbar^2). \tag{51}$$

Therefore, the corrected frequency at first order in $\hbar$ for $l=0$ is given by

$$\omega^2 = \omega^2_{\mathrm{Sch}} + \frac{3\hbar}{286720\pi M^2}((336\xi-241)\mathrm{Re}[\omega^2_{\mathrm{Sch}}] - 2i(336\xi-5)\mathrm{Im}[\omega^2_{\mathrm{Sch}}]) \tag{52}$$

One can see that, even if the geometry of the spacetime is drastically changed by quantum effects near to the horizon, they do not imply significant corrections to the physical observables in the exterior region.

B. Electromagnetic perturbations

Let us now study the propagation of electromagnetic waves on a general static and spherically symmetric metric given by $ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2$. The electromagnetic field F_{ab} satisfies the source-free Maxwell equations:

$$\nabla_a F^{ab} = 0, \qquad \nabla_a {}^*F^{ab} = 0, \tag{53}$$

where ${}^*F$ is the Hodge dual of F. The second equation is solved with $F_{ab} = A_{a,b} - A_{b,a}$, where A_a is the electromagnetic potential, and the problem is reduced to solve the first equation above for the vector field A_a. For a spherically symmetric background spacetime we can search for solutions by expanding A_a in the basis of 4-dimensional vector spherical harmonics $(Y_a)_{\ell m}$. Elements of this basis are classified according to their behavior under parity transformations. For axial/odd modes, which have parity $(-1)^{\ell+1}$, the electromagnetic potential can be expanded as

$$A_a^-(t,r,\theta,\phi) = \sum_{\ell,m}\begin{bmatrix} 0 \\ 0 \\ \frac{a^{lm}(t,r)}{\sin\theta}\partial_\phi Y_{lm} \\ -a^{lm}(t,r)\sin\theta\partial_\theta Y_{lm}\end{bmatrix}, \tag{54}$$

for some (gauge-invariant) coefficients $a^{lm}(t,r)$. Using this ansatz one can check that there is only one nontrivial independent equation from $\nabla_a F^{ab} = 0$. For a static spacetime we can further separate $a_{\ell m} = e^{-i\omega t}\Psi^-_{\ell m}(r)$, and the resulting equation can be written as

$$F^2 {\Psi^-}''_{\ell m} + FF'{\Psi^-}'_{\ell m} + (\omega^2 - V_l)\Psi^-{}_{\ell m} = 0, \tag{55}$$

where

$$F(r) = \sqrt{-\frac{g_{tt}}{g_{rr}}}, \tag{56}$$

$$V_l(r) = -g_{tt}(r)\frac{l(l+1)}{r^2}. \tag{57}$$

Again, introducing the tortoise coordinate by $\partial_{r^*} = F(r)\partial_r$, one recovers the usual Regge-Wheeler form of the equation, $\partial^2_{r_*}\Psi^-{}_{\ell m} + (\omega^2 - V_\ell)\Psi^-{}_{\ell m} = 0$. In particular, for a Schwarzschild background we recover the usual expression.

For polar/even modes, which have parity $(-1)^\ell$, the electromagnetic potential can be expanded as

$$A_a^+(t,r,\theta,\phi) = \sum_{\ell,m}\begin{bmatrix} f^{lm}(t,r)Y_{lm} \\ h^{lm}(t,r)Y_{lm} \\ k^{lm}(t,r)\partial_\theta Y_{lm} \\ k^{lm}(t,r)\partial_\phi Y_{lm}\end{bmatrix}, \tag{58}$$

for some coefficients f^{lm}, h^{lm}, k^{lm}. However, these coefficients are gauge-dependent. Let us introduce the three gauge-invariant combinations $\Psi^{+} = \sqrt{-g_{tt}g_{rr}}\frac{r^2}{\ell(\ell+1)}\times(\partial_t h^{lm} - \partial_r f^{lm})$, $\Psi_{1,\ell m} = f^{lm} - \partial_t k^{lm}$ and $\Psi_{2,\ell m} = h^{lm} - \partial_r k^{lm}$ (these combinations are essentially the field components F_{tr}, $F_{t\phi}$, $F_{r\phi}$, respectively; the rest of the field components are redundant). Using Maxwell equations one can conclude, after some work, that $\Psi^{+}_{\ell m}$ satisfies the same equation (55) as the axial solution $\Psi^{-}_{\ell m}$, and the rest of the field variables are determined from it: $\Psi_1 = -\frac{\partial_r\Psi^+}{g_{rr}} + \frac{\partial_r(g_{tt}g_{rr})}{2g_{tt}g_{rr}^2}\Psi^+$ and $\Psi_2 = \frac{\partial_t\Psi^+}{g_{tt}}$. One can easily check that these results fully solve the system of equations $\nabla_a F^{ab} = 0$, and the whole problem reduces to solve (55) with suitable boundary conditions.

