Character degrees, blocks and defect groups

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Contents

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Resumen

Sea G un grupo finito y p un primo. Uno de los principales temas de interés en la teoría de representaciones de grupos finitos es la interacción entre invariantes *globales* e invariantes *locales*. Cuando hablamos de propiedades globales normalmente nos referimos a propiedades de todo el grupo, y cuando hablamos de propiedades locales nos referimos a propiedades de ciertos p-subgrupos no triviales y de sus normalizadores. Frecuentemente, los invariantes globales están contenidos en la tabla de caracteres de G, y los subgrupos locales son p-subgrupos de Sylow y sus normalizadores. El ejemplo paradigmático de esta interacción en teoría de representaciones es la siguiente conjetura, propuesta por J. McKay en 1972.

Conjetura (McKay). *Si P es un p-subgrupo de Sylow de G, entonces G y $\mathbf{N}_G(P)$ tienen el mismo número de caracteres irreducibles cuyo grado no es divisible por p.*

Con frecuencia, estas conjeturas admiten generalizaciones que incorporan p-bloques de Brauer, el objeto principal de interés de esta tesis. Dado un bloque B con grupo de defecto D, entonces la correspondencia de Brauer asocia naturalmente a B un bloque b de $\mathbf{N}_G(D)$. En 1975, J. L. Alperin propuso una generalización de la conjetura de McKay incorporando la correspondencia de Brauer. Esta generalización constituye actualmente uno de los principales problemas en la teoría de representaciones modulares.

Conjetura (Alperin–McKay). *Sea B es un p-bloque de G con grupo de defecto D y sea b su correspondiente de Brauer en $\mathbf{N}_G(D)$. Los bloques B y b contienen el mismo número de caracteres de altura cero.*

Sumando esta igualdad sobre los bloques cuyo grupo de defecto es un p-subgrupo de Sylow, recuperamos la conjetura de McKay. Generalizaciones como la conjetura de Alperin–McKay son ejemplos de problemas que tratan de establecer conexiones entre propiedades de p-bloques y propiedades de sus grupos de defecto, un tema clásico en la teoria de representaciones modulares de grupos finitos. Otros ejemplos notables de estas conexiones se pueden encontrar en conjeturas de R. Brauer, E. C. Dade o G. R. Robinson.

Si B es un p-bloque de G, denotamos por $k(B) = |\operatorname{Irr}(B)|$, es decir, el número de caracteres ordinarios irreducibles que pertenecen a B. La famosa conjetura $k(B)$ de Brauer propone que si D es un grupo de defecto de B, entonces $k(B)$ debería estar acotado por el número de elementos de D. Esta conjetura se ha demostrado para grupos p-resolubles en [**GMRS04**], después de que, en 1962, H. Nagao redujera este caso de la conjetura a acotar clases de conjugación en un producto semidirecto. Sin embargo, la conjetura $k(B)$ sigue abierta en general, incluso para bloques principales. Notablemente, es una de las pocas conjeturas importantes en la teoría de representaciones modulares que todavía no ha sido reducida a grupos simples (más adelante explicaremos a qué nos referimos por *reducción a grupos simples*).

La conjetura de Héthelyi–Külshammer es de alguna forma una versión dual a la conjetura $k(B)$. En [**HK00**] se demostró que un grupo resoluble cuyo orden es divisible por p tiene al menos $2\sqrt{p-1}$ clases de conjugación. En este mismo artículo, se conjetura que lo mismo ocurre para grupos de orden divisible por p en general (esto fue finalmente resuelto por A. Maróti en [**Mar16**]). También se propuso una cota inferior para el número de caracteres en un bloque. Más concretamente, la conjetura de Héthelyi–Külshammer propone que si $k(B) > 1$ entonces $k(B) \geqslant 2\sqrt{p-1}$. Recientemente esta conjetura ha sido demostrada para el bloque principal en [**HS22**], pero para bloques arbitrarios todavía se desconoce incluso para grupos resolubles.

En el primer capítulo de esta tesis centramos nuestra atención en el problema 21 de la famosa lista de problemas de R. Brauer.

Problema (Problema 21 de Brauer). *Sea n un número natural. ¿Existe un número finito de clases de isomorfismo de grupos cuyo orden es potencia de un primo que ocurren como grupos de defecto de bloques que contienen exactamente n caracteres ordinarios irreducibles?*

En otras palabras, Brauer pregunta si es posible acotar superiormente en función de n el orden de un grupo de defecto de un bloque con n caracteres. El problema 21 de Brauer ha sido demostrado para grupos resolubles en [**Kül89**] y para p-resolubles en [**Kül90**]. Posteriormente, usando este resultado y la solución de E. Zelmanov al problema restringido de Burnside, se demostró que el problema 21 de Brauer tenía una solución afirmativa asumiendo la conjetura de Alperin–McKay en [**KR96**], pero en ningún caso se da una clasificación de los grupos de defecto posibles para ningún n fijado. De hecho, para grupos resolubles, la cota obtenida para $|D|$ cuando $k(B) = 2$ es mayor que 10^{22}, y para grupos p-resolubles la cota es mucho mayor. Sin embargo si $k(B) = 2$ entonces $|D| = 2$ en general, sin asumir p-resolubilidad.

En 1941, R. Brauer y C. Nesbitt demuestran que $k(B) = 1$ si y solo si $|D| = 1$. El siguiente avance en este problema tardaría 41 años en llegar, cuando J. Brandt demostró en [**Bra82**] que $k(B) = 2$ si y solo $|D| = 2$, como hemos mencionado antes. Para bloques principales, la clasificación de los grupos de defecto posibles que pueden ocurrir cuando $k(B_0(G)) \in \{3, 4\}$ se consiguió en [**KS21**] y cuando $k(B_0(G)) = 5$ en [**RSV21**], pero para bloques arbitrarios el problema sigue abierto incluso para bloques que contienen exactamente 3 caracteres ordinarios.

Aunque esta clasificación sigue siendo un problema extremadamente complicado, se demostró en [**KNST14**] que, asumiendo que B satisface la conjetura de Alperin–McKay, se tiene que $k(B) = 3$ si, y solo si $|D| = 3$. En esta tesis obtenemos el siguiente paso en esta dirección, demostrando que si $k(B) = 4$ entonces D debe ser isomorfo a $\mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_4$ o C_5, asumiendo que la conjetura de Alperin–McKay se satisface para B. Éste es el objetivo principal de la primera parte de esta tesis. Este resultado es el Teorema A del Capítulo 2 y aparece en [**MRS23**].

La segunta parte de esta tesis se centra en un invariante distinto. Dado un bloque B, en vez de considerar el número $k(B)$ de caracteres irreducibles en B, consideramos el conjunto $\mathrm{cd}(B)$ de grados de caracteres irreducibles en B. Aunque se ha demostrado que el conjunto de grados de un grupo no determina su resolubilidad [**Nav15**], hay conexiones entre el conjunto de grados de caracteres de un grupo resoluble y su longitud derivada. Esto se puede observar por ejemplo en la conjetura de Isaacs–Seitz, que propone que si G es resoluble entonces $|\mathrm{cd}(G)| \geqslant \mathrm{dl}(G)$, donde $\mathrm{dl}(G)$ denota la longitud derivada del grupo resoluble G. El teorema de Taketa (anterior a la conjetura) prueba que esto es cierto para los llamados M-grupos, una clase de grupos entre nilpotentes y resolubles. Además, I. M. Isaacs y D. S. Passman probaron la conjetura para grupos resolubles con hasta 3 grados distintos. Actualmente, se conoce solo hasta grupos resolubles con 5 grados distintos gracias a resultados de S. C. Garrison y M. L. Lewis.

En [**M21**], G. Navarro propuso una posible conexión nueva entre los caracteres de un bloque y sus grupos de defecto. Si B es un bloque con grupo de defecto D y $\mathrm{dl}(D)$ es la longitud derivada del grupo (resoluble) D, entonces Navarro formuló la siguiente pregunta.

Problema (Navarro). *¿Es cierto que* $|\mathrm{cd}(B)| \geqslant \mathrm{dl}(D)$*?*

El teorema de Taketa da una respuesta afirmativa a la pregunta de Navarro para p-grupos. Además, esta pregunta tiene una respuesta afirmativa para bloques nilpotentes (definidos en [**BP80**]). Si el bloque B tiene un único grado, entonces se demostró en [**OT83**] que B es nilpotente y por tanto la pregunta también tiene una respuesta afirmativa. Si $\mathrm{cd}(B) = \{m, n\}$ entonces, usando la reciente prueba de la conjetura de altura cero de Brauer [**MNST22**] y asumiendo la conjetura de Malle–Navarro sobre bloques nilpotentes [**MN11**] obtendríamos otra vez una respuesta afirmativa para B. En el caso del bloque principal de grupos resolubles, la pregunta de Navarro tendría una respuesta afirmativa como consecuencia de la conjetura de Isaacs–Seitz. El Teorema D de [**GMS22**] demuestra que también tiene una respuesta afirmativa para cualquier p-bloque del grupo simétrico $\mathfrak{S}_n$, el grupo alternado $\mathfrak{A}_n$ y el grupo general lineal $\mathrm{GL}_n(q)$, donde q es una potencia de p.

Nuestro objetivo en el Capítulo 3 es verificar ciertos casos de la pregunta de Navarro. En concreto, demostramos que la pregunta tiene una respuesta afirmativa para bloques principales con 3 grados de caracteres como máximo. Como consecuencia, usando cotas conocidas sobre la longitud derivada de p-grupos, obtenemos que la pregunta de Navarro tiene una respuesta positiva para bloques principales de grupos cuyos p-subgrupos de Sylow tienen tamaño menor o igual a p^{21}. Los resultados mencionados hasta ahora son el Teorema C, el Teorema D y el Corolario E del Capítulo 3 y aparecen en [**M21**] y [**GMS22**].

La demostración de estos resultados se hace mediante lo que llamamos una *reducción a grupos simples*. Uno de los mayores proyectos en la historia de las matemáticas es la llamada *Clasificación de grupos finitos simples*. Durante varias décadas y a lo largo de centenares de publicaciones, se ha demostrado que cualquier grupo finito simple pertenece a una de las siguientes cuatro familias: los grupos cíclicos de orden primo, los grupos alternados $\mathfrak{A}_n$ con $n \geqslant 5$, los llamados *grupos de tipo Lie* y una familia de 26 grupos llamados *esporádicos*. Esta clasificación se consideró terminada en la década de 1980. Desde entonces, se ha utilizado una técnica muy potente para demostrar teoremas en teoría de representaciones: se reduce la prueba del teorema a probar ciertas condiciones (frecuemente más fuertes que el teorema original) para grupos simples. Después, intentamos aprovechar las sofisticadas teorías de representaciones de los grupos alternados y de tipo Lie para demostrar estas propiedades y así concluir la demostración de nuestro teorema original. Estas técnicas han servido para reducir muchas de las grandes conjeturas en teoría de representaciones ordinarias y modulares (entre ellas la conjetura de McKay [**IMN07**] y la de Alperin–McKay [**Spa13b**]). Nuestros teoremas en este capítulo también se reducen a grupos simples. En algunos casos, incluimos las demostraciones de grupos simples, que requieren explorar los caracteres de bloques principales de grupos simples y sus extensiones a los grupos de automorfismos.

La siguiente generalización de la pregunta de Navarro ha sido propuesta por A. Jaikin-Zapirain.

Problema (Jaikin-Zapirain). *¿Es cierto que* $|\mathrm{cd}(B)| \geqslant |\mathrm{cd}(D)|$*?*

Por el momento no hemos encontrado ningún contraejemplo a la pregunta de Jaikin-Zapirain. Sin embargo, también se verifica para bloques nilpotentes, y por tanto se verifica en varios casos que hemos mencionado anteriormente.

También es importante mencionar que la pregunta de Navarro es en cierta forma semejante a una conjetura propuesta en [**Mor04**] por A. Moretó. La conjetura es la siguiente.

Conjetura (Moretó). *Si B es un bloque de G con grupo de defecto D entonces la longitud derivada de un grupo de D está acotada en función del número de alturas distintas de caracteres en un bloque B.*

Si denotamos por $\mathrm{ht}(B)$ el conjunto de alturas del grupo B, una versión más fuerte de la conjetura de Moretó sería la cota $|\mathrm{ht}(B)| \geqslant |\,\mathrm{dl}(D)|$. Además, esta cota también daría una respuesta afirmativa a la pregunta de Navarro, dado que caracteres del mismo grado tienen la misma altura. Esta cota se verifica para ciertas familias de grupos importantes, como grupos simétricos, alternados o el grupo general lineal en característica p, pero es en general falsa y en la Sección 3.6 damos un contraejemplo. Esta cota para las familias de grupos mencionadas es [**GMS22**, Theorem D]. El caso del grupo general lineal aparece en [**Mor04**].

Finalmente, consideramos dos resultados clásicos que muestran la gran cantidad de información sobre la estructura de grupos que se puede leer en conjuntos de grados de sus caracteres. El clásico teorema de J. G. Thompson sobre p-complementos normales dice que si todos los grados de caracteres irreducibles no lineales de un grupo G son divisibles por el primo p entonces G tiene un p-complemento normal. Un teorema de I. M. Isaacs y S. D. Smith mejoró el resultado de Thompson demostrando que es suficiente que esta propiedad se cumpla para los grados del p-bloque principal. Cerramos el Capítulo 3 con una versión de estos resultados, donde tomamos en consideración un primo q distinto de p. En concreto, probamos que si todos los grados del p-bloque principal de un grupo G son potencias de q entonces G es p-resoluble y $G/\mathbf{O}_{p'}(G)$ tiene un q-complemento abeliano normal. La demostración es mucho más elemental que la de los otros resultados principales de este capítulo y no depende de la Clasificación de grupos finitos simples. Esto es el Teorema F y también apareció en [**M21**].

El último capítulo de esta tesis estudia la existencia de biyecciones entre conjuntos de caracteres que respetan divisibilidad de grados. Un ejemplo paradigmático es la *correspondencia de Glauberman*. Si un p-grupo P actua por automorfismos sobre un grupo K cuyo orden no es divisible por p, entonces existe una biyección *natural* entre el conjunto $\mathrm{Irr}_P(K)$ de caracteres irreducibles de K que son P-invariantes y el conjunto de caracteres irreducibles de $C = \mathbf{C}_K(P)$, el subgrupo de puntos fijados por P. De hecho, si $\chi \in \mathrm{Irr}_P(K)$, G. Glauberman demostró que la restricción de χ a C se puede escribir como

$$\chi_C = e\chi^* + p\Delta$$

donde e es un entero no divisible por p y Δ es un carácter (no necesariamente irreducible) de C que no contiene a χ^*. Precisamente, χ^* es el correspondiente de Glauberman de χ. Si $\chi \in \mathrm{Irr}_P(K)$ se corresponde con $\chi^* \in \mathrm{Irr}(C)$, es cierto que $\chi^*(1)$ divide a $\chi(1)$, aunque este hecho tiene una demostración excepcionalmente sofisticada que se consiguió en [**Gec20**] tras su reducción a grupos simples en [**HT94**]. Además, se puede usar la correspondencia de Glauberman para probar la conjetura de McKay en el caso en que el grupo G tiene un p-complemento normal (esto se puede encontrar al principio del capítulo 9 de [**Nav18**]).

Tanto la conjetura de McKay como la de Alperin–McKay predicen que hay conjuntos de caracteres irreducibles con el mismo número de elementos, y por tanto es natural preguntar qué propiedades pueden cumplir las biyecciones entre estos conjuntos. Esto ha sido una de las claves de la reducción a grupos simples de estas conjeturas. En ambas reducciones, se demostró que las conjeturas se verifican si se puede demostrar para grupos simples que existen biyecciones verificando que *lo que ocurre por encima* de caracteres que se corresponden en esta biyección es, en cierto sentido, igual (esta igualdad se describe usando la sofisticada teoría de *central character triple isomorphisms* y *block character triple isomorphisms*). Un tiempo después de la demostración de ambas reducciones, se ha conseguido demostrar en [**NS14**] y [**Ros23**] que, de hecho, si estas biyecciones existen para grupos simples, entonces existen para todo grupo finito.

Como hemos mencionado antes, en el Capítulo 4 abordamos el problema de encontrar biyecciones que respeten la divisibilidad de grados de caracteres, en concreto en las conjeturas de McKay y Alperin–McKay. Este problema tiene una respuesta negativa en general: en el grupo $\mathfrak{A}_5$ para $p = 5$ no se puede encontrar esta biyección con divisibilidad de grados ni para la conjetura de McKay ni para la de Alperin–McKay. En efecto, sus caracteres de grado no divisible por 5 tienen grados $1, 3, 3$ y 4, pero los del normalizador de un 5-subgrupo de Sylow tienen grados $1, 1, 2, 2$. Además, estos son precisamente los caracteres del 5-bloque principal de este grupo.

Sin embargo, en [**Tur07**] se construyó una biyección de estas características para la conjetura de McKay en grupos resolubles. Esto fue extendido a grupos p-resolubles en [**Riz19**] asumiendo la divisibilidad de grados en la correspondencia de Glauberman, un hecho que se demostró a posteriori. En el Capítulo 4 demostramos que es posible encontrar una biyección con divisibilidad de grados en la conjetura de Alperin–McKay para grupos p-resolubles.

En general, se sabe poco sobre la dimensión de un bloque (en [**Lin18**, Sección 10.1] se exponen algunos de los resultados conocidos). Brauer demostró que si B es un bloque de G con grupo de defecto D y su correspondiente de Brauer en $\mathbf{N}_G(D)$ es b, entonces la p-parte de $\dim(B)$ es divisible por la p-parte de $\dim(b)$. Sin embargo, todavía no se sabe ni siquiera si en general $\dim(c) \leqslant \dim(B)$ donde c es un bloque de un subgrupo $H \leqslant G$ que induce B. Sorprendentemente, las técnicas que usamos en este capítulo nos permiten demostrar que la dimensión de un bloque B es divisible por la dimension de su correspondiente de Brauer en grupos p-resolubles. El 2-bloque principal de $\mathfrak{A}_5$ sirve como contraejemplo de este hecho fuera de los grupos p-resolubles, ya que $\dim(B) = 44$ y $\dim(b) = 12$. Los resultados mencionados son el Teorema G y el Teorema H del Capítulo 4 y aparecerán en [**MR22**].

Guion de la tesis

El Capítulo 1 contiene los resultados básicos sobre teoría de caracteres ordinarios, caracteres de Brauer y bloques de Brauer que necesitaremos en esta tesis. Para caracteres ordinarios, nuestras referencias son [**Isa06**] y [**Nav18**], y para caracteres de Brauer y bloques usamos [**Nav98**]. Incluimos una discusión de la teoria de Fong de bloques de grupos p-resolubles, ya que es completamente esencial para los resultados de los Capítulos 2 y 4. Ya que en el Capítulo 2 nos es necesario tratar con grupos de permutaciones primitivos, incluimos una sección de preliminares con las definiciones importantes y las propiedades y clasificaciones necesarias.

En el Capítulo 2 demostramos que si un bloque contiene exactamente 4 caracteres irreducibles y satisface la conjetura de Alperin–McKay entonces sus grupos de defecto tienen orden 4 o 5. El resultado clave es demostrar que el resultado se cumple cuando los grupos de defecto son normales. Esta demostración contiene dos ingredientes clave: el estudio de grupos con un número pequeño de caracteres proyectivos y el uso de clasificaciones de grupos de permutaciones primitivos afines de rango pequeño.

Por *caracteres proyectivos* nos referimos a caracteres de G cuya restricción contiene el mismo carácter G-invariante θ de un subgrupo normal N (a veces decimos que estos caracteres *están sobre* θ). Usando la teoría de Fong de bloques de grupos p-resolubles, podemos reducir el resultado principal a estudiar grupos con pocos caracteres sobre un carácter de un subgrupo normal en concreto. Usamos un famoso resultado de R. B. Howlett e I. M. Isaacs y resultados de R. Higgs [**Hig88**] que demuestran que si $|\operatorname{Irr}(G|\theta)| \leqslant 2$ entonces G/N debe ser resoluble y además los caracteres que están sobre θ tienen el mismo grado. Ésto último nos permite generalizar un resultado de F. DeMeyer y G. Janusz que relaciona la condición $|\operatorname{Irr}(G|\theta)| \leqslant 2$ con el número de caracteres en $\operatorname{Irr}(PN|\theta)$, donde P es un p-subgrupo de Sylow de G. Higgs ya demostró este resultado que generalizamos, pero en el lenguaje de las representaciones proyectivas. En esta tesis, aportamos una demostración puramente carácter-teorética.

Usando caracteres proyectivos y una fórmula clásica de Brauer de conteo de caracteres conseguimos reducir el problema a poder asumir que cierto cociente de nuestro grupo es un grupo de permutaciones primitivo cuyo rango es menor o igual que 3, y empleamos clasificaciones de estos grupos de D. S. Passman [**Pas68**] y D. A. Foulser [**Fou69**] (en el caso de rango 3, podemos asumir resolubilidad usando el teorema de Higgs mencionado anteriormente). Nuestra demostración incluye un estudio caso por caso de las posibles estructuras que aparecen en los resultados de Passman y Foulser. Frecuentemente esto necesita cálculos con el software [**GAP**] y argumentos extremadamente ad-hoc. En concreto, obtenemos un resultado sobre la estructura de ciertos subgrupos de Sylow de un caso que llamamos el *caso imprimitivo* en la clasificación de Foulser, que combinamos con la generalización del teorema de DeMeyer y Janusz mencionada en el párrafo anterior para encontrar una contradicción.

Una vez demostramos el caso de defecto normal, obtenemos nuestro resultado principal estudiando el posible número de caracteres de altura cero de un bloque con 4 caracteres. Como asumimos que nuestro bloque satisface la conjetura de Alperin–McKay, el número de caracteres de altura cero coincide con el de su correspondiente de Brauer, y podemos usar resultados de dominación de bloques y los casos de defecto normal para concluir nuestro teorema. Tanto en el caso de defecto normal como en la prueba del resultado principal usamos la profunda teoría de bloques con grupo de defecto cíclico de Dade [**Dad66**]. Merece la pena mencionar que los resultados de este capítulo dependen de la Clasificación de grupos finitos simples. En efecto, usamos los resultados de Howlett–Isaacs y Higgs, que dependen de la Clasificación. Si intentásemos evitar estos resultados, necesitaríamos clasificaciones de grupos de permutaciones primitivos que también dependen de la Clasificación. En cualquier caso, parece que una prueba de este resultado independiente de la Clasificación está fuera de alcance, de momento.

En el Capítulo 3 demostramos que la pregunta de Navarro tiene una respuesta afirmativa para bloques con 3 grados de caracteres como máximo. Tratamos los casos de 2 y 3 grados por separado, ya que en el caso que $|\mathrm{cd}(B_0(G))| \leqslant 2$ es posible demostrar que G es p-resoluble y hallar más información estructural del grupo, mientras que en el caso $|\mathrm{cd}(B_0(G))| = 3$ esto ya no es posible (una vez más, esto viene ilustrado por el ejemplo $\mathfrak{A}_5$, cuyo 2-bloque principal tiene 3 grados).

En la Sección 3.2 obtenemos una reducción del Teorema C a grupos simples, usando resultados de la teoría de Clifford. Demostramos estos resultados para grupos simples en la Sección 3.3. En la Sección 3.4 obtenemos una reducción del Teorema D a grupos simples. En este caso, la reducción es más sofisticada, y necesitamos usar resultados de inducción tensorial de caracteres. Incluimos una demostración de los resultados de grupos simples para grupos alternados en la Sección 3.5. Finalizamos el capítulo con comentarios sobre el Problema B y una demostración para $\mathrm{GL}_n(q)$ y $\mathrm{SL}_n(q)$ en la Sección 3.6. Demostramos el Teorema F en la Sección 3.7. A diferencia de las demostraciones de los Teoremas C y D, la demostración del Teorema F no depende de la Clasificación de grupos finitos simples y es mucho más elemental.

Finalmente, en el Capítulo 4 demostramos que existe una biyección entre los conjuntos de caracteres de altura cero de un bloque B con grupo de defecto D y su correspondiente de Brauer en $\mathbf{N}_G(D)$ con divisibilidad de grados de caracteres en grupos p-resolubles. Podemos ver este resultado como una versión refinada de la conjetura de Alperin–McKay para esta familia de grupos.

En la Sección 4.2 incluimos demostraciones de lo que serán los ingredientes principales de nuestra biyección. En primer lugar, obtenemos una consecuencia de los resultados de A. Turull en [**Tur08**] y [**Tur09**] sobre ciertos isomorfismos de *character triples* que ocurren *por encima de* la correspondencia de Glauberman. Estas propiedades están íntimamente relacionadas con las biyecciones de las reducciones de las conjeturas de McKay y Alperin–McKay que hemos mencionado anteriormente. Esta consecuencia que obtenemos a partir de los resultados de Turull será el punto clave para el último paso de nuestra demostración, y está basada en ideas de A. Laradji [**Lar14**]. Además, combinamos la correspondencia de Fong–Reynolds con la teoría de Fong de bloques de grupos p-resolubles, lo cual será necesario para reducir nuestro problema al caso en que el bloque tiene defecto maximal. Otro ingrediente para esta reducción es la mejora de un teorema de Navarro sobre el número de p-subgrupos de Sylow en grupos p-resolubles (ver [**Nav03**]). Este resultado fue demostrado por Navarro y lo incluimos en esta tesis con su permiso.

En la Sección 4.3 demostramos la existencia de la biyección en una serie de proposiciones que tratan algunos casos por separado. El objetivo fundamental de la mayoría de estas proposiciones es demostrar que podemos asumir que G no tiene p-subgrupos normales no triviales. Llegados a ese punto, usamos nuestra consecuencia de los resultados de Turull para concluir la demostración.

En la Sección 4.4 demostramos la divisibilidad de dimensiones en correspondientes de Brauer, aprovechando la versión de la correspondencia de Fong–Reynolds para grupos p-resolubles y la mejora del teorema de Navarro, que han sido demostrados en la Sección 4.2.

Curiosamente, a pesar de que los teoremas principales de este capítulo son para grupos p-resolubles, dependen también de la Clasificación de grupos finitos simples, ya que ambos usan el profundo resultado de Geck [**Gec20**] sobre la correspondencia de Glauberman.

Concluimos este capítulo con una sección breve de consecuencias interesantes de nuestro Teorema G para bloques nilpotentes y la correspondencia de Dade–Glauberman–Nagao.

Resum

Siga G un grup finit i p un nombre primer. Un dels principals temes d'interés en la teoria de representacions de grups finits és la interacció entre invariants *globals* i invariants *locals*. Quan parlem de propietats globals, normalment ens referim a propietats de tot el grup, i quan parlem de propietats locals ens referim a propietats de certs p-subgrups no trivials i els seus normalitzadors. Freqüentment els invariants globals es troben a la taula de caràcters de G, i els subgrups locals són p-subgrups de Sylow i els seus normalitzadors. L'exemple paradigmàtic d'aquesta interacció en teoria de representacions és la següent conjectura, proposada per J. McKay a 1972.

CONJECTURA (McKay). *Si P és un p-subgrup de Sylow de G, aleshores G i $\mathbf{N}_G(P)$ tenen el mateix nombre de caràcters irreductibles de grau no divisible per p.*

Freqüentment, estes conjectures admeten generalitzacions que incorporen p-blocs de Brauer, l'objecte principal d'interés d'aquesta tesi. Donat un bloc B amb grup de defecte D, aleshores la correspondència de Brauer associa naturalment a B un bloc b de $\mathbf{N}_G(D)$. A 1975, J. L. Alperin va proposar una generalització de la conjectura de McKay incorporant la correspondència de Brauer. Esta generalització constitueix actualment un dels principals problemes de la teoria de representacions modulars.

CONJECTURA (Alperin–McKay). *Siga B un p-bloc de G amb grup de defecte D i siga b el seu corresponent de Brauer en $\mathbf{N}_G(D)$. Els blocs B i b contenen el mateix nombre de caràcters d'altura zero.*

Sumant aquesta igualtat sobre els blocs que tenen un p-subgrup de Sylow com a grup de defecte, recuperem la igualtat proposada a la conjectura de McKay. Generalitzacions com la conjetura d'Alperin–McKay són exemples de problemes que intenten establir connexions entre propietats de p-blocs i propietats dels seus grups de defecte, un tema clàssic a la teoria de representacions modulars de grups finits. Altres exemples notables d'aquestes connexions es poden trobar a conjectures de R. Brauer, E. C. Dade o G. R. Robinson.

Si B és un p-bloc de G, denotem per $k(B) = |\operatorname{Irr}(B)|$, és a dir, el nombre de caràcters ordinaris irreductibles que pertanyen a B. La famosa conjectura

$k(B)$ de Brauer proposa que si D és un grup de defecte de B, aleshores $k(B)$ està fitat superiorment per el nombre d'elements de D. Aquesta conjectura s'ha pogut provar per a grups p-resolubles a [**GMRS04**], després de que, en 1962, H. Nagao reduira aquest cas de la conjectura a un problema de classes de conjugació en un producte semidirecte. No obstant això, la conjectura $k(B)$ segueix oberta en general, inclús per a blocs principals. Notablement, és una de les poques conjectures importants de la teoria de representacions modulars que encara no ha sigut reduïda a grups simples (més endavant explicarem què volem dir per *reducció a grups simples*).

La conjectura de Héthelyi–Külshammer és, en certa forma, una versió dual a la conjectura $k(B)$. En [**HK00**] es va provar que un grup resoluble d'ordre divisible per p tenia com a mínim $2\sqrt{p-1}$ classes de conjugació. En el mateix article, es conjectura que la mateixa fita deuria ser certa per a qualsevol grup finit d'ordre divisible per p (finalment esta conjectura va ser resolta per A. Maróti a [**Mar16**]). També es va proposar una fita inferior per al nombre de caràcters en un bloc. Més concretament, la conjectura de Héthelyi–Külshammer proposa que si $k(B) > 1$ aleshores $k(B) \geqslant 2\sqrt{p-1}$. Recentment s'ha aconseguit provar aquesta fita per al bloc principal en [**HS22**], però per a blocs arbitrarisencara es desconeix inclús per a grups resolubles.

En tot cas, en el primer capítol d'esta tesi centrem la nostra atenció en el problema 21 de la famosa llista de problemes de Brauer.

Problema (Problema 21 de Brauer). *Siga n un nombre natural. Existeix un nombre finit de classes d'isomorfisme de grups d'ordre potència d'un nombre primer que ocorren com a grups de defecte de blocs que contenen exactament n caràcters ordinaris irreductibles?*

En altres paraules, Brauer pregunta si és possible fitar superiorment en funció de n l'ordre d'un grup de defecte d'un bloc amb n caràcters. El problema 21 de Brauer ha sigut demostrat per a grups resolubles en [**Kül89**] i per a p-resolubles en [**Kül90**]. Posteriorment, utilitzant este resultat i la solució de E. Zelmanov al problema restringit de Burnside, es va demostrar que el problema 21 de Brauer tenia una solució afirmativa assumint la conjectura d'Alperin–McKay, però en cap cas es va donar una classificació dels possibles grups de defecte per a cap n fixat. De fet, per a grups resolubles, la fita donada per a $|D|$ quan $k(B) = 2$ és major que 10^{22}, i per a grups p-resolubles és molt més gran. Malgrat això, si $k(B) = 2$ aleshores $|D| = 2$ en general, sense assumir p-resolubilitat.

En 1941 R. Brauer i C. Nesbitt proven que $k(B) = 1$ si i sols si $|D| = 1$. El següent pas en aquest problema tardaria 41 anys en arribar, quan J. Brandt prova en [**Bra82**] que $k(B) = 2$ si i sols si $|D| = 2$, com hem esmentat abans. Per a blocs principals, la classificació dels grups de defecte possibles quan $k(B_0(G)) \in \{3, 4\}$ es va aconseguir a [**KS21**] i quan $k(B_0(G)) = 5$ a [**RSV21**], però per a blocs arbitraris, el problema segueix obert inclús per a blocs amb exactament 3 caràcters ordinaris.

Encara que esta classificació segueix sent un problema extremadament complicat, es va demostrar en [**KNST14**] que, assumint que B satisfà la conjectura d'Alperin–McKay, es té que $k(B) = 3$ si i sols si $|D| = 3$. En aquesta tesi obtenim el següent pas en esta direcció, provant que si $k(B) = 4$ aleshores D ha de ser isomorf a $\mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_4$ o C_5, assumint la conjectura d'Alperin–McKay per a B. Aquest és l'objectiu principal de la primera part de la tesi i és el Teorema B del Capítol 2, que va apareixer a [**MRS23**].

En la segona part d'esta tesi ens centrem en un invariant distint. Donat un bloc B, en compte de considerar el nombre $k(B)$ de caràcters irreductibles en B, considerem el conjunt $\mathrm{cd}(B)$ de graus de caràcters en B. Encara que s'ha demostrat que el conjunt de graus d'un grup no determina la seua resolubilitat [**Nav15**], hi ha connexions entre el conjunt de graus d'un grup resoluble i la seua longitud derivada. Esta connexió es pot observar per exemple a la conjectura de Isaacs–Seitz, que proposa que si G és resoluble aleshores $|\mathrm{cd}(G)| \geqslant \mathrm{dl}(G)$, on $\mathrm{dl}(G)$ denota la longitud derivada del grup resoluble G. El teorema de Taketa (anterior a la conjectura) prova aquesta desigualtat per als anomenats M-grups, una classe de grups entre els nilpotents i els resolubles. A més, Isaacs i Passman van provar la conjectura per a grups resolubles amb 3 graus distints com a molt. Actualment, la conjectura només es coneix fins a grups resolubles amb 5 graus distints gràcies a resultats de S. C. Garrison i M. L. Lewis.

En [**M21**], G. Navarro va proposar una possible connexió nova entre caràcters d'un bloc i els seus grups de defecte. Si B es un bloc amb grup de defecte D i $\mathrm{dl}(D)$ és la longitud derivada del grup (resoluble) D, aleshores Navarro va formular la següent pregunta.

PROBLEMA (Navarro). *És cert que* $|\mathrm{cd}(B)| \geqslant \mathrm{dl}(D)$*?*

El teorema de Taketa dona una resposta afirmativa a la pregunta de Navarro per a p-grups. A més, la pregunta té una resposta afirmativa per a blocs nilpotents (definits en [**BP80**]). Si el bloc B té un únic grau, aleshores es va provar a [**OT83**] que B es nilpotent i per tant la pregunta també té una resposta afirmativa. Si $\mathrm{cd}(B) = \{m, n\}$ aleshores utilitzant la recent prova de la conjectura d'altura zero de Brauer [**MNST22**] i assumint la conjectura de Malle–Navarro sobre blocs nilpotents [**MN11**] obtindríem altra vegada una resposta afirmativa per a B. En el cas del bloc principal, la pregunta de Navarro tindría una resposta afirmativa per a grups resolubles assumint la conjectura de Isaacs–Seitz. En [**GMS22**, Theorem D] es demostra que també hi ha una resposta afirmativa per a qualsevol p-bloc del grup simètric $\mathfrak{S}_n$, el grup alternat $\mathfrak{A}_n$ i el grup general lineal $\mathrm{GL}_n(q)$ on q és una potència de p.

El nostre objectiu al Capítol 3 és verificar certs casos de la pregunta de Navarro. Concretament, demostrem que la pregunta té una resposta afirmativa per a blocs principals amb 3 graus de caràcters com a màxim. Com a conseqüència, utilitzant fites conegudes per a la longitud derivada de p-grups, obtenim que la pregunta de Navarro té una resposta positiva per a blocs principals de grups amb p-subgrups de Sylow de tamany menor o igual a p^{21}. Els resultats esmentats fins ara són el Teorema C, el Teorema D i el Corol·lari E del capítol 3 i aparegueren en [**M21**] i [**GMS22**].

La prova d'aquests resultats es fa amb el que anomenem una *reducció a grups simples*. Un dels majors projectes de la història de les matemàtiques és la *Classificació de grups finits simples*. Després de dècades i amb centenars de publicacions s'aconseguí provar que un grup finit simple pertany a una de les següents quatre famílies: els grups cíclics d'ordre primer, els grups alternats $\mathfrak{A}_n$ per a $n \geqslant 5$, els anomenats *grups de tipus Lie* i una família de 26 grups anomenats *esporàdics*. Des d'aleshores, s'ha trobat una tècnica molt potent per a provar teoremes en teoria de representacions: es reduix la prova del teorema a provar certes condicions (freqüentment més fortes que el teorema original) per a grups simples. Després utilitzem les sofisticades teories de representacions dels grups alternats i de tipus Lie per a provar estes condicions i així concloure la prova del nostre teorema original. Estes tècniques han funcionat per a reduir moltes de les grans conjectures en la teoria de representacions ordinàries i modulars (entre elles la conjectura de McKay [**IMN07**] i la de Alperin–McKay [**Spa13b**]). Els nostres teoremes d'este capítol també es redueixen a grups simples. En alguns casos, incloem les proves de grups simples, que requereixen l'estudi de caràcters en blocs principals de grups simples i les seues extensions a grups d'automorfismes.

La següent generalització a la pregunta de Navarro ha sigut proposada per A. Jaikin-Zapirain.

Problema. *És cert que* $|\mathrm{cd}(B)| \geqslant |\mathrm{cd}(D)|$*?*

De moment, no tenim cap contraexemple a la pregunta de Jaikin-Zapirain. Encara així, sabem que també es verifica per a blocs nilpotents, i per tant per als casos esmentats anteriorment.