The fields $\Psi^{\pm}_{\ell m}$ constitute the two fundamental degrees of freedom per spacetime point of the electromagnetic field. Notice that both fields satisfy exactly the same dynamical equation even when the quantum corrections considered in this paper are included, leading in particular to the usual phenomenon of isospectrality [41]. This could have been guessed in advance from the electric-magnetic duality symmetry of the source-free Maxwell equations [42], since Ψ^+ plays the role of the electric field while Ψ^- represents the magnetic degree of freedom.

For the perturbative corrected Schwarzschild metric provided in Sec. III, the first order correction in $\hbar$ to the potential yields

$$V(r) = V_1(r) - \frac{\hbar l(l+1)}{5040\pi M^2 r^7}\left(\frac{2M}{f(r)}(21M^2r^2 + 40M^3 r - 14M^4 - 36Mr^3 + 9r^4) + 9r^4(r-3M)\log(f(r))\right) + O(\hbar^2), \tag{59}$$

where $V_1 = f(r)\frac{l(l+1)}{r^2}$ is the potential for electromagnetic perturbations on the Schwarzschild metric, and $f(r) = 1 - \frac{2M}{r}$. As in the scalar case, for $r \to r_0$ the effective potential acquires a nonzero residual value of order $\sqrt{\hbar} = \ell_p$, which is not present in the classical case.

Finally, let us analyze the quantum corrections to the light ring frequencies of electromagnetic perturbations, again using the WKB approximation described above. For the Schwarzschild metric the maximum of the Regge-Wheeler potential is located at $r = 3M$, and therefore using the expression (45) one obtains

$$\omega^2_{\rm Sch} = \frac{l(l+1)}{27M^2} - i\frac{2\sqrt{l(l+1)}}{27M^2}\left(n + \frac{1}{2}\right). \tag{60}$$

For our (perturbatively) corrected spacetime, at first order in $\hbar$ we obtain the maximum of the potential (59) at $r = 3M + \frac{\hbar}{90720\pi M}(243\log(3) - 20)$. Substituting in (45) and expanding in Taylor series we obtain the following expression for the quantum correction to the light ring frequencies:

$$\omega^2 = \omega^2_{\rm Sch} + \frac{\hbar}{17010\pi M^2}(-13{\rm Re}[\omega^2_{\rm Sch}] + 11i{\rm Im}[\omega^2_{\rm Sch}]). \tag{61}$$

Again, the quantum effects of vacuum polarization do not lead to significant, observable corrections.

VI. SUMMARY AND FINAL COMMENTS

The theory of test quantum fields in a given gravitational background is widely regarded as a useful and fruitful framework for exploring quantum fluctuations enhanced by gravity. This theory can be further used to analyze the backreaction of these quantum effects on the spacetime background by looking at the semiclassical Einstein's equations (1). Solving these equations is, however, a very elusive problem and only in very highly symmetric situations one can carry out the computation in a manageable way. A good example are conformally flat spacetimes with conformal matter fields. In this case $\langle T_{ab}\rangle$ is essentially characterized by the conformal anomaly. Another relevant example emerges in two-dimensional dilaton-gravity models coupled to conformal matter. The conformal anomaly in two dimensions fully determines the quantum stress tensor for a given choice of the vacuum state, thus allowing us to solve analytically the semiclassical backreaction equations for a relevant class of two-dimensional models [17].