També és important la següent conjectura de A. Moretó [**Mor04**], que es sembla en certa forma a la pregunta de Navarro. La conjectura és la següent.

Conjectura. *Si B és un bloc de G amb grup de defecte D aleshores la longitud derivada de D està fitada en funció del nombre d'altures distintes de caràcters del bloc B.*

Si denotem per $\mathrm{ht}(B)$ el conjunt d'altures del grup B, una versió més forta de la conjectura de Moretó seria la fita $|\mathrm{ht}(B)| \geqslant |\,\mathrm{dl}(D)|$. A més, aquesta fita també donaria una resposta afirmativa a la pregunta de Navarro, ja que caràcters amb graus iguals tenen la mateixa altura. Aquesta fita es verifica per a algunes famílies importants de grups, com els grups simètrics, els grups alternats i el general lineal en característica p, però és falsa en general i a la Secció 3.6 donem un contraexemple. La prova per a les famílies de grups que hem dit abans és [**GMS22**, Theorem D]. El cas del grup general lineal apareix en [**Mor04**].

Finalment considerem dos resultats clàssics on es pot observar la gran quantitat d'informació estructural d'un grup que es pot llegir al conjunt de graus dels seus caràcters. Un clàssic teorema de J. G. Thompson prova que si tots els graus de caràcters no lineals són divisibles per p aleshores G té un p-complement normal. Un teorema de I. M. Isaacs i S. D. Smith va millorar el resultat de Thompson provant que és suficient que la propietat dels graus es verifique al p-bloc principal. Tanquem el Capítol 3 amb una versió d'estos resultats, on considerem un primer q distint de p. En concret, provem que si tots els graus del p-bloc principal d'un grup G són potències de p aleshores G és p-resoluble i $G/\mathbf{O}_{p'}(G)$ té un q-complement abelià normal. La prova és molt més elemental que la dels altres resultats principals i no depén de la Classificació de grups finits simples. És el Teorema F i també aparegué en [**M21**].

A l'últim capítol d'esta tesi s'estudia la existència de bijeccions entre conjunts de caràcters que respecten divisibilitat de graus. Un exemple paradigmàtic és la *correspondència de Glauberman*. Si un p-grup actua per automorfismes sobre un grup K d'ordre no divisible per p, aleshores existeix una bijecció *natural* entre el conjunt $\mathrm{Irr}_P(K)$ de caràcters irreductibles de K que són P-invariants i el conjunt de caràcters irreductibles de $C = \mathbf{C}_K(P)$, el subgrup de punts fixats per P. De fet, si $\chi \in \mathrm{Irr}_P(K)$, G. Glauberman va provar que la restricció de χ a C es pot escriure com

$$\chi_C = e\chi^* + p\Delta$$

on e és un enter no divisible per p i Δ és un caràcter (no necessàriament irreductible) de C que no conté χ^*. Aquest χ^* és el corresponent de Glauberman de χ. Si $\chi \in \mathrm{Irr}_P(K)$ es correspon amb $\chi^* \in \mathrm{Irr}(C)$ aleshores $\chi^*(1)$ divideix $\chi(1)$, encara que esta divisibilitat té una prova excepcionalment sofisticada, que es va aconseguir a [**Gec20**] després de la seua reducció a grups simples en [**HT94**]. A més, es pot utilitzar la correspondència de Glauberman per a provar la conjectura de McKay en el cas que G té un p-complement normal (veure el principi del capítol 9 de [**Nav18**]).

Tant la conjectura de McKay com la de Alperin–McKay proposen que hi ha certs conjunts de caràcters amb el mateix nombre d'elements, i per tant és natural preguntar-se quines propietats poden verificar les bijeccions entre aquests conjunts. Esta ha sigut una de les claus de la reducció a grups simples d'estes conjectures. En ambdues reduccions, es va provar que la conjectura es verifica si, per a grups simples, podem trobar bijeccions verificant que *el que passa per damunt* de caràcters que es corresponen es, en cert sentit, igual (esta igualtat es descriu més precisament amb el llenguatge de *central character triple isomorphisms* i *block character triple isomorphisms*). Un temps després de les proves d'estes reduccions, s'ha aconseguit provar en [**NS14**] i [**Ros23**] que de fet, si estes bijeccions existeixen per a tot grup simple, aleshores existeixen per a tot grup finit.

Com hem dit abans, al Capítol 4 tractem el problema de trobar bijeccions que respecten la divisibilitat de graus de caràcters, en concret per a les conjectures de McKay i Alperin–McKay. Este problema té una resposta negativa en general: al grup $\mathfrak{A}_5$ per a $p = 5$ no es pot trobar esta bijecció ni per a la conjectura de McKay ni per a la de Alperin–McKay. En efecte, els graus no divisibles per 5 dels seus caràcters són $1, 3, 3$ i 4, mentre que els del normalitzador d'un 5-subgrup de Sylow són $1, 1, 2, 2$. A més, estos són precissament els caràcters del 5-bloc principal d'este grup.

No obstant, en [**Tur07**] es va construir una bijecció amb estes característiques per a la conjectura de McKay en grups resolubles. A [**Riz19**] es va extendre aquest resultat a grups p-resolubles, assumint la divisibilitat de graus en la correspondència de Glauberman, un fet que es va demostrar a posteriori. Al Capítol 4 demostrem que és possible trobar una bijecció amb divisibilitat de graus per a la conjectura d'Alperin–McKay en grups p-resolubles.

En general es sap molt poc sobre la dimensió d'un bloc ([**Lin18**, Secció 10.1] exposa alguns dels resultats coneguts). Brauer va provar que si B és un bloc amb grup de defecte D i el seu corresponent de Brauer en $\mathbf{N}_G(D)$ és b, aleshores la p-part de $\dim(B)$ és divisible per la p-part de $\dim(b)$. Malgrat això, no es sap ni tant se vol si $\dim(c) \leqslant \dim(B)$ on c és un bloc d'un subgrup H que indueix B. Sorprenentment, les tècniques que utilitzem a aquest capítol ens permeten provar que la dimensió del bloc B és divisible per la dimensió del seu corresponent de Brauer en grups p-resolubles. El 2-bloc principal de $\mathfrak{A}_5$ és un contraexemple que mostra que este fet és fals fora dels grups p-resolubles, ja que $\dim(B) = 44$ i $\dim(b) = 12$. Els resultats que hem esmentat són el Teorema G i el Teorema H del Capítol 4 i apareixeràn en [**MR22**].

Guió de la tesi

El Capítol 1 conté els resultats bàsics sobre teoria de caràcters ordinaris, caràcters de Brauer i blocs de Brauer que necessitarem a aquesta tesi. Per a caràcters ordinaris, les nostres referències són [**Isa06**] i [**Nav18**], i per a caràcters de Brauer i blocs utilitzem [**Nav98**]. Incloem una discussió de la teoria de Fong sobre blocs p-resolubles, ja que serà fonamental per als resultats dels Capítols 2 i 4. Ja que al Capítol 2 necessitarem tractar amb grups de permutacions primitius, incloem una secció de preliminars amb les definicions i propietats bàsiques, així com les classificacions que ens seràn necessàries.

Al Capítol 2 provem que si un bloc conté exactament 4 caràcters irreductibles i satisfà la conjectura d'Alperin–McKay, aleshores els seus grups de defecte tenen ordre 4 o 5. El resultat clau és la prova del resultat quan assumim que els grups de defecte són normals. Esta prova conté dos ingredients clau: l'estudi de grups amb un nombre menut de caràcters projectius i l'ús de classificacions de grups de permutacions primitius afins de rang menut.

Per *caràcters projectius* ens referim a caràcters de G la restricció dels quals conté al mateix caràcter G-invariant θ d'un subgrup normal N (solem dir que aquests caràcters *estàn sobre* θ). Utilitzant la teoria de Fong de blocs de grups p-resolubles, podem reduir el resultat principal a estudiar grups amb pocs caràcters sobre un caràcter d'un subgrup normal en concret. Utilitzem un resultat famós de R. B. Howlett i I. M. Isaacs i resultats de R. Higgs [**Hig88**] que proven que si $|\operatorname{Irr}(G|\theta)|$ aleshores G/N és resoluble i els caràcters que estàn sobre θ tenen el mateix grau. Este fet ens permet generalitzar un resultat de F. DeMeyer i G. Janusz que relaciona la condició $|\operatorname{Irr}(G|\theta)| \leqslant 2$ amb el nombre de caràcters en $\operatorname{Irr}(PN|\theta)$, on P és un p-subgrup de Sylow de G. Higgs ja va provar aquesta generalització, però en el llenguatge de representacions projectives. En esta tesi, donem una prova completament caràcter-teorètica.

Utilitzant caràcters projectius i una fórmula de Brauer per a contar caràcters, podem reduir el problema a poder assumir que cert quocient del nostre grup és un grup de permutacions primitiu amb rang menor o igual a 3, i emprem classificacions d'aquests grups de D. S. Passman [**Pas68**] i D. A. Foulser [**Fou69**] (en el cas de rang 3, podem assumir resolubilitat utilitzant el teorema de Higgs que hem esmentat abans). La nostra prova inclou un estudi cas per cas de les possibles estructures que apareixen als resultats de Passman i Foulser. Freqüentment, es necessiten càlculs amb el software [**GAP**] i arguments extremadament ad-hoc. En concret, obtenim un resultat sobre la estructura de certs subgrups de Sylow d'un cas que anomenem el *cas imprimitiu* en la classificació de Foulser, que combinem amb la generalització del teorema de DeMeyer i Janusz per a trobar la contradicció.

Una vegada hem provat el cas de defecte normal, obtenim el nostre resultat principal estudiant el possible nombre de caràcters d'altura zero a un bloc amb 4 caràcters. Com assumim que el nostre bloc satisfà la conjectura d'Alperin–McKay, el nombre de caràcters d'altura zero en B coincideix amb el del seu corresponent de Brauer, i podem utilitzar resultats de dominació de blocs i els casos de defecte normal per a concloure la prova del teorema. Tant en el cas de defecte normal com a la prova del resultat principal, fem ús de la profunda teoria de blocs amb grup de defecte cíclic de Dade [**Dad66**]. És interessant observar que els resultats d'aquest capítol depenen de la Classificació de grups finits simples. En efecte, utilitzem resultats de Howlett–Isaacs i Higgs que depenen de la Classificació. Si intentem evitar aquests resultats, necessitem classificacions de grups de permutacions primitius que també depenen de la Classificació. En qualsevol cas, sembla que una prova d'este resultat independent de la Classificació està fora d'abast, per ara.

Al Capítol 3 provem que la pregunta de Navarro té una resposta afirmativa per a blocs amb 3 graus de caràcters com a màxim. Tractem els casos de 2 i 3 graus per separat, ja que quan $|\mathrm{cd}(B_0(G))| \leqslant 2$ és possible provar que G és p-resoluble i obtindre més informació estructural, mentre que en el cas $|\mathrm{cd}(B_0(G))| = 3$ no és possible (una vegada més, el grup $\mathfrak{A}_5$ il·lustra este fet, ja que el seu 2-bloc principal té 3 graus).

En la Secció 3.2 obtenim una reducció del Teorema C a grups simples, utilitzant resultats de la teoria de Clifford. Provem estos resultats per a grups simples en la Secció 3.3. En la Secció 3.4 obtenim una reducció del Teorema D a grups simples. En aquest cas, la reducció es més sofisticada, i necesitem fer ús de resultats d'inducció tensorial de caràcters. Incloem una prova dels resultats de grups simples per a grups alternats en la Secció 3.5. Tanquem el capítol amb uns comentaris sobre el Problema B i una prova per a $\mathrm{GL}_n(q)$ i $\mathrm{SL}_n(q)$ en la Secció 3.6. Provem el Teorema F en la Secció 3.7. A diferència de les proves dels Teoremes C y D, la prova del Teorema F no depén de la Classificació de grups finits simples i és molt més elemental.

Finalment, al Capítol 4 provem que existeix una bijecció entre els conjunts de caràcters d'altura zero d'un bloc B amb grup de defecte D i el seu corresponent de Brauer a $\mathbf{N}_G(D)$ amb divisibilitat de graus de caràcters en grups p-resolubles. Podem veure aquest resultat com una versió refinada de la conjectura d'Alperin–McKay per a esta família de grups.

En la Secció 4.2 s'inclouen proves dels que seràn els ingredients principals de la nostra bijecció. En primer lloc, obtenim una conseqüència dels resultats d' A. Turull en [**Tur08**] i [**Tur09**] sobre certs isomorfismes de *character triples* que ocorren *per damunt de* la correspondència de Glauberman. Estes propietats estàn íntimament relacionades amb les bijeccions de les reduccions de les conjectures de McKay i Alperin–McKay que hem esmentat abans. La conseqüència que obtenim a partir dels resultats de Turull serà clau al pas final de la nostra prova, i està basada en idees de A. Laradji [**Lar14**]. A més, combinem la correspondència de Fong–Reynolds amb la teoria de Fong de blocs p-resolubles, lo qual ens permetrà reduir el nostre problema al cas en que el bloc té defecte maximal. Altre ingredient per a esta reducció és la millora d'un teorema de Navarro sobre el nombre de p-subgrups de Sylow a grups p-resolubles (veure [**Nav03**]). Esta millora va ser provada per Navarro i la incloem a la tesi amb el seu permís.

En la Secció 4.3 provem la existència de la bijecció en una sèrie de proposicions que tracten alguns casos per separat. L'objectiu principal de la majoria d'estes proposicions és provar que podem assumir que G no té p-subgrups normals no trivials. Arribats ací, utilitzem la conseqüència dels resultats de Turull per a obtindre la prova.

En la Secció 4.4 provem la divisibilitat de dimensions en corresponents de Brauer, aprofitant la versió de la correspondència de Fong–Reynolds per a grups p-resolubles i la millora del teorema de Navarro, que hem provat a la Secció 4.2.

Curiosament, malgrat que els teoremes principals d'este capítol són per a grups p-resolubles, depenen també de la Classificació de grups finits simples, ja que necessitem fer ús del sofisticat resultat de Geck [**Gec20**] sobre la correspondència de Glauberman.

Finalitzem el capítol amb una breu secció de conseqüències interessants del nostre Teorema G per a blocs nilpotents i la correspondència de Dade–Glauberman–Nagao.

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Introduction

Let G be a finite group and let p be a prime. One of the main areas of interest in the representation theory of finite groups is the interaction between *global* and *local* properties. By *global* we usually mean properties regarding the whole group G, and by *local* we mean properties of certain nontrivial p-subgroups and their normalizers. Frequently, our global properties are contained in the character table of G and the local subgroups are the Sylow p-subgroups and their normalizers. The paradigmatic example of this type of interaction is the McKay conjecture.

Conjecture (McKay). *If P is a Sylow p-subgroup of G, then G and $\mathbf{N}_G(P)$ have the same number of irreducible characters whose degree is not divisible by p.*

These conjectures often admit generalizations to Brauer p-blocks, which will be the focus of this thesis. Given a block B with defect group D, then the Brauer correspondence associates a block b of $\mathbf{N}_G(D)$ to B. In 1975, J. L. Alperin proposed a generalization of the McKay conjecture by considering blocks under the Brauer correspondence. This generalization constitutes one of the main problems in modular representation theory.

Conjecture (Alperin–McKay). *Let B be a p-block of G and let b be its Brauer correspondent. Then B and b have the same number of height-zero irreducible characters.*

Generalizations such as the Alperin–McKay conjecture fit into the classical theme in modular representation theory of establishing connections between properties of blocks and their defect groups, and can shed a light on the true nature of the original conjectures. Other notable examples of connections between properties of blocks and their defect groups can be found in conjectures of R. Brauer, E. C. Dade or G. R. Robinson.

If B is a p-block of G, we denote by $k(B) = |\operatorname{Irr}(B)|$, that is, the number of irreducible complex characters that belong to B. The famous $k(B)$-conjecture states that if D is a defect group of B then we should have that $k(B) \leqslant |D|$. This conjecture was proved to hold for p-solvable groups in [**GMRS04**] after H. Nagao reduced this case to a problem on conjugacy classes of a semidirect product. However, the $k(B)$-conjecture remains open in general, even for principal blocks. Notably, it is one of the few main conjectures in modular representation theory that still has no reduction to simple groups (we will explain later what we mean by a *reduction to simple groups*).

The Héthelyi–Külshammer conjecture can be seen as a dual version of the $k(B)$-conjecture. In [**HK00**], it was shown that a solvable group whose order is divisible by p has at least $2\sqrt{p-1}$ conjugacy classes. In the same paper, the authors conjecture that this bound is valid for any finite group (this was finally proved by A. Maróti in [**Mar16**]). Furthermore, a lower bound on the number of characters in a block was also conjectured. More precisely, they conjecture that if $k(B) > 1$ then $k(B) \geqslant 2\sqrt{p-1}$. This has been recently proved for the principal block in [**HS22**], but for arbitrary blocks it is not known even for solvable groups. However, the problem to which we turn our attention in the first chapter of this thesis is Brauer's problem 21.

Question (Brauer's Problem 21). *Given a natural number n, are there only finitely many isomorphism classes of prime-power order groups that occur as defect groups of blocks with exactly n irreducible characters?*

In other words, Brauer asks if it is possible to find an upper bound for the order of the defect group of some block in terms of the number n of irreducible characters in this block. Brauer's problem 21 was proven to hold for solvable groups in [**Kül89**] and for p-solvable groups in [**Kül90**]. Using this and E. Zelmanov's solution to the restricted Burnside problem, it was proven to follow from the Alperin–McKay conjecture in [**KR96**], although no classification of the possible defect groups is given for any fixed n. In fact, for solvable groups, the bound obtained for $|D|$ when $k(B) = 2$ is greater than 10^{22}, and it gets even larger in p-solvable groups. However, if $k(B) = 2$ then $|D| = 2$ in general.

In 1941, R. Brauer and C. Nesbitt proved that $k(B) = 1$ if and only if $|D| = 1$. The next step appeared 41 years later, when J. Brandt proved that $k(B) = 2$ if and only if $|D| = 2$ [**Bra82**]. If B is the principal block, which we denote by $B_0(G)$, the classification of the possible defect groups was achieved when $k(B_0(G)) \in \{3, 4\}$ in [**KS21**] and when $k(B_0(G)) = 5$ in [**RSV21**]. However, for arbitrary blocks, it still remains open to classify the defect groups of blocks with 3 irreducible characters.

Even though this classification remains a very difficult problem, it was proved in [**KNST14**] that, assuming the Alperin–McKay conjecture holds for B, we have that $k(B) = 3$ if and only if $|D| = 3$. In this thesis we obtain the next step in this direction, proving that if $k(B) = 4$ then $D \cong \mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_4$ or C_5, assuming that the Alperin–McKay conjecture holds for B. This is the main goal in the first part of this thesis. The main result mentioned is Theorem A of Chapter 2 and appears in [**MRS23**].

The second part of this thesis is concerned with a different block invariant. Instead of considering the number $k(B)$ of irreducible characters in B, we now focus on the set $\mathrm{cd}(B)$ of degrees of characters in the block B. Although the set of character degrees of a group does not determine its solvability [**Nav15**], there is some connection between sets of character degrees and the derived length of solvable groups. This is illustrated for example by the Isaacs–Seitz conjecture, which posits that if G is solvable then $|\mathrm{cd}(G)| \geqslant \mathrm{dl}(G)$, where $\mathrm{dl}(G)$ denotes the derived length of the solvable group G. Taketa's theorem (proved long before the conjecture) proves this bound for the so called M-groups, a family of groups between nilpotent and solvable. The Isaacs–Passman theorems prove the conjecture for groups with at most 3 character degrees. Currently, the Isaacs–Seitz conjecture is known to hold for solvable groups with up to 5 character degrees, by work of S. C. Garrison and M. L. Lewis.

In [**M21**], G. Navarro proposed a new possible connection between the characters in a block and their defect groups. Let B be a p-block of G with defect group D with derived length $\mathrm{dl}(D)$.

Question (Navarro). *Is it true that* $|\mathrm{cd}(B)| \geqslant \mathrm{dl}(D)$*?*

Navarro's question has a positive answer for p-groups by Taketa's theorem. It also holds for nilpotent blocks as defined in [**BP80**]. If a block B has only one character degree, then by the main result of [**OT83**] we have that B is nilpotent and thus the question also holds in this case. If $\mathrm{cd}(B) = \{m, n\}$ then a positive answer for B would follow using the recent proof of Brauer's height zero conjecture [**MNST22**] and assuming a conjecture of G. Malle and G. Navarro on nilpotent blocks [**MN11**]. Furthermore, for principal blocks of solvable groups, an affirmative answer to Navarro's question would follow from the Isaacs–Seitz conjecture. By Theorem D of [**GMS22**], it also holds for all p-blocks of $\mathfrak{S}_n$, $\mathfrak{A}_n$ and $\mathrm{GL}_n(q)$ where q is a power of p.

Our focus in Chapter 3 is on the verification of certain cases of Navarro's question. More precisely, we prove that his question has a positive answer for principal blocks with at most 3 character degrees. In particular, by using well-known bounds on the derived length of p-groups, we verify that Navarro's question has a positive answer for principal blocks of groups whose Sylow p-subgroups have size at most p^{21}. The results mentioned so far constitute Theorem C, Theorem D, Corollary E in Chapter 3 and appeared in [**M21**] and [**GMS22**].

The proof of these results is done via a *reduction to simple groups*. One of the largest projects in the history of mathematics is what is known as the *Classification of finite simple groups*. After several decades and hundreds of publications, it has been possible to prove that a finite simple group belongs to one of the following four families: the cyclic groups of prime order, the alternating groups $\mathfrak{A}_n$ for $n \geqslant 5$, the so called *groups of Lie type*, and a family of 26 groups called the *sporadic* simple groups. This classification was finished in the decade of the 1980's. Since then, many theorems in the representation theory of finite groups have been proved using a very powerful technique: we reduce the proof of a theorem to a certain set of conditions (frequently stronger than the original statement) which have to be checked for every simple group. Then, we take advantage of the very sophisticated theory of representations of alternating groups and groups of Lie type to check these questions and prove our original theorem. These techniques have worked wonderfully to reduce many of the main conjectures in ordinary and modular representation theory (such as the McKay conjecture [**IMN07**] and the Alperin–McKay conjecture [**Spa13b**]). Our theorems in this chapter also reduce to simple groups. We include the proofs for simple groups in some cases, which require a study of characters in principal blocks of simple groups and their extensions to automorphism groups.

It is worth mentioning that A. Jaikin-Zapirain has proposed the following stronger version of Navarro's question.

QUESTION (Jaikin-Zapirain). *Is it true that* $|\mathrm{cd}(B)| \geqslant |\mathrm{cd}(D)|$*?*

At the time of this writing we have no counterexamples for Jaikin-Zapirain's question. However, it also holds for nilpotent blocks and thus it holds in the cases mentioned in the previous paragraph.

It is also important to mention that Navarro's question is similar to a conjecture posed in [**Mor04**] by A. Moretó.

CONJECTURE (Moretó). *If B is a block of G with defect group D then the derived length of D is bounded above in terms of the number of distinct heights of characters in B.*

If $\mathrm{ht}(B)$ denotes the set of heights in B, then a stronger version of Moretó's bound would be $|\mathrm{ht}(B)| \geqslant \mathrm{dl}(D)$. This bound would also give a positive answer to Navarro's question, using that characters of the same degree have equal heights. This bound holds for several important families of groups, such as symmetric groups, alternating groups and general linear groups in characteristic p, but it is false in general and in section 3.6 we give a counterexample. For the families of groups that we mentioned, this is proved in [**GMS22**, Theorem D], although the case of the general linear group appears in [**Mor04**].

Finally, we consider two classical results in character theory which show how much information on the structure of a group can be read from its set of character degrees. J. G. Thompson's theorem on normal p-complements states that if every nonlinear irreducible character has degree divisible by p then G has a normal p-complement. This was improved by I. M. Isaacs and S. D. Smith by proving that it is sufficient to check the condition on character degrees in the principal p-block. We end the chapter with a version of these results on normal complements, where we consider a prime $q \neq p$. More precisely, we prove that if every character degree in the principal p-block is a power of q then G is p-solvable and $G/\mathbf{O}_{p'}(G)$ has an abelian normal q-complement. Its proof is much more elementary in nature and it does not depend on the classification of finite simple groups. This is Theorem F and also appeared in [**M21**].

The final chapter is concerned with the existence of bijections between certain character sets that respect degree divisibility. A paradigmatic example of this is the *Glauberman correspondence*. If a p-group P acts by automorphisms on a group K of order not divisible by p, then there is a *natural* bijection between the set $\mathrm{Irr}_P(K)$ of irreducible characters that are P-invariant and the irreducible characters of $C = \mathbf{C}_K(P)$, the fixed point subgroup. In fact if $\chi \in \mathrm{Irr}_P(K)$ then G. Glauberman proved that the restriction of χ to C can be written as

$$\chi_C = e\chi^* + p\Delta$$

where e is an integer coprime to p and Δ is a (not necessarily irreducible) character of C that does not contain χ^*. This χ^* is the Glauberman correspondent of χ. If $\chi \in \mathrm{Irr}_P(K)$ corresponds to $\chi^* \in \mathrm{Irr}(C)$ then it turns out that $\chi^*(1)$ divides $\chi(1)$, although this fact has an exceptionally deep proof (see [**Gec20**]) after its reduction to simple groups in [**HT94**]. Moreover, the Glauberman correspondence can be used to prove the McKay conjecture in the case that G has a normal p-complement (this can be found at the beginning of Chapter 9 of [**Nav18**]).

Since both the McKay and Alperin–McKay conjectures suggest that two sets have the same number of elements, it is a natural question to ask if there are bijections with *nice* properties. This has been one of the key ideas of the reductions to simple groups of these conjectures. In both reductions, it was proved that the conjectures hold if it is possible to prove that, for simple groups, there are bijections such that *what happens above* corresponding characters is, in some sense, equal (this is described using the sophisticated theory of *central character triple isomorphisms* and *block character triple isomorphisms*). After these reductions, it has been proved in [**NS14**] and [**Ros23**] that in fact, if these bijections exist for simple groups, then they exist for every finite group.

As we mentioned before, in Chapter 4 we explore the problem of finding bijections which respect degree divisibility in the McKay and Alperin–McKay conjectures. It turns out that this is false in general, and the group $\mathfrak{A}_5$ for $p = 5$ is a counterexample. Indeed, its characters of degree not divisible by 5 have degrees $1, 3, 3, 4$ and the character degrees of the normalizer of a Sylow 5-subgroup have degrees $1, 1, 2, 2$. Further, these are precisely the characters in the principal 5-block of this group.

However, in solvable groups, it was proved in [**Tur07**] that it is possible to construct such bijections for the McKay conjecture. This was extended to p-solvable groups in [**Riz19**] by assuming that there is degree divisibility in the

Glauberman correspondence, a fact that was proved afterwards. We prove in Chapter 4 that this can also be done for Alperin–McKay correspondences in p-solvable groups.

In general, very little is known about the dimension of a block ([**Lin18**, Section 10.1] shows some of the known results). Brauer proved that if B is a block of G with defect group D and its Brauer correspondent in $\mathbf{N}_G(D)$ is b then the p-part of $\dim(b)$ divides the p-part of $\dim(B)$. However, it is not clear even if $\dim(c) \leqslant \dim(B)$ where c is a block of a subgroup $H \leqslant G$ inducing B. Surprisingly the techniques used in this chapter allow us to prove that the dimension of a block is divisible by the dimension of its Brauer correspondent in p-solvable groups. This is again false outside p-solvable groups, as shown by $\mathfrak{A}_5$ for $p = 2$, whose principal block has dimension 44 and its Brauer correspondent has dimension 12. These are Theorem G and Theorem H of Chapter 4 and will appear in [**MR22**].

Structure of the work

Chapter 1 contains the basic results on character theory, Brauer characters and blocks that we will need for this work. For ordinary characters we use [**Isa06**] and [**Nav18**], and for Brauer characters and blocks we use [**Nav98**]. We include a brief discussion of some of Fong's results on the block theory of p-solvable groups, which is essential for the results in Chapters 2 and 4. Since we need to deal with primitive permutation groups in Chapter 2, we also add a section discussing the important definitions and properties and the necessary classifications.

In Chapter 2 we prove that if a block contains exactly 4 irreducible characters and it satisfies the Alperin–McKay conjecture then its defect groups have order 4 or 5. Here, the key result is proving that the result holds whenever the defect groups are normal. The proof of this result contains two main ingredients: the study of groups with a small number of projective characters, and the classification of affine primitive permutation groups of small rank.

By *projective characters* we mean characters of G whose restriction contains the same G-invariant character θ of a normal subgroup N (we sometimes say that these characters *lie over* θ). Using Fong's theory of blocks of p-solvable groups, we are able to reduce the main result to the study of groups with few characters lying over a particular character of a particular normal subgroup. We make use of an important theorem of R. B. Howlett and I. M. Isaacs and results of R. Higgs [**Hig88**] that prove that if $|\operatorname{Irr}(G|\theta)| \leqslant 2$ then G/N is solvable and the characters in $\operatorname{Irr}(G|\theta)$ all have the same degree. This last part allows us to generalize a result of F. DeMeyer and G. Janusz which relates this condition on $|\operatorname{Irr}(G|\theta)|$ with the number of characters in $\operatorname{Irr}(PN|\theta)$ where P is a Sylow subgroup of G. This was already proved by Higgs but in the language of projective representations. We include a character-theoretic proof.

By using a well known character-counting formula of Brauer and further reductions, we end up assuming that our group is a solvable primitive permutation group of low rank, and use a classification of such groups due to D. S. Passman [**Pas68**] and D. A. Foulser [**Fou69**]. Our proof of the normal defect case includes a case-by-case analysis of the groups in the results of Passman and Foulser to exclude every possibility that contradicts the conclusion of the theorem. This analysis often needs GAP calculations and some very ad-hoc arguments. More precisely, we obtain a result on certain Sylow subgroups of a case we call the *imprimitive case* in Foulser's classification, which we combine with our generalization of the DeMeyer–Janusz theorem to find a contradiction.

From this key result of the normal defect case, we obtain our main result by studying the possible number of height zero characters in a block with four irreducible characters. By assuming the Alperin–McKay conjecture, we know that the number of height zero characters in our block agrees with the number of height zero characters in its Brauer correspondent, and we can use results of block domination and the normal defect cases to conclude the proof of our theorem. Both the normal defect case and the main result use Dade's deep theory of blocks with cyclic defect [**Dad66**]. It is worth mentioning that these results depend on the classification of finite simple groups. Indeed, we use the results of Howlett–Isaacs and Higgs that depend on the classification. If we try to avoid these, we would require classifications of primitive permutation groups that rely on the classification. In any case, it seems that a classification-free proof of these results is currently out of reach.

In Chapter 3 we prove that Navarro's question has a positive answer for principal blocks with at most three character degrees. We treat the cases $|\mathrm{cd}(B_0(G))| \leqslant 2$ and $|\mathrm{cd}(B_0(G))| = 3$ separately, since in the first case we can even prove that G is p-solvable and find even more structural information, whereas in the second case p-solvability is no longer guaranteed (once again this is illustrated by the example $\mathfrak{A}_5$ whose principal 2-block has 3 character degrees).

In Section 3.2 we reduce Theorem C to simple groups. The proof of the necessary results on simple groups is done in 3.3. In Section 3.4 we reduce Theorem D to simple groups using some results on tensor induced characters, and provide a proof for alternating groups in Section 3.5. We end the chapter with some remarks on Question B in Section 3.6, including a proof for $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$. The proof of Theorem F is done in Section 3.7.

Finally, in Chapter 4, we prove that there is a correspondence between height zero characters in Brauer correspondent blocks such that the degree of a character is divisible by the degree of its correspondent in p-solvable groups, which can be seen as a refinement of the Alperin–McKay conjecture for these groups.

In Section 4.2 we provide proofs for the main ingredients of this correspondence. First, we derive a consequence of results of A. Turull in [**Tur08**] and [**Tur09**] on certain character triple isomorphisms *above* the Glauberman correspondence. These properties are closely related to the bijections in the reduction theorems of the McKay and Alperin–McKay conjectures. The consequence that we obtain from Turull's results is key for the final step of our proof and it is based on ideas of Laradji [**Lar14**]. We also extend the Fong–Reynolds correspondence for p-solvable groups in order to reduce our problem to blocks of maximal defect, and finally we include a proof of a stronger version of Navarro's result on the number of Sylow p-subgroups in subgroups of p-solvable groups from [**Nav03**]. This last result was proved by Navarro, and we include it in this thesis with his kind permission.

In Section 4.3 we prove the existence of this correspondence in a series of propositions that handle different cases separately. The largest part of the proof is devoted to showing that we may assume that G has no nontrivial normal p-subgroups, and at that point we can use Turull's results.

Finally, in Section 4.4 we prove the divisibility property of dimensions of Brauer correspondents, which takes full advantage of Navarro's result and the extended Fong–Reynolds correspondence proved in Section 4.2.

Interestingly, even though all the results in this chapter are for p-solvable groups, they depend on the classification of finite simple groups as well. The classification is necessary to use Geck's deep result [**Gec20**] on the Glauberman correspondence.

We end the chapter with some remarks on interesting consequences of our Theorem G for nilpotent blocks and the Dade–Glauberman–Nagao correspondence.

CHAPTER 1

Preliminaries

In this thesis, our group theoretical notation follows [**Isa08**].

1.1. Basics on character theory

For ordinary characters, we follow [**Isa06**] and [**Nav18**].

Let G be a finite group and let F be a field. We denote by FG the set

$$FG = \{\sum_{g \in G} a_g g \mid a_g \in F\}.$$

The structure of F-vector space is given to FG by

$$f \sum_{g \in G} a_g g = \sum_{g \in G} f a_g g$$

for $f \in F$ and

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g.$$

We may identify the elements $g \in G$ with the sum $\sum_{h \in G} a_h h$ where $a_g = 1$ and $a_h = 0$ for all $h \neq g$, and in fact the elements of G form a basis for FG. Finally, to define multiplication in FG we use the product in G for the elements of the basis and extend linearly to FG. Thus FG is an F-algebra, which we call the **group algebra**.

An F**-representation** of G is a group homomorphism

$$\mathcal{X} : G \to \mathrm{GL}_n(F).$$

The integer n is the **degree** of $\mathcal{X}$. By extending linearly, we obtain an algebra homomorphism $FG \to \mathrm{Mat}_n(F)$. Conversely, an algebra homomorphism $FG \to \mathrm{Mat}_n(F)$ defines an F-representation of G by restriction. Two F-representations $\mathcal{X}$ and $\mathcal{Y}$ are said to be **similar** if there exists $M \in \mathrm{GL}_n(F)$ such that $\mathcal{X}(g) = M^{-1}\mathcal{Y}(g)M$ for all $g \in G$.

We say that an F-representation $\mathcal{X}$ is **irreducible** if it is not similar to a representation of G which can be written in block form as

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

The F**-character** afforded by an F-representation $\mathcal{X}$ is defined by $\chi(g) = \mathrm{tr}(\mathcal{X}(g))$ for all $g \in G$, where $\mathrm{tr}(A)$ denotes the trace of the square matrix A. We say that an F-character is **irreducible** if it is afforded by some irreducible F-representation. The integer $\chi(1)$ is the **degree** of χ (which is precisely the degree of $\mathcal{X}$). It follows easily from the definitions that F-characters are constant on the conjugacy classes of G. Furthermore, similar F-representations afford the same F-character.

If $\mathcal{Y}$ and $\mathcal{Z}$ are F-representations of G, then the map defined as

$$g \mapsto \mathcal{X}(g) = \begin{pmatrix} \mathcal{Y}(g) & 0 \\ 0 & \mathcal{Z}(g) \end{pmatrix}$$

is an F-representation of G which we call the **sum** of $\mathcal{Y}$ and $\mathcal{Z}$. It follows easily that sums of F-characters are F-characters.

Now if $A \in \mathrm{Mat}_n(F)$ and $B \in \mathrm{Mat}_m(F)$ then we define the **Kronecker product** of matrices $A \otimes B \in \mathrm{Mat}_{nm}(F)$ by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

It turns out that if $\mathcal{X}$ and $\mathcal{Y}$ are F-representations of G of degrees n and m respectively, $\mathcal{X} \otimes \mathcal{Y}$ defined by

$$(\mathcal{X} \otimes \mathcal{Y})(g) = \mathcal{X}(g) \otimes \mathcal{Y}(g)$$

for all $g \in G$ is an F-representation of G of degree nm. Furthermore, if $\mathcal{X}$ affords χ and $\mathcal{Y}$ affords ψ then $\mathcal{X} \otimes \mathcal{Y}$ affords $\chi\psi$, so products of characters are also characters.

From now on we fix $F = \mathbb{C}$. We denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters (we often omit the word *complex* and just say irreducible character). We also denote by $\mathrm{Char}(G)$ the set of characters of G. The map

$$\begin{aligned} 1_G : G &\to \mathbb{C}^\times \\ g &\mapsto 1 \end{aligned}$$

is a character of G, called the **trivial** or **principal** character. A character of degree 1 is called a **linear** character. Clearly, linear characters are irreducible. We denote by $\mathrm{Lin}(G)$ the set of linear characters of G. Notice that if λ is a linear character then λ is a group homomorphism $\lambda : G \to \mathbb{C}^\times$. It is easy to check that $\mathrm{Lin}(G)$ is a group with multiplication given by $(\lambda\mu)(g) = \lambda(g)\mu(g)$ for all $g \in G$.