In this paper we have reanalyzed the four-dimensional problem from scratch, focusing on static and spherically-symmetric backgrounds. The general expressions given in [14] for the renormalized stress tensor, when the quantum field lives in static and spherically symmetric spacetimes, represent a very significant progress, but they are still quite involved and unpractical to solve the semiclassical equations. One way to simplify the problem is to restrict ourselves to conformal matter and take advantage of the trace anomaly. However, those assumptions (spherical symmetry, staticity and conformal matter) are still not sufficient to reduce the problem to a manageable form, in sharp contrast with the effective two-dimensional case [18–20]. To overcome this difficulty we have introduced an extra simplifying assumption, suggested by well-known results in the fixed Schwarzschild background. Since we are mainly interested in the behavior of the geometry in the very near horizon region $r \sim 2M$ (in the macroscopic vicinity of $2M$ one does not expect any significant modification of the classical Schwarzschild geometry) we have assumed the exact relation between $\langle p_t\rangle$ and $\langle p_r\rangle$ in the vicinity of the classical horizon (suggested by the results in the fixed background approach). Our findings appear to be essentially insensitive of this assumption. More precisely, we have numerically checked that the (nonperturbative) backreaction solution obtained with other restrictions (such as $\langle p_t \rangle = 0$) are qualitatively similar to those described in Sec. III. Furthermore, our results do not depend on the particular form of the conformal matter either (for a massless Dirac field we have obtained results similar to those for a scalar field).

One remarkable property of the semiclassical backreaction solution obtained in Secs. III and IV is that the radial function can never reach 0 (where the classical curvature singularity is located), but rather it has a minimum on a time-like surface. This mimics the throat of an (asymmetric) wormhole, and it is located at $r_0 \approx 2M + \mathcal{O}(\sqrt{\hbar})$, where the red-shift function reaches a very small but nonzero value.[4] Beyond this bouncing surface for the radial function we have found a null curvature singularity at a finite proper-time distance (of order $\mathcal{O}(\sqrt{\hbar})$ from the throat). The overall physical picture qualitatively agrees with the results obtained from the purely two-dimensional approach. This indicates that the two-dimensional approach could be more accurate than it could be expected.

The global picture obtained from this semiclassical framework differs drastically from its counterpart in classical general relativity, specially regarding the black hole interior region. Strictly speaking, here the classical horizon disappears and it is replaced by a bouncing timelike surface, beyond which a null curvature singularity emerges immediately. The underlying reason for this seems to be rooted in the singular behavior of the renormalized stress tensor at the classical horizon obtained in the fixed background approach. In light of these results, it looks as if the original singular behavior of the stress-tensor in the classical horizon manifests itself in the metric in the form of a curvature singularity as a result of the backreaction. We regard this singularity as a side effect of the assumption of a pure vacuum solution. The presence of matter could tame the singularity if vacuum polarization effects continue to be relevant (as suggested by the results in [23]) and allow the formation of ECO's. However in this case the maximum compactness of these objects is bounded by $2M/r_0 \sim 1 - 0.01686\sqrt{\hbar}/(2M)$. This bound is a direct consequence of the fact that the exterior geometry of ECOs has to be described by the external portion of our semiclassical solution, and not by the classical Schwarzschild metric.

We have also analyzed potential physical implications of the quantum corrected geometry in the exterior region. In Sec. V we have analyzed in detail the scalar and electromagnetic perturbations, paying special attention to the so-called "light ring frequencies," which are the relevant observables in the ringdown of binary black holes. We have evaluated the corrected light ring frequencies using our predictions for the semiclassical metric, and they differ from their classical counterpart by corrections of order $\mathcal{O}(\hbar/M^2)$. Somewhat not surprising, the drastic modification of the metric around the classical horizon does not lead to observable corrections on these observables, since these frequencies are determined by the spacetime curvature around the light-ring. To really probe the quantum corrections around the classical horizon geometry one would need to compute the proper BH quasinormal mode frequencies of the system, which would most likely differ nonperturbatively from the classical BH QNMs. However these observables require the specification of boundary conditions at the center or surface of the quantum object in question, and this is out of the scope of the present paper.

We plan to extend this work in several directions. Apart from computing the QNM frequencies above, our goal is to analyze the inclusion of collapsing matter and the impact of the time-dependent phase on the backreaction effects. This is indeed a very difficult problem in the four-dimensional arena and requires a separate study.

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Note added.—After the communication of this paper another work appeared [43] which confirms some of our conclusions, giving further support to our results.

APPENDIX: NATURE OF THE SINGULAR POINT $r = r_0$

In this appendix will analyze in detail the nature of the singular point $r = r_0$ obtained in Sec. III and will prove that it is a coordinate singularity. We will also see why this singular point does not define a classical horizon.