We denote by $\mathrm{cf}(G)$ the set of maps $G \to \mathbb{C}$ that are constant on the conjugacy classes of G (we call these maps **class functions**). It is easy to see that $\mathrm{cf}(G)$ is a $\mathbb{C}$-vector space.

We can define a hermitian inner product in $\mathrm{cf}(G)$ as follows. If α and β are class functions, then

$$[\alpha, \beta] = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}$$

where $\overline{\beta(g)}$ denotes the complex conjugate of $\beta(g)$. The following is a fundamental theorem in character theory.

THEOREM 1.1. *The set* $\mathrm{Irr}(G)$ *is an orthonormal basis of* $\mathrm{cf}(G)$. *In particular,* $|\mathrm{Irr}(G)|$ *equals the number of conjugacy classes of* G.

PROOF. This is [**Isa06**, Theorem 2.8] and the First Orthogonality Relation ([**Isa06**, Corollary 2.14]). □

If χ is a character of G, then by Theorem 1.1 we may write

$$\chi = \sum_{\psi \in \mathrm{Irr}(G)} [\chi, \psi]\psi.$$

The irreducible characters that verify $[\chi, \psi] \neq 0$ are called the **irreducible constituents** of χ, and $[\chi, \psi]$ is the **multiplicity** of ψ in χ.

Let $\mathrm{Cl}(G) = \{K_1, \ldots, K_k\}$ be the set of conjugacy classes of G. Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ and let $g_j \in K_j$. Another consequence of Theorem 1.1 is that the square matrix

$$X(G) = (\chi_i(g_j))$$

is regular. We call $X(G)$ the **character table** of G.

We end this section with the definition of the determinant of a character. Let χ be a character of G afforded by a representation $\mathcal{X}$. Since $\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$ is a group homomorphism, we define the **determinant** $\det(\chi)$ by

$$\det(\chi)(g) = \det(\mathcal{X}(g))$$

and it is easy to see that $\det(\chi) \in \mathrm{Lin}(G)$. The **determinantal order** $o(\chi)$ of χ is the order of $\det(\chi)$ in the group $\mathrm{Lin}(G)$.

1.1.1. Induction and restriction of characters. Let H be a subgroup of G and $\alpha \in \mathrm{cf}(H)$. Define the map $\dot{\alpha} : G \to \mathbb{C}$ as $\dot{\alpha}(x) = \alpha(x)$ if $x \in H$ and $\dot{\alpha}(x) = 0$ otherwise.

Definition 1.2. *The **induced** class function α^G is defined by*

$$\alpha^G(x) = \frac{1}{|H|} \sum_{g \in G} \dot{\alpha}(gxg^{-1}).$$

It follows directly from the definition that α^G is a class function of G, and that $\alpha^G(1) = |G : H|\alpha(1)$.

Definition 1.3. *If $\alpha \in \mathrm{cf}(G)$, then the **restriction** of α to H is defined by*

$$\alpha_H(x) = \alpha(x)$$

for all $x \in H$.

Again, it is straightforward to check that α_H is a class function of H, and that if $\alpha \in \mathrm{Char}(G)$ then $\alpha_H \in \mathrm{Char}(H)$. The following easy result gives a relation between induction and restriction.

Theorem 1.4 (Frobenius reciprocity). *Let $\alpha \in \mathrm{cf}(H)$ and $\beta \in \mathrm{cf}(G)$. Then $[\alpha^G, \beta] = [\alpha, \beta_H]$.*

Proof. See [**Isa06**, Lemma 5.2]. □

It follows that if $\alpha \in \mathrm{Char}(H)$ then $\alpha^G \in \mathrm{Char}(G)$.

Theorem 1.5. *Suppose that G is a finite group, $N \lhd G$ and $H \leqslant G$ such that $NH = G$ and let $M = H \cap N$. Then restriction defines a bijection $\mathrm{Irr}(G/N) \to \mathrm{Irr}(H/M)$.*

Proof. See [**Nav18**, Theorem 1.18]. □

1.1.2. Factor groups.

Definition 1.6. *Let χ be a character of G. Then the **kernel** of χ is*

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}.$$

A character with trivial kernel is said to be **faithful**.

Lemma 1.7. *Let $\mathcal{X}$ be a representation of G affording the character χ. Then $\ker(\chi) = \ker(\mathcal{X})$.*

Proof. See [**Isa06**, Lemma 2.19]. □

Theorem 1.8. *Let G be a finite group and $N \lhd G$.*

(i) *If $\chi \in \mathrm{Char}(G)$ and $N \subseteq \ker(\chi)$ then $\chi(ng) = \chi(g)$ for all $g \in G$, $n \in N$.*

(ii) *Let $\mathcal{A} = \{\chi \in \mathrm{Char}(G) \mid N \subseteq \ker(\chi)\}$. If $\chi \in \mathcal{A}$ then the map $\tilde{\chi} : G/N \to \mathbb{C}$ given by*
$$\tilde{\chi}(Ng) = \chi(g)$$
is a well-defined character of G/N.

(iii) *The map $\chi \mapsto \tilde{\chi}$ is a bijection $\mathcal{A} \to \mathrm{Char}(G/N)$ satisfying $[\chi, \psi] = [\tilde{\chi}, \tilde{\psi}]$.*

Proof. This is [**Nav18**, Theorem 1.11]. □

In this thesis, we view $\mathrm{Irr}(G/N)$ as a subset of $\mathrm{Irr}(G)$, by identifying the characters χ and $\tilde{\chi}$ from the previous theorem. Notice that, if G' denotes the commutator subgroup of G, we have $\mathrm{Lin}(G) = \mathrm{Irr}(G/G')$.

1.1.3. Normal subgroups. Let $N \lhd G$ and $\theta \in \mathrm{cf}(N)$. If $g \in G$ then we define θ^g by $\theta^g(n) = \theta(gng^{-1})$ for all $n \in N$. If $\theta \in \mathrm{Char}(N)$ is afforded by the representation $\mathcal{X}$ then $\theta^g \in \mathrm{Char}(N)$ is the character afforded by the representation $\mathcal{X}^g$ defined by $\mathcal{X}^g(n) = \mathcal{X}(gng^{-1})$ for all $n \in N$. In this case, we say that θ and θ^g are **conjugate** characters in G. We have that G acts on $\mathrm{Char}(N)$ and $\mathrm{Irr}(N)$ by conjugation.

Theorem 1.9 (Clifford). *Let $\chi \in \mathrm{Irr}(G)$ and let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. Let $\{\theta_1, \ldots, \theta_t\}$ be the set of different G-conjugates of θ. Then*
$$\chi_N = e \sum_{i=1}^{t} \theta_i,$$
where $e = [\chi_N, \theta]$.

Proof. This is [**Isa06**, Theorem 6.2] □

Let $\theta \in \mathrm{Irr}(N)$. We denote by $\mathrm{Irr}(G|\theta)$ the set of irreducible characters of G whose restriction to N contains θ as an irreducible constituent. Notice that by Frobenius reciprocity, $\mathrm{Irr}(G|\theta)$ consists of the irreducible constituents of θ^G.

Notice that from Theorem 1.9 it follows that $e = [\chi_N, \theta]$, t and $\theta(1)$ all divide $\chi(1)$.

If $\theta \in \mathrm{Irr}(N)$ we denote by $G_\theta = \{g \in G \mid \theta^g = \theta\}$ the stabilizer of θ in G (also sometimes known as the **inertia subgroup** of θ in G). By the orbit-stabilizer theorem $|G : G_\theta| = t$, using the notation of the previous theorem. The following result relates $\mathrm{Irr}(G_\theta|\theta)$ and $\mathrm{Irr}(G|\theta)$ and often allows us to assume that θ is G-invariant.

THEOREM 1.10 (Clifford correspondence). *Let $\theta \in \mathrm{Irr}(N)$. Then the map $\psi \mapsto \psi^G$ defines a bijection $\mathrm{Irr}(G_\theta|\theta) \to \mathrm{Irr}(G|\theta)$.*

PROOF. This is [**Isa06**, Theorem 6.11]. □

In the situation above, we say $\psi \in \mathrm{Irr}(G_\theta|\theta)$ and $\psi^G \in \mathrm{Irr}(G|\theta)$ are **Clifford correspondents**.

Suppose that $\chi \in \mathrm{Irr}(G)$ is such that $\chi_N = \theta \in \mathrm{Irr}(N)$. Then we say that χ is an **extension** of θ to G, or that θ **extends** to G. In this situation, we have full control of the set $\mathrm{Irr}(G|\theta)$ via the following correspondence.

THEOREM 1.11 (Gallagher correspondence). *Suppose $\chi \in \mathrm{Irr}(G)$ is such that $\chi_N = \theta \in \mathrm{Irr}(N)$. Then the map*

$$\begin{aligned} \mathrm{Irr}(G/N) &\to \mathrm{Irr}(G|\theta) \\ \beta &\mapsto \beta\chi \end{aligned}$$

is a well-defined bijection.

PROOF. This is [**Isa06**, Theorem 6.17]. □

There is a situation when we can guarantee that a character of a normal subgroup extends.

THEOREM 1.12. *Let $\theta \in \mathrm{Irr}(N)$ be G-invariant and assume that G/N is cyclic. Then θ extends to G.*

PROOF. This is [**Nav18**, Theorem 5.1]. □

1.1.4. The Glauberman correspondence.

DEFINITION 1.13. *Assume a group A acts on a group G. Assume further that for any $x, y \in G$, $a \in A$ we have $(xy)^a = x^a y^a$. Then we say that A **acts by automorphisms** on G.*

In this case, for all elements $a \in A$, the map defined by

$$\begin{aligned} \sigma_a : G &\to G \\ g &\mapsto g^a \end{aligned}$$

is an automorphism of G. A classical example of such an action is a group acting by conjugation on a normal subgroup. By [**Nav18**, Theorem 2.1] it follows that A acts on $\mathrm{Irr}(G)$ by

$$\chi^a(g) = \chi(g^{a^{-1}}).$$

When $|A|$ and $|G|$ are coprime and A acts by automorphisms on G, then there is a *natural* correspondence $\mathrm{Irr}_A(G) \to \mathrm{Irr}(\mathbf{C}_G(A))$ known as the **Glauberman–Isaacs correspondence** (here $\mathrm{Irr}_A(G)$ denotes the set of irreducible characters of G fixed by A).

By using the odd order theorem, either A or G is solvable. G. Glauberman proved that this correspondence exists assuming that A is solvable, and I. M. Isaacs proved that it exists when G is solvable. In his PhD Thesis, T. R. Wolf proved that both correspondences coincide when both A and G are solvable.

We state here the p-group case of the Glauberman correspondence, which is what we will need in Chapter 4.

THEOREM 1.14 (Glauberman correspondence). *Suppose that A is a p-group that acts by automorphisms on a finite group G whose order is not divisible by p. Let $C = \mathbf{C}_G(A)$. Then for all $\chi \in \mathrm{Irr}_A(G)$ we have*

$$\chi_C = e\chi^* + p\Delta$$

where $\chi^* \in \mathrm{Irr}(C)$, p does not divide e and Δ is a character of C or zero and such that $[\Delta, \chi^*] = 0$. The map $\chi \mapsto \chi^*$ is a bijection.

PROOF. This is [**Nav18**, Theorem 2.9]. □

So the Glauberman correspondence associates to a character $\chi \in \mathrm{Irr}_A(G)$ the unique irreducible constituent of the restriction χ_C that occurs with multiplicity not divisible by p. The following deep result was proved by M. Geck after a reduction to simple groups due to B. Hartley and A. Turull.

THEOREM 1.15 (Geck). *Suppose that A is a p-group that acts by automorphisms on a finite group G whose order is not divisible by p. Let $C = \mathbf{C}_G(A)$ and let $^* : \mathrm{Irr}_A(G) \to \mathrm{Irr}(C)$ be the Glauberman correspondence. Then $\chi^*(1)$ divides $\chi(1)$.*

PROOF. This was reduced to simple groups in [**HT94**] and proved in [**Gec20**]. □

We mentioned in the introduction that the Glauberman correspondence is in some sense *natural*. We now show a consequence of this.

Lemma 1.16. *Let $L \lhd G$ be a p'-subgroup and let D be a p-subgroup of G. Denote by $C = \mathbf{C}_L(D)$ and let $^* : \mathrm{Irr}_D(L) \to \mathrm{Irr}(C)$ be the D-Glauberman correspondence. If $\theta \in \mathrm{Irr}_D(L)$ then $\mathbf{N}_G(D)_\theta = \mathbf{N}_G(D)_{\theta^*}$.*

Proof. First notice that $C \lhd \mathbf{N}_G(D)$. Let $\theta \in \mathrm{Irr}_D(L)$ and write $\theta_C = e\theta^* + p\Delta$, as in Theorem 1.14, so that e is not divisible by p. If $g \in \mathbf{N}_G(D)$ then

$$(\theta^g)_C = e(\theta^*)^g + p\Delta^g \tag{1.1.1}$$

where, if $\Delta \neq 0$ then Δ^g denotes the character

$$\Delta^g = \sum_{\psi \in \mathrm{Irr}(C)} [\Delta, \psi]\psi^g$$

and otherwise $\Delta^g = 0$. Now if $g \in \mathbf{N}_G(D)_\theta$ then by 1.1.1, $(\theta^*)^g$ is a constituent of θ_C with multiplicity not divisible by p. By Theorem 1.14 necessarily $(\theta^*)^g = \theta^*$ and we conclude that $g \in \mathbf{N}_G(D)_{\theta^*}$.

Conversely, if $g \in \mathbf{N}_G(D)_{\theta^*}$ then by 1.1.1 we see that θ_C and $(\theta^g)_C$ both contain θ^* as an irreducible constituent with multiplicity not divisible by p. By Theorem 1.14 we conclude that $\theta^g = \theta$ and $g \in \mathbf{N}_G(D)_\theta$, as desired. □

1.1.5. Character triples.

Definition 1.17. *If $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$ is G-invariant, we say (G, N, θ) is a* ***character triple****.*

The theory of character triples is a very strong tool for studying the set $\mathrm{Irr}(G|\theta)$, whenever $\theta \in \mathrm{Irr}(N)$ is G-invariant. The key result is that we may find an *isomorphic* character triple with nice properties. But, what exactly do we mean by *isomorphic* in this case? In the following definition, we denote by $\mathrm{Char}(G|\theta)$ the set of characters of G whose restriction to N is a multiple of θ.

Definition 1.18 (Isaacs). *Let (G, N, θ) and (H, M, φ) be character triples and suppose that $^* : G/N \to H/M$ is an isomorphism of groups. If $N \leqslant U \leqslant G$, denote by U^* the unique subgroup $M \leqslant U^* \subseteq H$ with $U^*/M = (U/N)^*$. If $\beta \in \mathrm{Char}(U/N)$ then β^* denotes the corresponding character of U^*/M. Now assume that for each subgroup $N \leqslant U \leqslant G$ there is a bijection*

$$^* : \mathrm{Irr}(U|\theta) \to \mathrm{Irr}(U^*|\varphi)$$

which we extend linearly to a map $\mathrm{Char}(U|\theta) \mapsto \mathrm{Char}(U^*|\varphi)$. We say that * is a ***character triple isomorphism*** *if for every $N \leqslant V \leqslant U \leqslant G$, $\chi \in \mathrm{Irr}(U|\theta)$, $\beta \in \mathrm{Irr}(U/N)$ the following holds:*

(i) $(\chi_V)^* = (\chi^*)_{V^*}$,

(ii) $(\chi\beta)^* = \chi^*\beta^*$.

LEMMA 1.19. *If* $^*: (G, N, \theta) \to (H, M, \varphi)$ *is a character triple isomorphism, then for all* $\chi \in \mathrm{Irr}(G|\theta)$ *we have*

$$\frac{\chi(1)}{\theta(1)} = \frac{\chi^*(1)}{\varphi(1)}.$$

PROOF. This is [**Isa06**, Lemma 11.24]. □

The key result is the following.

THEOREM 1.20. *Every character triple* (G, N, θ) *is isomorphic to a character triple* (G^*, N^*, θ^*) *where* N^* *is central in* G^* *and* θ^* *is linear and faithful.*

PROOF. See [**Nav18**, Corollary 5.9]. □

We mentioned at the beginning of this section that character triples provide a strong tool for analyzing the set $\mathrm{Irr}(G|\theta)$. We illustrate this by providing some results which can be proved by using character triple isomorphisms, and which will be used in later chapters.

THEOREM 1.21. *Let* $\theta \in \mathrm{Irr}(N)$. *Then* $\chi(1)/\theta(1)$ *divides* $|G:N|$ *for all* $\chi \in \mathrm{Irr}(G|\theta)$.

PROOF. This is [**Nav18**, Theorem 5.12]. □

THEOREM 1.22. *Let* $\theta \in \mathrm{Irr}(N)$ *be* G*-invariant. Then* θ *extends to* G *if and only if* θ *extends to* P *for every Sylow subgroup* P/N *of* G/N.

PROOF. This is [**Nav18**, Theorem 5.10]. □

THEOREM 1.23. *Let* $\theta \in \mathrm{Irr}(N)$ *be* G*-invariant, and assume*

$$\gcd(o(\theta)\theta(1), |G:N|) = 1.$$

Then θ *extends to* G. *In particular, this happens if* $\gcd(|N|, |G:N|) = 1$.

PROOF. This is [**Nav18**, Corollary 6.2]. □

1.1.6. Fully ramified characters. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Continuing with the analysis of the set $\mathrm{Irr}(G|\theta)$, we explore the situation where $\mathrm{Irr}(G|\theta) = \{\chi\}$. In this case, it is said that θ is **fully ramified** in G.

LEMMA 1.24. *Let $\theta \in \mathrm{Irr}(N)$ be G-invariant and let $\chi \in \mathrm{Irr}(G|\theta)$. The following are equivalent:*

(i) *θ is fully ramified in G,*

(ii) *$\chi_N = e\theta$ where $e^2 = |G : N|$,*

(iii) *$\chi(g) = 0$ for every $g \in G \backslash N$.*

PROOF. This is [**Nav18**, Lemma 8.2]. □

Notice that by the previous result, if θ is fully ramified in G then $|G : N|$ is a square.

THEOREM 1.25 (DeMeyer–Janusz). *Let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Then θ is fully ramified in G if and only if θ is fully ramified in P for every $P/N \in \mathrm{Syl}_p(G)$ for any prime p dividing $|G|$.*

PROOF. This is [**Nav18**, Theorem 8.3] and [**Nav18**, Problem 8.1]. □

Using Gallagher's character counting formula (see [**Nav18**, Section 5.5]) we can obtain the following result.

LEMMA 1.26. *Let $N \lhd G$ and suppose that $\theta \in \mathrm{Irr}(N)$ is G-invariant. If θ is fully ramified in G and $N < G$ then G/N has no cyclic self-centralizing subgroup.*

PROOF. This is [**Nav18**, Problem 8.3]. □

In 1964, Iwahori and Matsumoto conjectured that if $\theta \in \mathrm{Irr}(N)$ is fully ramified in G then G/N had to be solvable. One of the first applications of the classification of finite simple groups to character theory was the proof of this conjecture, due to Howlett and Isaacs.

THEOREM 1.27 (Howlett–Isaacs). *Let $\theta \in \mathrm{Irr}(N)$ be G-invariant. If θ is fully ramified in G then G/N is solvable.*

PROOF. See [**Nav18**, Theorem 8.13]. □

1.2. Brauer characters and blocks

Richard Brauer introduced the concepts of Brauer characters and blocks as a way to connect representations in characteristic 0 and in prime characteristic. We discuss some basic properties of these objects in this section. Our reference is [**Nav98**].

1.2.1. Brauer characters. Let $\mathbf{R}$ denote the ring of algebraic integers in $\mathbb{C}$. If $\chi \in \mathrm{Irr}(G)$ and $g \in G$ then by elementary linear algebra $\chi(g)$ is a sum of roots of unity and therefore irreducible characters take values in $\mathbf{R}$. Let p be a fixed prime and choose a maximal ideal M of $\mathbf{R}$ containing p. Let $F = \mathbf{R}/M$, which is a field of characteristic p, and consider the canonical ring epimorphism $^* : \mathbf{R} \to F$. We set

$$\mathcal{S} = \{\frac{r}{s} \mid r \in \mathbf{R}, s \in \mathbf{R}\backslash M\} \subseteq \mathbb{C}.$$

We have that $\mathcal{S}$ is a ring with $\mathbf{R} \subseteq \mathcal{S}$. We extend $^* : \mathbf{R} \to F$ in the natural way, i.e.

$$\left(\frac{r}{s}\right)^* = r^*(s^*)^{-1}.$$

Let

$$\mathbf{U} = \{\xi \in \mathbb{C} \mid \xi^m = 1 \text{ for some integer } m \text{ not divisible by } p\} \subseteq \mathbf{R}.$$

Lemma 1.28. *The restriction of * to $\mathbf{U}$ defines an isomorphism $\mathbf{U} \to F^\times$ of multiplicative groups. Also, F is algebraically closed.*

Proof. See [**Nav98**, Lemma 2.1]. □

In this thesis, we denote by G^0 the set of p-regular elements of G. Suppose $\mathcal{X} : G \to \mathrm{GL}_n(F)$ is an F-representation of G. If $g \in G^0$ then the eigenvalues of $\mathcal{X}(g)$ lie in $F^\times$. By Lemma 1.28, these eigenvalues are of the form $\xi_1^*, \dots, \xi_n^*$ for unique $\xi_1, \dots, \xi_n \in \mathbf{U}$. We say the **Brauer character** of G afforded by $\mathcal{X}$ is the map $\varphi : G^0 \to \mathbb{C}$ defined by $\varphi(g) = \xi_1 + \cdots + \xi_n$. We remark that φ is uniquely determined up to choice of the maximal ideal M. We say that φ is **irreducible** if $\mathcal{X}$ is irreducible, and denote the set of irreducible Brauer characters of G by $\mathrm{IBr}(G)$. Brauer characters are also sometimes called **modular characters**.

Exactly as it happens with ordinary characters, sums and products of Brauer characters are Brauer characters.

Theorem 1.29. *If p does not divide $|G|$, then $\mathrm{IBr}(G) = \mathrm{Irr}(G)$.*

Proof. This is [**Nav98**, Theorem 2.12]. □

Let $\mathrm{cf}(G^0)$ be the set of class functions on the conjugacy classes of G contained in G^0.

Theorem 1.30. *The set* $\mathrm{IBr}(G)$ *is a basis for* $\mathrm{cf}(G^0)$*. Moreover,* $\varphi \in \mathrm{cf}(G^0)$ *is a Brauer character if and only if* φ *is a linear combination of the elements of* $\mathrm{IBr}(G)$ *with nonnegative integral coefficients.*

Proof. This is [**Nav98**, Theorem 2.3] and [**Nav98**, Theorem 2.10]. □

Now if $\chi \in \mathrm{Irr}(G)$ we denote by χ^0 the restriction of χ to G^0. By [**Nav98**, Corollary 2.9] χ^0 is a Brauer character of G. By Theorem 1.30 we may write

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi$$

for unique nonnegative integers $d_{\chi\varphi}$. These integers are called the **decomposition numbers**.

As we did with ordinary characters, if $N \lhd G$ and $\eta \in \mathrm{IBr}(N)$ then we denote by $\mathrm{IBr}(G|\eta)$ the set of irreducible Brauer characters of G whose restriction to N contains η as a constituent.

Definition 1.31. *Let* φ *be a Brauer character afforded by the* F*-representation* $\mathcal{X}$ *of* G*. Then* $\ker(\varphi) = \ker(\mathcal{X})$*.*

As happens with ordinary characters, if $N \lhd G$ then we view $\mathrm{IBr}(G/N)$ as the subset of $\mathrm{IBr}(G)$ of Brauer characters that contain N in their kernel.

Theorem 1.32. *Let* $\mathcal{X}$ *be an irreducible* F*-representation of* G*. Then* $\mathbf{O}_p(G) \subseteq \ker(\mathcal{X})$*.*

Proof. See [**Nav98**, Lemma 2.32]. □

It follows that $\mathrm{IBr}(G/\mathbf{O}_p(G)) = \mathrm{IBr}(G)$.

1.2.2. Blocks. Let $K \in \mathrm{Cl}(G)$ and let R be any commutative ring with 1. We denote by $\hat{K} = \sum_{x \in K} x \in RG$. In fact, it is easy to see that $\hat{K} \in \mathbf{Z}(RG)$ and the set $\{\hat{K} \mid K \in \mathrm{Cl}(G)\}$ is a basis for $\mathbf{Z}(RG)$ (see for example [**Isa06**, Theorem 2.4]).

Let $\chi \in \mathrm{Irr}(G)$

$$\omega_\chi(\hat{K}) = \frac{|K|\chi(x_K)}{\chi(1)}$$

where $x_K \in K$. A fundamental fact of character theory is that $\omega_\chi(\hat{K}) \in \mathbf{R}$ for all $K \in \mathrm{Cl}(G)$ (see [**Isa06**, Theorem 3.7]). This is the key fact that allows the connection between characteristic 0 and characteristic p representation theory.

Indeed, set $\lambda_\chi(\hat{K}) = \omega_\chi(\hat{K})^* \in F$. By viewing $\hat{K} \in \mathbf{Z}(FG)$ and extending λ_χ linearly, $\lambda_\chi : \mathbf{Z}(FG) \to F$ is an algebra homomorphism.

Now let $\varphi \in \mathrm{IBr}(G)$ and let $\mathcal{X} : G \to \mathrm{GL}_n(F)$ be an F-representation affording φ. We extend $\mathcal{X} : FG \to \mathrm{Mat}_n(F)$ linearly and obtain an algebra homomorphism. By Schur's lemma, $\mathcal{X}(\hat{K})$ is a scalar matrix for all $K \in \mathrm{Cl}(G)$. Set $\lambda_\varphi(\hat{K}) \in F$ to be the scalar such that $\mathcal{X}(\hat{K}) = \lambda_\varphi(\hat{K}) I_n$ and again $\lambda_\varphi : \mathbf{Z}(FG) \to F$ defines an algebra homomorphism by extending linearly.

We are now ready to define the main object of this section (and of this thesis).

Definition 1.33. *The p-**blocks** of G are the equivalence classes in $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ under the relation $\chi \sim \psi$ if $\lambda_\chi = \lambda_\psi$ for $\chi, \psi \in \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$.*

We often omit the prime p whenever there is no ambiguity and refer to p-blocks as blocks. We write $\mathrm{Bl}(G)$ for the set of p-blocks of G.

If B is a p-block we denote $\mathrm{Irr}(B) = \mathrm{Irr}(G) \cap B$ and $\mathrm{IBr}(B) = \mathrm{IBr}(G) \cap B$. We write $\lambda_B = \lambda_\psi$ for any $\psi \in B$. We write $k(B) = |\mathrm{Irr}(B)|$ and $l(B) = |\mathrm{IBr}(B)|$.

Theorem 1.34. *If $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(G)$ are such that $d_{\chi\varphi} \neq 0$ then $\lambda_\chi = \lambda_\varphi$.*

Proof. See [**Nav98**, Theorem 3.3]. □

As a consequence, $\mathrm{IBr}(B)$ is precisely the set of constituents of the Brauer characters χ^0 for all $\chi \in \mathrm{Irr}(B)$. It follows from Theorem 1.34 and [**Nav98**, Corollary 2.11] that both $k(B) > 0$ and $l(B) > 0$.

We digress a little bit to state a different characterization of blocks. We will say $\chi, \psi \in \mathrm{Irr}(G)$ are **linked** if there is some $\varphi \in \mathrm{IBr}(G)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}$. Linking defines a graph whose vertices are the set $\mathrm{Irr}(G)$ and where we connect two vertices if the corresponding characters are linked. This graph is known as the **Brauer graph**.

Theorem 1.35. *If $B \in \mathrm{Bl}(G)$, then $\mathrm{Irr}(B)$ is a single connected component of the Brauer graph.*

Proof. This is [**Nav98**, Lemma 3.5]. □

Theorem 1.36 (Weak block orthogonality). *Let B be a p-block of G. Let $x \in G^0$ and $y \in G \backslash G^0$. Then*

$$\sum_{\chi \in \mathrm{Irr}(B)} \chi(x)\overline{\chi(y)} = 0.$$

Proof. See [**Nav98**, Corollary 3.7]. □

There is a particularly important block called the **principal block**. It is denoted by $B_0(G)$ and it is the unique block of G that contains the trivial character 1_G.

Throughout this thesis, if n is a positive integer, we will write n_p for the p**-part** of n, that is, the largest power of the prime p that divides n.

Write $|G|_p = p^a$ and let B be a block of G. We define the **defect** of a block B to be the nonnegative integer $\mathrm{d}(B)$ such that

$$p^{a-\mathrm{d}(B)} = \min\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B)\}.$$

By definition, if $\chi \in \mathrm{Irr}(B)$ then there exists a nonnegative integer h_χ such that

$$p^{a-\mathrm{d}(B)+h_\chi} = \chi(1)_p.$$

We say h_χ is the **height** of χ.

1.2.3. Defect groups. In this section, we show how to associate a G-conjugacy class of p-subgroups, called the **defect groups**, to a block (see Definition 1.40).

Let $\chi \in \mathrm{Irr}(G)$. The **primitive central idempotent** associated to χ is

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g \in \mathbf{Z}(\mathbb{C}G).$$

To see that these are central, notice that we may write

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{K\in \mathrm{Cl}(G)}\chi(x_K^{-1})\hat{K}$$

where $x_K \in K$.

Recall the definition of $\mathcal{S}$ at the beginning of Section 1.2.

Theorem 1.37. *Let B be a block of G. Then*

$$f_B = \sum_{\chi\in\mathrm{Irr}(B)} e_\chi \in \mathbf{Z}(\mathcal{S}G).$$

Proof. This is [**Nav98**, Corollary 3.8]. □

We extend $^* : \mathbf{Z}(\mathcal{S}G) \to \mathbf{Z}(FG)$ by

$$\left(\sum_{K\in\mathrm{Cl}(G)} s_K\hat{K}\right)^* = \sum_{K\in\mathrm{Cl}(G)} s_K^*\hat{K}$$

and write

$$e_B = (f_B)^* \in \mathbf{Z}(FG).$$

Theorem 1.38. *Let $B, B' \in \mathrm{Bl}(G)$. Then the following hold.*

(i) $e_B e_{B'} = \delta_{BB'} e_B$,

(ii) $1 = \sum_{B \in \mathrm{Bl}(G)} e_B$,

(iii) $\lambda_B(e_{B'}) = \delta_{BB'}$

(iv) *the set* $\{\lambda_B \mid B \in \mathrm{Bl}(G)\}$ *is the set of all algebra homomorphisms* $\mathbf{Z}(FG) \to F$.

PROOF. See [**Nav98**, Theorem 3.11] and the preceding comments. □

Write $e_B = \sum_{K \in \mathrm{Cl}(G)} a_B(K)\hat{K}$. By using (iii) of the previous result we have that

$$1 = \sum_{K \in \mathrm{Cl}(G)} a_B(K)\lambda_B(\hat{K}).$$

It follows that there exists some conjugacy class $K \in \mathrm{Cl}(G)$ such that

$$\lambda_B(\hat{K}) \neq 0 \neq a_B(K).$$

In this situation, we say that K is a **defect class** for B.

THEOREM 1.39. *Let* K *and* L *be defect classes for the block* B *and let* $x_K \in K$, $x_L \in L$. *Let* $P \in \mathrm{Syl}_p(\mathbf{C}_G(x_K))$ *and* $Q \in \mathrm{Syl}_p(\mathbf{C}_G(x_L))$. *Then* P *and* Q *are* G*-conjugate.*

PROOF. This is [**Nav98**, Corollary 4.5]. □

It is this last result that allows us to define the defect groups of a block.

DEFINITION 1.40. *Let* B *be a block with defect class* K, *let* $x_K \in K$ *and* $P \in \mathrm{Syl}_p(\mathbf{C}_G(x_K))$. *The defect groups of* B *are the* G*-conjugates of* P.

THEOREM 1.41. *Let* B *be a block with defect* $\mathrm{d}(B)$ *and let* D *be a defect group of* B. *Then*

$$|D| = p^{\mathrm{d}(B)}.$$

PROOF. This is [**Nav98**, Theorem 4.6]. □

Notice for instance that $\chi(1)$ is not divisible by p for some $\chi \in \mathrm{Irr}(B)$ if and only if the defect groups of B are the Sylow p-subgroups of G. This is the case of the principal block, for instance.

THEOREM 1.42. *Let* $B \in \mathrm{Bl}(G)$ *and let* D *be a defect group of* B. *Then* $\mathbf{O}_p(G) \subseteq D$.

PROOF. See Theorem [**Nav98**, Theorem 4.8], □

1.2.4. Block induction. There are different definitions of block induction in the literature, although they coincide in the important cases. The one we use is due to Brauer.

If $H \leqslant G$ and $b \in \mathrm{Bl}(H)$, we extend $\lambda_b : \mathbf{Z}(FH) \to F$ to an F-linear map $\lambda_b^G : \mathbf{Z}(FG) \to F$ by

$$\lambda_b^G(\hat{K}) = \lambda_b \left(\sum_{x \in K \cap H} x \right)$$

(here the sum is zero if $K \cap H = \varnothing$). Notice that $K \cap H$ is the union of conjugacy classes of H, so

$$\sum_{x \in K \cap H} x \in \mathbf{Z}(FH)$$

and λ_b^G is well defined.

The map λ_b^G may not be an algebra homomorphism. However, if it is an algebra homomorphism then by Theorem 1.38(iv) we have that there is a unique block $b^G \in \mathrm{Bl}(G)$ such that $\lambda_b^G = \lambda_{b^G}$. In this case we say b^G is defined and that it is the **induced block**.

THEOREM 1.43. *Suppose that P is a p-subgroup of G and let $H \leqslant G$ be a such that $P\mathbf{C}_G(P) \subseteq H \subseteq \mathbf{N}_G(P)$. If $b \in \mathrm{Bl}(H)$ then b^G is defined. Moreover if B is any block of G then $B = b^G$ for some $b \in \mathrm{Bl}(H)$ if and only if P is contained in some defect group of B.*

PROOF. See [**Nav98**, Theorem 4.14]. □

Let P be a p-subgroup of G. We denote by $\mathrm{Bl}(G|P)$ the set of blocks of G with defect group P.

THEOREM 1.44 (Brauer's first main theorem). *The map*

$$\begin{aligned} \mathrm{Bl}(\mathbf{N}_G(P)|P) &\to \mathrm{Bl}(G|P) \\ b &\mapsto b^G \end{aligned}$$

is a bijection.

PROOF. See [**Nav98**, Theorem 4.17]. □

In the situation of Brauer's first main theorem, we say b and b^G are **Brauer correspondent blocks**. The famous **Alperin–McKay conjecture** states that Brauer correspondent blocks should have the same number of characters of height zero.

LEMMA 1.45. *Let $H \leqslant L \leqslant G$ and assume $b \in \mathrm{Bl}(H)$ is such that b^L is defined. Then b^G is defined if and only if $(b^L)^G$ is defined, and in this case $b^G = (b^L)^G$.*

Proof. Since b^L is defined, there is some block $B \in \mathrm{Bl}(L)$ such that $b^L = B$ and then $\lambda_B = \lambda_b^L$. If $K \in \mathrm{Cl}(G)$ then

$$\lambda_B^G(\hat{K}) = \lambda_B\left(\sum_{x \in K \cap L} x\right) = \lambda_b^L\left(\sum_{x \in K \cap L} x\right) = \lambda_b\left(\sum_{x \in K \cap H} x\right) = \lambda_b^G(\hat{K})$$

and the result follows. □

The following result relates irreducible induction of characters with block induction.

Theorem 1.46. *Let $H \leqslant G$, $b \in \mathrm{Bl}(H)$ and assume $\psi \in \mathrm{Irr}(b)$ is such that $\psi^G \in \mathrm{Irr}(G)$. Then b^G is defined and $\psi^G \in \mathrm{Irr}(b^G)$.*

Proof. This is [**Nav98**, Corollary 6.2]. □

A natural question comes up: suppose $H \leqslant G$ and $b \in \mathrm{Bl}(H)$ such that b^G is defined. When is it possible that $b^G = B_0(G)$? Brauer's third main theorem answers this question.

Theorem 1.47 (Brauer's third main theorem). *Let $H \leqslant G$ and $b \in \mathrm{Bl}(H)$. Assume b^G is defined. Then $b^G = B_0(G)$ if and only if $b = B_0(H)$.*

Proof. This is [**Nav98**, Theorem 6.7]. □

1.2.5. Blocks and normal subgroups. Let $N \lhd G$. We now discuss the relation between blocks of G and the blocks of G/N and N. Recall that we identify (ordinary and Brauer) characters of G/N with characters of G that contain N in their kernel.

Definition 1.48. *Let $B \in \mathrm{Bl}(G)$ and $\overline{B} \in \mathrm{Bl}(G/N)$. Then we say B **dominates** $\overline{B}$ if $\mathrm{Irr}(\overline{B}) \cap \mathrm{Irr}(B) \neq \varnothing$.*

It turns out that if B dominates $\overline{B}$ then $\overline{B} \subseteq B$ (see the discussion before [**Nav98**, Theorem 7.6]). In general, $B \in \mathrm{Bl}(G)$ may not dominate any blocks of G/N (of course this happens when B contains no characters with N in their kernel). The following is an extremely useful result when working with factor groups.