Let us consider a general metric of the form

$$ds^2 = -G(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2, \tag{A1}$$

with $G(r) > 0$ and $F(r) > 0$. Its corresponding curvature scalar is given by

[4] We note that the power in the dependence on $\hbar$ is different from that obtained in the approach of Ref. [24], for which $r_0 \approx 2M + \mathcal{O}(\hbar)$. Furthermore, we also have discrepancies in the analytic form of the metric components.

$$R = \frac{4G^2(rF' + F - 1) + rG(G'(rF' + 4F) + 2rFG'') - r^2F(G')^2}{2r^2G^2}. \tag{A2}$$

In our case $F(r_0) = 0$ at the singular point, but $G(r_0) \neq 0$ and their derivatives are not divergent, so the scalar curvature is finite at this point, and therefore $r = r_0$ is a coordinate singularity. This statement can also be inferred from a perturbative analysis of the semiclassical Einstein's equations $R = 8\pi\langle T^a_a\rangle$, since at first order the trace does not diverge $[\langle T^a_a\rangle = \frac{\hbar M^2}{60\pi^2 r^6} + O(\hbar^2)]$.

Another way to confirm this, and to assess the impact of the quantum-vacuum polarization on the classical Schwarzschild geometry, is by analyzing the Kretschmann curvature scalar. For a static and spherically symmetric metric, the explicit expression can be simplified considerably if we use the TOV equations. It reads

$$K(r) = 16\left(-\frac{8\pi m(r)\langle\rho(r)\rangle}{r^3} + \frac{3m(r)^2}{r^6} + 4\pi^2[2\langle p(r)\rangle\langle\rho(r)\rangle + 3\langle p(r)\rangle^2 + 3\langle\rho(r)\rangle^2]\right) \tag{A3}$$

Since the renormalized pressure and density are of order $\hbar/f^2$ near the singular point (i.e. numerically of order ~ 1 since $f(r_0) \sim \sqrt{\hbar}$), we can see that the Kretschmann scalar does not diverge. In particular, by substituting the perturbative solution at first order in $\hbar$ into this expression we obtain

$$K(r) = \frac{48M^2}{r^6} + \frac{\hbar}{105\pi r^9}\left(\frac{2M}{r^2 f(r)^2}\left(728M^4 - 818M^3r + 212M^2r^2 + 27Mr^3 - 9r^4\right) - 9r^3\log\left(1 - \frac{2M}{r}\right)\right) + O(\hbar^2). \tag{A4}$$

As mentioned above, near the singular point the leading correction to the Kretschmann scalar behaves as $\hbar/f^2$, which tends to $O(1)$ at this point. Notice that, as compared to the classical Schwarzschild value, the Kretschmann scalar is expected to receive corrections that are of order $O(\hbar^0)$ in a neighborhood of the singular point, meaning that quantum corrections may be significant for the nearby geometry despite the tiny value of $\hbar$.

If we substitute the equation of state (10) in (A3), we see that the terms that include the trace anomaly are of order $\hbar$ near the singular point, so for a conformal quantum field we can further approximate the Kretschmann scalar as

$$K(r) \sim 16\left(-\frac{8\pi m(r)\langle\rho(r)\rangle}{r^3} + \frac{3m(r)^2}{r^6} + 16\pi^2\langle\rho(r)\rangle^2\right) \tag{A5}$$

As mentioned above, this coordinate singularity does not define a classical horizon. To check this explicitly, it is useful to switch to Eddington-Finkelstein coordinates. Defining the generalized tortoise coordinate as $dr_*^2 = G^{-1}F^{-1}dr^2$ and the advanced time as $v := t + r_*$, the metric (A1) can be expressed as

$$ds^2 = -G(r)dv^2 + 2F^{-1/2}(r)G^{1/2}(r)dvdr + r^2d\Omega^2. \tag{A6}$$

Notice that $2F^{-1/2}G^{1/2}dvdr = -(-ds^2 - Gdv^2 + r^2d\Omega^2)$. Therefore for causal ($ds^2 \leq 0$) and future-directed ($dv > 0$) curves, $dr < 0$ is only possible if $G(r) < 0$. If there were a critical point where $G(r) = 0$, it would define a one-way membrane for radial ($d\Omega = 0$) null geodesics, i.e. a horizon. But in our case $G(r) > 0$ for all $r \geq r_0$, so there is no horizon in this spacetime.

As a side remark, notice that in sharp contrast to the Schwarzschild metric where $F(r) = G(r)$, the Eddington-Finkelstein coordinates are not useful to penetrate across the coordinate singularity $r = r_0$, because the metric in these coordinates is not regular. We discuss the question of how to extend the metric across $r = r_0$ in Sec. IV.

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