Theorem 1.49. *Let $N \lhd G$ and write $\overline{G} = G/N$.*

(i) *Suppose that $\overline{B} \subseteq B$ where $\overline{B} \in \mathrm{Bl}(\overline{G})$ and $B \in \mathrm{Bl}(G)$. If $\overline{D}$ is a defect group of $\overline{B}$, then there is a defect group P of B such that $\overline{D} \subseteq PN/N$.*

(ii) *If N is a p-group, then every block $B \in \mathrm{Bl}(G)$ contains a block $\overline{B} \in \mathrm{Bl}(\overline{G})$ with defect groups P/N where P is a defect group of B.*

(iii) *If N is a p'-group and $\overline{B} \subseteq B$ where $\overline{B} \in \mathrm{Bl}(\overline{G})$ and $B \in \mathrm{Bl}(G)$, then $\mathrm{Irr}(B) = \mathrm{Irr}(\overline{B})$, $\mathrm{IBr}(B) = \mathrm{IBr}(\overline{B})$ and the defect groups of $\overline{B}$ are the groups PN/N for every defect group P of B.*

PROOF. This is [**Nav98**, Theorem 9.9]. □

There is a particular situation in case (ii) of the previous result where more can be said.

THEOREM 1.50. *Suppose that G has a normal p-subgroup P such that $G/\mathbf{C}_G(P)$ is a p-group. If $B \in \mathrm{Bl}(G)$ then denote by $\overline{B} \in \mathrm{Bl}(G/\mathbf{C}_G(P))$ the unique block dominated by B (by Theorem 1.49(ii)). Then the map $B \mapsto \overline{B}$ is a bijection $\mathrm{Bl}(G) \to \mathrm{Bl}(G/\mathbf{C}_G(P))$ and $l(B) = l(\overline{B})$.*

PROOF. This is part of [**Nav98**, Theorem 9.10]. □

It is interesting to note that there is some compatibility between the Gallagher correspondence (Theorem 1.11) and blocks.

LEMMA 1.51. *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Suppose that $\chi_N = \theta \in \mathrm{Irr}(N)$. Let $\chi_1, \chi_2 \in \mathrm{Irr}(G|\theta)$ and write $\chi_i = \beta_i\chi$ where $\beta_i \in \mathrm{Irr}(G/N)$ for $i = 1, 2$. If β_1 and β_2 lie in the same block of G/N then χ_1 and χ_2 lie in the same block of G.*

PROOF. See [**Riz18**, Lemma 2.4]. □

If $B \in \mathrm{Bl}(G)$ then we define the **kernel** of B as

$$\ker(B) = \bigcap_{\chi \in \mathrm{Irr}(B)} \ker(\chi).$$

THEOREM 1.52. *Let $B \in \mathrm{Bl}(G)$ and $\chi \in \mathrm{Irr}(B)$. Then $\ker(B) = \mathbf{O}_{p'}(\ker(\chi))$. In particular, if $B = B_0(G)$ then $\ker(B) = \mathbf{O}_{p'}(G)$.*

PROOF. This is [**Nav98**, Theorem 6.10]. □

Moreover we can obtain the following consequence of Theorem 1.49 for principal blocks.

COROLLARY 1.53. *Let $N \lhd G$. Then $B_0(G/N) \subseteq B_0(G)$. If N is a p'-group, then $\mathrm{Irr}(B_0(G/N)) = \mathrm{Irr}(B_0(G))$.*

PROOF. This follows from the definition of block domination by noticing that $1_G \in B_0(G)$ contains N in its kernel and is identified with $1_{G/N} \in B_0(G/N)$. Now by Theorem 1.52, every irreducible character in $B_0(G)$ contains $\mathbf{O}_{p'}(G)$ in its kernel, and the second part follows. □

We move on now to discuss how blocks of G interact with blocks of the normal subgroup N. If $b \in \mathrm{Bl}(N)$ and $g \in G$ then $b^g = \{\psi^g \mid \psi \in b\}$ is a block of N.

DEFINITION 1.54. *We say $B \in \mathrm{Bl}(G)$ **covers** $b \in \mathrm{Bl}(N)$ if there is some $\chi \in B$ such that χ_N has an irreducible constituent in b.*

The standard definition of covering is a bit more technical, and we refer the reader to the comments before [**Nav98**, Theorem 9.1]. The fact that the standard definition of covering and ours is equivalent is [**Nav98**, Theorem 9.2]. In fact, this result gives yet another characterization.

THEOREM 1.55. *The block $B \in \mathrm{Bl}(G)$ covers $b \in \mathrm{Bl}(N)$ if and only if for every $\chi \in B$ we have that every irreducible constituent of χ_N lies in a G-conjugate of b.*

PROOF. See [**Nav98**, Theorem 9.2]. □

THEOREM 1.56. *Suppose that $B \in \mathrm{Bl}(G)$ covers $b \in \mathrm{Bl}(N)$. Then given $\theta \in b$ there is $\chi \in B$ over θ.*

PROOF. See [**Nav98**, Theorem 9.4]. □

THEOREM 1.57. *If G/N is a p-group and $b \in \mathrm{Bl}(N)$, then there is a unique block B of G covering b.*

PROOF. See [**Nav98**, Theorem 9.6]. □

Next we state a Clifford correspondence for blocks. If $b \in \mathrm{Bl}(N)$ we denote by $\mathrm{Bl}(G|b)$ the set of blocks of G covering b.

THEOREM 1.58 (Fong–Reynolds correspondence). *Let $b \in \mathrm{Bl}(N)$ and $T(b)$ the stabilizer of b in G.*

(i) *The map*

$$\begin{aligned} \mathrm{Bl}(T(b)|b) &\to \mathrm{Bl}(G|b) \\ B &\mapsto B^G \end{aligned}$$

is a bijection.

(ii) *If* $B \in \mathrm{Bl}(T(b)|b)$ *then* $\mathrm{Irr}(B^G) = \{\chi^G \mid \chi \in \mathrm{Irr}(B)\}$ *and* $\mathrm{IBr}(B^G) = \{\varphi^G \mid \varphi \in \mathrm{IBr}(B)\}$.

(iii) *Every defect group of* $B \in \mathrm{Bl}(T(b)|b)$ *is a defect group of* B^G.

(iv) *If* $B \in \mathrm{Bl}(T(b)|b)$ *and* $\chi \in \mathrm{Irr}(B)$*, then* $h_\chi = h_{\chi^G}$.

Proof. See [**Nav98**, Theorem 9.14]. □

Theorem 1.59. *Suppose that* $B \in \mathrm{Bl}(G|D)$ *covers* $b \in \mathrm{Bl}(N)$ *and that* $\mathbf{C}_G(D) \subseteq N$*. Then* b^G *is defined,* $b^G = B$ *and* B *is the unique block covering* b.

Proof. By [**Nav98**, Lemma 9.20] the block B is regular with respect to N (in the sense of [**Nav98**, p. 210]) and now the result follows from [**Nav98**, Theorem 9.19]. □

Next we state a result of Fong on the defect groups of covered blocks.

Theorem 1.60 (Fong). *Let* $N \lhd G$ *and let* $b \in \mathrm{Bl}(N)$ *be* G*-invariant. Suppose that* $B \in \mathrm{Bl}(G)$ *covers* b *and that* $\mathrm{d}(B) \geqslant \mathrm{d}(B')$ *for all blocks* $B' \in \mathrm{Bl}(G)$ *covering* b*. If* P *is a defect group of* B*, then* p *does not divide* $|G : PN|$ *and* $P \cap N$ *is a defect group of* b.

Proof. See [**Nav98**, Theorem 9.17]. □

We introduce now two important concepts when studying the behavior of blocks.

Definition 1.61. *Let* $B \in \mathrm{Bl}(G)$ *and let* P *be a defect group of* B*. Let* $b \in \mathrm{Bl}(P\mathbf{C}_G(P)|P)$ *be such that* $b^G = B$*. Then* b *is a* ***root*** *of* B.

Notice that roots always exist, by Theorem 1.43. This motivates the following definition.

Definition 1.62. *If* $B \in \mathrm{Bl}(G|P)$ *and* $b \in \mathrm{Bl}(P\mathbf{C}_G(P)|P)$ *is a root of* B*, write* $T(b)$ *for the stabilizer of* b *in* $\mathbf{N}_G(P)$*. Then* $|T(b) : P\mathbf{C}_G(P)|$ *is the* ***inertial index*** *of* B.

By [**Nav98**, Theorem 9.22] the inertial index of a block is always a p'-number.

We end this section by stating an important result of R. Knörr.

Theorem 1.63 (Knörr). *Let* $N \lhd G$*,* $b \in \mathrm{Bl}(N|Q)$ *and let* B *be a block of* G *covering* b*. Then there is a defect group* P *of* B *such that* $P \cap N = Q$.

Proof. This is [**Nav98**, Theorem 9.26]. □

1.2.6. Blocks of p-solvable groups. We include a short section discussing an important resut of Fong on the blocks of p-solvable groups. This is only a very small fraction of what is known about these blocks. The reader is referred to Chapter 10 of [**Nav98**] for a complete exposition of these results. We begin by stating the Fong–Swan theorem.

THEOREM 1.64 (Fong–Swan). *Let G be a p-solvable group, and let $\varphi \in \mathrm{IBr}(G)$. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi^0 = \varphi$.*

PROOF. This is [**Nav98**, Theorem 10.1]. □

Let G be a group of order not divisible by p. Then by Theorem 1.29, $\mathrm{IBr}(G) = \mathrm{Irr}(G)$. Now since $G^0 = G$ and $\mathrm{Irr}(G)$ is a basis for $\mathrm{cf}(G)$, this implies that $\chi, \psi \in \mathrm{Irr}(G)$ are linked if and only if $\chi = \psi$. It follows that the connected components of the Brauer graph are the vertices. By using Theorem 1.35, we have proved the following result.

THEOREM 1.65. *If G is a p'-group, then the p-blocks of G are the singletons $\{\chi\}$ where $\chi \in \mathrm{Irr}(G)$.*

Now we are ready to state Fong's theorem on blocks of p-solvable groups.

THEOREM 1.66 (Fong). *Suppose that G is p-solvable, let $K = \mathbf{O}_{p'}(G)$ and let $\theta \in \mathrm{Irr}(K)$ be G-invariant. Then there is a unique block B of G covering $\{\theta\}$. Further, $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$, $\mathrm{IBr}(B) = \mathrm{IBr}(G|\theta)$, and the defect groups of B are the Sylow p-subgroups of G.*

PROOF. This is [**Nav98**, Theorem 10.20]. □

It turns out that using Fong's theory, we can relate Brauer's first main theorem (Theorem 1.44) with Glauberman's correspondence in a fascinating way.

THEOREM 1.67. *Let G be p-solvable and let $K = \mathbf{O}_{p'}(G)$ and $P \in \mathrm{Syl}_p(G)$. Assume that $\varphi \in \mathrm{Irr}(K)$ is G-invariant and let $B \in \mathrm{Bl}(G)$ be the block covering $\{\varphi\}$. Write $C = \mathbf{C}_K(P)$ and let $\varphi^* \in \mathrm{Irr}(C)$ be the Glauberman correspondent of φ. Let $b \in \mathrm{Bl}(\mathbf{N}_G(P))$ be the Brauer correspondent block of B. Then $\mathrm{Irr}(b) = \mathrm{Irr}(\mathbf{N}_G(P)|\varphi^*)$.*

PROOF. See [**MW93**, Theorem 0.29]. □

1.2.7. Ordinary-modular character triples. We devote this section to the discussion of a refinement of the theory of character triples and character triple isomorphisms that will work particularly well for p-solvable groups.

Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant (so that (G, N, θ) is a character triple). Recall that θ^0 is a Brauer character of N. It may happen that $\theta^0 \in \mathrm{IBr}(N)$. In this case, we say that (G, N, θ) is an **ordinary-modular character triple**.

In the following definition, we use the notation $\mathrm{Br}(G|\eta)$ for the set of Brauer characters of G whose restriction is a multiple of $\eta \in \mathrm{IBr}(N)$.

Definition 1.68. *Let (G, N, θ) and (H, M, φ) be ordinary-modular character triples and assume that there is a character triple isomorphism*

$$^* : (G, N, \theta) \to (H, M, \varphi)$$

in the sense of Definition 1.18. If $N \leqslant U \leqslant G$, denote by U^* the unique subgroup $M \leqslant U^* \subseteq H$ with $U^*/M = (U/N)^*$. If $\beta \in \mathrm{IBr}(U/N)$ then β^* denotes the corresponding Brauer character of U^*/M. Now assume that for each subgroup $N \leqslant U \leqslant G$ there is a bijection

$$^* : \mathrm{IBr}(U|\theta^0) \to \mathrm{IBr}(U^*|\varphi^0)$$

which we extend linearly to a map $\mathrm{Br}(U|\theta^0) \to \mathrm{Br}(U^*|\varphi^0)$. We say * is an ***ordinary-modular character triple isomorphism*** *if for every $N \leqslant V \leqslant U \leqslant G$, $\psi \in \mathrm{IBr}(U|\theta^0)$, $\beta \in \mathrm{IBr}(U/N)$ and $\chi \in \mathrm{Irr}(U|\theta)$ the following holds:*

(i) $(\psi_V)^* = (\psi^*)_{V^*}$,

(ii) $(\psi\beta)^* = \psi^*\beta^*$

(iii) $(\chi^*)^0 = (\chi^0)^*$.

Essentially, the previous definition extends the definition of character triple isomorphism by adding the condition that the character triples are also isomorphic *for Brauer characters* and that these isomorphisms are compatible. The following is the key result.

Theorem 1.69. *Suppose (G, N, θ) is an ordinary-modular character triple, and that $\theta(1)_p = |N|_p$. Then there exists an isomorphic ordinary-modular character triple (H, M, φ) where φ is linear and faithful and M is a p'-group.*

Proof. This is [**Nav98**, Problem 8.13]. □

Notice that whenever N is a normal p'-subgroup of G, then any character triple (G, N, θ) is an ordinary-modular character triple that satisfies the conditions of the previous result.

1.3. Primitive permutation groups

This section is entirely group-theoretical. We introduce the notion of primitive permutation group, their properties and some of the classifications that are needed for the results in Chapter 2.

1.3.1. Primitive actions. Suppose that G acts transitively on a set Ω and let $\Delta \subseteq \Omega$. We denote by $\Delta \cdot g = \{\alpha \cdot g \mid \alpha \in \Delta\}$. We say that Δ is a **set of imprimitivity** if for every $g \in G$ we have that either $\Delta \cdot g = \Delta$ or $\Delta \cdot g \cap \Delta = \varnothing$.

The sets Δ defined above are sometimes called *blocks* in the literature (for instance, in [**Isa08**]), although we steer clear of this word to avoid confusion. The terminology we employ here appears for example in [**KS03**]. The following example appears in [**Isa08**, 8B].

Example 1.70. *Suppose $H \leqslant G$ and let $\Omega = \{Hg \mid g \in G\}$, so that G acts transitively by right multiplication on Ω. Assume that $H \leqslant K \leqslant G$, and let $\Delta = \{Hk \mid k \in K\}$. If $g \in G$ then $\Delta \cdot g = \{Hkg \mid k \in K\}$. Now $Hx \in \Delta \cdot g$ if and only if there is some $k \in K$ such that $Hx = Hkg$, which happens if and only if Hx is contained in the coset Kg. In particular, $(\Delta \cdot g) \cap \Delta = \varnothing$ if $Kg \neq K$ and $\Delta \cdot g = \Delta$ if $Kg = K$. It follows that Δ is a set of imprimitivity.*

Every singleton is a set of imprimitivity, and so is Ω. These are sometimes called the **trivial** sets of imprimitivity.

Definition 1.71. *We say that a transitive action of G on Ω is **primitive** if the only sets of imprimitivity are the singletons and Ω. Otherwise, the action is **imprimitive**.*

If the action of G on Ω is primitive and faithful then we say G is a **primitive permutation group**.

In Example 1.70, notice that if H is a maximal subgroup of G, then the sets of imprimitivity constructed would be trivial. In fact, we have the following result.

Theorem 1.72. *Let G be a group acting transitively on a set Ω which contains more than one point. Let $H = G_\alpha$, for $\alpha \in \Omega$. Then the action of G on Ω is primitive if and only if H is a maximal subgroup of G.*

Proof. See [**Isa08**, Corollary 8.14]. □

Definition 1.73. *Let G be a group acting primitively on a set Ω of n elements. Then the **degree** of the action is n.*

1.3.2. The O'Nan–Scott theorem. Originally, the O'Nan–Scott theorem classified maximal subgroups of symmetric groups. In [**Cam81**], Cameron recognized the strength of this result as it classified the possible structures of finite primitive permutation groups. However, both the original theorem and Cameron's survey omitted a case. This was later corrected by Liebeck, Praeger and Saxl in [**LPS88**].

Before stating this theorem, we briefly discuss a family of primitive permutation groups which will be our main focus. Let G be a primitive permutation group acting on a set Ω of size n. Let V be the socle of G, that is, the subgroup generated by the minimal normal subgroups of G. If V is p-elementary abelian of size p^k, then we say that G is an **affine primitive permutation group** on Ω. In this case Ω can be identified with V (so $n = p^k$) and G is a subgroup of the affine group $V \rtimes \mathrm{GL}(V)$. Also $G_\alpha = G \cap \mathrm{GL}(V)$ for any $\alpha \in V$.

Theorem 1.74 (O'Nan–Scott, Liebeck–Praeger–Saxl). *Let G be a finite group acting primitively on a set Ω of n elements. Then one of the following holds.*

(i) *G is an affine group.*

(ii) *G is almost simple.*

(iii) *The socle of G is a direct product of at least two isomorphic nonabelian simple groups.*

Proof. See [**LPS88**, Section 2] □

We are exclusively interested in case (i) of Theorem 1.74 (notice that this is the only case possible when G is solvable). However, it is worth pointing out that in case (iii), there are three possible actions of G on Ω, which are described in [**LPS88**, Section 1]. One of these had been omitted before.

1.3.3. Affine primitive permutation groups of low rank. The main focus of this section is to state classifications of certain affine primitive permutation groups. We begin by introducing some important concepts.

Definition 1.75. *Let G be a primitive permutation group acting on Ω. The **rank** of G is the number of orbits of G_α acting on Ω, for any $\alpha \in \Omega$.*

By part (i) of Theorem 1.74, if G is a primitive permutation group of affine type with socle V, then V is p-elementary abelian of order p^k and $G = V \rtimes H$ where H is the stabilizer of the trivial element of V, and $H \leqslant \mathrm{GL}(k, p)$. Then the rank of G is the number of orbits of H on V. We will show a classification of groups of this form. Before stating this classification, we need to introduce a family of groups.

The group V can be identified with the additive group of Galois field $\mathbb{F}_{p^k}$. The **semilinear group** $\Gamma(V)$ is the group of maps $V \to V$ defined by

$$\Gamma(V) = \{x \mapsto ax^\sigma \mid a \in V\backslash\{0\}, \sigma \in \mathrm{Gal}(\mathbb{F}_{p^k}/\mathbb{F}_p)\}.$$

Notice that $\Gamma(V)$ is a metacyclic group. Indeed, it contains the normal subgroup

$$\Gamma_0(V) = \{x \mapsto ax \mid a \in V\backslash\{0\}\}$$

which is cyclic and isomorphic to the multiplicative group of $\mathbb{F}_{p^k}^\times$, and

$$\Gamma(V)/\Gamma_0(V) \cong \mathrm{Gal}(\mathbb{F}_{p^k}/\mathbb{F}_p)$$

is cyclic of order k. We may use the notation $\Gamma(p^k) = \Gamma(V)$.

Primitive permutation groups are usually classified in terms of their rank. Notice, for example, that if the rank of G is 2 then G is **doubly transitive** in the sense of [**Isa08**, 8A]. In Chapter 2, when dealing with affine primitive permutation groups, we will be limited to groups of rank 2 and 3, and to p-solvable groups. In the first case, we will use a famous theorem of Passman.

THEOREM 1.76 (Passman). *Let G be a p-solvable affine primitive permutation group of rank 2 with socle V of size p^k, and let H be the stabilizer of the trivial element of V in G. Then one of the following happens.*

(i) $H \leqslant \Gamma(p^k)$,

(ii) *G is solvable and* $p^k \in \{3^2, 5^2, 7^2, 11^2, 23^2, 3^4\}$,

(iii) *G is nonsolvable and* $p^k \in \{11^2, 19^2, 29^2, 59^2\}$.

PROOF. This is [**Pas68**, Theorem 1]. □

We remark that the solvable case of Theorem 1.76 was proved by Huppert.

Cases (ii) and (iii) of Theorem 1.76 are described in Table 1.3.3 when $p \neq 3$ (which is what we will need in Chapter 2) and is extracted from [**Sam14**, Table 15.1] (although we complete the table by including the ID in the last row). We give the order of the stabilizer H of the trivial element of V in G and the [**GAP**] ID in the library of primitive groups.

In the case of rank 3, we will even be able to assume that the group is solvable, thus using the following classification.

THEOREM 1.77 (Foulser). *Let G be a solvable (and thus affine) primitive permutation group of rank 3. Let V be its socle, and let H be the stabilizer of the trivial element of V in G, so that $G = V \rtimes H$. Then one of the following happens:*

(i) *(the affine case)* $H \leqslant \Gamma(V)$,

TABLE 1. Cases (ii) and (iii) of Theorem 1.76 when $p \neq 3$

p^k	$\lvert H\rvert$	GAP Primitive Group ID
5^2	24	$(5^2, 15)$
	48	$(5^2, 18)$
	96	$(5^2, 19)$
7^2	48	$(7^2, 25)$
	144	$(7^2, 29)$
11^2	120	$(11^2, 39)$
	240	$(11^2, 42)$
	120	$(11^2, 56)$
	600	$(11^2, 57)$
19^2	1080	$(19^2, 86)$
23^2	528	$(23^2, 59)$
29^2	840	$(29^2, 106)$
	1680	$(29^2, 119)$
59^2	3480	$(59^2, 84)$

(ii) *(the exceptional case) H stabilizes no nontrivial subgroup of V and*

$$|V| \in \{2^6, 3^4, 13^2, 17^2, 19^2, 3^6, 29^2, 31^2, 47^2, 7^4\},$$

(iii) *(the imprimitive case) there exists a decomposition $V = V_1 \times V_2$ such that V_i is a subgroup of V and each V_i is a set of imprimitivity of the action of H on V. Moreover, H has two orbits on each V_i.*

PROOF. See [**Fou69**, Theorem 1.1]. □

The statement is not given with as much information as in their original classification, but with what is needed for the purposes of Chapter 2. For instance, Cases 2(a), 2(b) and 2(c) of the original result are unified here as Case(ii)(a) and some orbit sizes are omitted.

In the imprimitive case (case (iii) of Theorem 1.77), the subgroups V_1 and V_2 are sometimes called the imprimitivity spaces. Let $K = \mathbf{N}_H(V_1)$. Then $|H : K| = 2$ so $K = \mathbf{N}_H(V_2)$. Then $K \cong (A \times B) \rtimes \mathsf{C}_2$ where $H \cong A \times B$ and $A \cong H/\mathbf{C}_H(V_1) \cong B$ (see [**MW93**, Lemma 2.8] for more general information for the imprimitive case).

CHAPTER 2

The blocks with four irreducible characters

2.1. Introduction

Suppose that G is a finite group, p is a prime, and B is a Brauer p-block of G with defect group D. In [**BF59**], R. Brauer and W. Feit proved that

$$k(B) \leqslant \frac{1}{4}|D|^2 + 1.$$

Besides this bound, very little is known about the number of irreducible complex characters in the block B. Brauer's $k(B)$-conjecture asserts that $k(B)$ should be bounded by the size of the defect groups of B. However, this conjecture has not even been reduced to finite simple groups.

In this chapter we are interested in the problem of classifying the defect groups of blocks in terms of the number $k(B)$. As mentioned in the introduction, this is a hard problem. R. Brauer and C. Nesbitt proved in 1941 that if $k(B) = 1$ then $|D| = 1$ (see [**Nav98**, Theorem 3.18]). The next step came 41 years later, when J. Brandt proved in [**Bra82**] that $k(B) = 2$ if and only if $|D| = 2$, and at the present time there is no classification for $k(B) > 2$. It is however a consequence of the Alperin–McKay conjecture that $k(B) = 3$ if and only if $|D| = 3$, as shown in [**KNST14**], but no proof without assuming this open conjecture is yet available. Although the cases where B is a principal block and $k(B) = 4$ or 5 have been recently solved in [**KS21**] and [**RSV21**], the nonprincipal block cases remain open.

It is well-known that many blocks with $k(B) = 4$ have defect groups with $|D| = 4$ or 5 (for instance $2.\mathfrak{A}_5$ for $p = 5$, or $2.\mathfrak{S}_5$ for $p = 2$), but it is not known if these are the only possibilities, even assuming the Alperin–McKay conjecture. The following is the main result of this chapter.

Theorem A. *Suppose that B is a Brauer p-block of a finite group G with defect group D. Assume that $k(B) = 4$. If the Alperin–McKay conjecture holds for B, then $|D| = 4$ or $|D| = 5$.*

We prove Theorem A by studying finite groups with a small number of projective characters, a problem of interest on its own. This constitutes the main part of this chapter.

Finally, we would like to remark that this result can be seen as a contribution to Brauer's Problem 21, as stated in the introduction. We care to remark that for p-solvable groups, this problem was already solved by B. Külshammer in [**Kül90**], and shown to follow from the Alperin–McKay conjecture in [**KR96**] but without giving the exact bound on $|D|$. In this chapter we give this bound for blocks with four irreducible characters.

The results in this chapter appeared in [**MRS23**].

2.2. Preliminary results

We begin by stating a few necessary results. The first one is a small part of Dade's deep theory of blocks with cyclic defect groups. Recall that we denote by $l(B)$ the number of irreducible Brauer characters in B. In this chapter, we write $k_0(B)$ for the number of height zero ordinary characters in the p-block B.

Theorem 2.1 (Dade). *Let B be a p-block of G with cyclic defect group D. Then*

$$k_0(B) = k(B) = l(B) + \frac{|D| - 1}{l(B)}$$

and $l(B)$ is the inertial index of B (see Definition 1.62).

Proof. This is part of [**Dad66**, Theorem 1]. □

For the primes 2 and 3, P. Landrock was able to obtain useful results on the number of irreducible characters of height zero.

Theorem 2.2 (Landrock). *Let B be a p-block of G with defect group D, and write $|D| = p^d$.*

(i) *If $p = 2$ and $d > 1$ then*

  (a) $k_0(B) \equiv 0 \bmod 4$,

  (b) if there is no $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_p = \frac{|G|_p}{p^{d-1}}$, then

$$k_0(B) \equiv 2^d \bmod 8.$$

(ii) *If $p = 3$ then $k_0(B) \equiv 0 \bmod 3$.*

Proof. The case $p = 2$ is included in [**Lan81**, Corollary 1.3], and the case $p = 3$ is [**Lan81**, Corollary 1.6]. □

The characters satisfying $\chi(1)_p = \frac{|G|_p}{p^{d-1}}$ are precisely those which have height 1. If D is abelian then no such characters exist, by the main result of [**KM13**].

Next we include a summary of the known classifications of defect groups of blocks with a small number of irreducible characters.

THEOREM 2.3. *Let B be a p-block of G with defect group D. Then*

(i) $k(B) = 1$ *if and only if* $|D| = 1$,

(ii) $k(B) = 2$ *if and only if* $|D| = 2$,

(iii) *if the Alperin–McKay conjecture holds for B then $k(B) = 3$ if and only if $|D| = 3$,*

(iv) *if $k(B) = 4$ and $l(B) = 1$ then $|D| = 4$.*

PROOF. The case $k(B) = 1$ was proved by Brauer and Nesbitt, and a proof can be found in [**Nav98**, Theorem 3.18]. The case of $k(B) = 2$ is the main result of [**Bra82**]. If $k(B) = 3$ and the Alperin–McKay conjecture holds for B then this was proved in [**KNST14**, Theorem 4.1], and the converse follows from Theorem 2.1. The final case is done in [**Kül84**]. □

Next we state a remarkable result which will allow us to assume our block has maximal defect when dealing with blocks with normal defect groups.

THEOREM 2.4 (Reynolds). *Let $B \in \mathrm{Bl}(G|D)$ and assume $D \lhd G$. Then Brauer's height zero conjecture holds for B. Furthermore, there is a finite group M and a block $B' \in \mathrm{Bl}(M)$ such that*

(i) *there are bijections $\mathrm{Irr}(B) \to \mathrm{Irr}(B')$ and $\mathrm{IBr}(B) \to \mathrm{IBr}(B')$, where corresponding characters have proportional degrees and equal heights,*

(ii) *D is isomorphic to a defect group P of B', $P \in \mathrm{Syl}_p(M)$ and $P \lhd M$.*

PROOF. This is [**Rey63**, Theorem 6]. □

In this chapter, we use a fundamental concept in the theory of projective representations: the **Schur multiplier**. We refer the reader to [**Isa08**, Section 5A] for the basic definitions and properties.

LEMMA 2.5. *Let G be a finite group. Suppose that $N \lhd G$ and assume $\lambda \in \mathrm{Irr}(N)$ is G-invariant and linear. Let $o(\lambda)$ be the order of λ as an element of $\mathrm{Lin}(N)$. If every Sylow p-subgroup of G/N has trivial Schur multiplier whenever p divides $o(\lambda)$ then λ extends to G.*

PROOF. By [**Isa06**, Theorem 6.26] and [**Isa06**, Theorem 11.7]. □

When dealing with Frobenius complements (see [**Isa08**, Section 6A]) the previous lemma is particularly useful.

LEMMA 2.6. *Let G be a Frobenius complement and let $P \in \mathrm{Syl}_p(G)$. Then P is cyclic or generalized quaternion. In particular P has trivial Schur multiplier.*

PROOF. By [**Isa08**, Theorem 6.10] and [**Isa08**, Theorem 6.11] we have that P is cyclic or generalized quaternion, and by [**Isa08**, Corollary 5.4] and [**Isa08**, Problem 5A.7] its Schur multiplier is trivial. ☐

THEOREM 2.7 (Higgs). *Let G be a finite group, $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. If $\mathrm{Irr}(G|\theta) = \{\alpha, \beta\}$ then $\alpha(1) = \beta(1)$ and G/N is solvable.*

PROOF. See [**Hig88**]. ☐

It is worth mentioning that Theorem 2.7 depends on the Classification of Finite Simple Groups. Before proving a version of Theorem 1.25, we need the following elementary fact.

LEMMA 2.8. *Let $\chi \in \mathrm{Irr}(G)$ and $Z \leqslant \mathbf{Z}(G)$. Then $\chi(1)^2 \leqslant |G : Z|$.*

PROOF. This is [**Isa06**, Corollary 2.30]. ☐

LEMMA 2.9. *Let $Z \lhd G$ and let $\lambda \in \mathrm{Irr}(Z)$ be G-invariant. Suppose that $\lambda^G = e_1\chi_1 + e_2\chi_2$ for some $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ and $e_1, e_2 \in \mathbb{N}$. If p is an odd prime dividing the order of G/Z and $Q/Z \in \mathrm{Syl}_p(G/Z)$, then $\lambda^Q = d\eta$ for some $\eta \in \mathrm{Irr}(Q)$ and $d \in \mathbb{N}$. In particular, Q is nonabelian.*

PROOF. Using that character triple isomorphisms preserve the number $|\mathrm{Irr}(G|\lambda)|$ and the structure of G/N, using Theorem 1.20, there is no loss in assuming Z central. Since $\chi_1(1) = \chi_2(1)$ by Theorem 2.7 and $(\chi_i)_Z = e_i\lambda$ for $i = 1, 2$, we have that $e_1 = \chi_1(1) = \chi_2(1) = e_2$ so $\lambda^G = e_1(\chi_1 + \chi_2)$. Also observe that $|G : Z| = \lambda^G(1) = 2e_1\chi_1(1) = 2\chi_1(1)^2$. Now write $\psi = \chi_1 + \chi_2$. Since ψ vanishes on $G \backslash Z$ we have that $\psi_Q = d\lambda^Q$ where

$$d = \frac{2\chi_1(1)}{|Q : Z|}.$$

If $\eta \in \mathrm{Irr}(Q|\lambda)$ then

$$[\psi_Q, \eta] = [d\lambda^Q, \eta] = d\eta(1) \in \mathbb{Z},$$

so $d^2\eta(1)^2 \in \mathbb{Z}$. Now,

$$d^2\eta(1)^2 = \frac{4\chi_1(1)^2}{|Q : Z|^2}\eta(1)^2 = \frac{2|G : Q|\eta(1)^2}{|Q : Z|} \in \mathbb{Z}$$

and we conclude that $|Q : Z|$ divides $\eta(1)^2$. By Lemma 2.8 we have that $\eta(1)^2 \leqslant |Q : Z|$ so $\eta(1)^2 = |Q : Z|$, and then Lemma 1.24 implies that $\mathrm{Irr}(Q|\lambda) = \{\eta\}$ as wanted. ☐

The following lemma will be used when dealing with groups appearing in Case (iii) of Theorem 1.77. The definition and basic properties of the semilinear group $\Gamma(V)$ of an $\mathbb{F}_p$-vector space V are given before Theorem 1.76.

Recall that a prime t is called a **primitive prime divisor** for (p, a) if t divides $p^a - 1$ but t does not divide $p^j - 1$ for $1 \leqslant j < a$. By a well-known result by Zsigmondy (see [**MW93**, Theorem 6.2] for instance), such a prime always exists except when $a = 6$ and $p = 2$, or $a = 2$ and $p + 1$ is a power of 2.

Lemma 2.10. *Let K be a finite group and let $Z \subseteq \mathbf{Z}(K)$. Suppose that there exist $H/Z \leqslant K/Z$ with $|K : H| = 2$ and $A/Z \leqslant K/Z$ is isomorphic to a subgroup of a semilinear group $\Gamma(V)$, where V is an $\mathbb{F}_p$-vector space ($p \notin \{2, 3\}$ prime) of dimension a, such that $H/Z = A/Z \times B/Z$ and $(A/Z)^g = B/Z$ for every $g \in K \backslash H$. Suppose further that A/Z acts transitively on $V \backslash \{0\}$. Then there is an odd prime divisor t of $p^a - 1$ such that K has abelian Sylow t-subgroups.*

Proof. We use the bar notation, so write $\overline{K} = K/Z$, $\overline{A} = A/Z$, $\overline{B} = B/Z$ and so on. Notice that since $\overline{A}$ acts transitively on $V \backslash \{0\}$, we have that $p^a - 1$ divides $|\overline{A}|$, and so does every prime divisor of $p^a - 1$.

Suppose there is a primitive prime divisor t of $p^a - 1$. Let $\overline{T_1}$ be a Sylow t-subgroup of $\overline{A}$ and let $\overline{z} \in \mathbf{Z}(\overline{T_1})$ of order t. Let $\overline{T_0} = \langle \overline{z} \rangle$. By [**CDPS16**, Lemma 3.7] we have that

$$\overline{T_1} \leqslant \mathbf{C}_{\overline{A}}(\overline{T_0}) = \overline{A} \cap \Gamma_0(V) = \mathbf{F}(\overline{A})$$

where $\mathbf{F}(\overline{A})$ denotes the Fitting subgroup of $\overline{A}$. Since $\Gamma_0(V)$ is cyclic we have that $\overline{T}_1$ is cyclic and normal in $\overline{A}$.

Let $x \in T_1$ such that $\langle \overline{x} \rangle = \overline{T_1}$ and let $\overline{g} \in \overline{K} - \overline{H}$, so $\overline{g}$ is an involution. Let $\overline{y} = \overline{x}^{\overline{g}} \in \overline{B}$ and notice that $\langle \overline{y} \rangle = \overline{T_2}$ is the Sylow t-subgroup of $\overline{B}$. Now $\overline{T} = \overline{T_1} \times \overline{T_2}$, the Sylow t-subgroup of $\overline{H}$, is abelian. Write $\overline{T} = T/Z$, so $T' \subseteq Z$. We note that it is enough to show that T is abelian. Suppose to the contrary that T is not abelian. We have $1 \neq [x, y] \in T' \subseteq Z$ but since $\overline{g}$ is an involution and $x^g = yz_1$, $y^g = xz_2$ for some $z_1, z_2 \in Z$, then

$$[x, y]^g = [x^g, y^g] = [y, x] = [x, y]^{-1} \neq [x, y],$$

a contradiction. Thus we may assume that there does not exist a primitive prime divisor for $p^a - 1$. Since $p \notin \{2, 3\}$ we have that $(p, a) = (2^m - 1, 2)$ for some integer $m > 2$. Then

$$p^2 - 1 = (p + 1)(p - 1) = 2^{m+1}(2^{m-1} - 1)$$

and hence there is an odd prime q dividing $p^2 - 1$. Since $|\overline{A} : \overline{A} \cap \Gamma_0(V)| \leqslant 2$, we have that t divides $|\overline{A} \cap \Gamma_0(V)|$. Hence $\overline{A}$ has a normal and cyclic Sylow t-subgroup. Now, we proceed as above. □

Recall that we denote by $\mathrm{Bl}(G)$ the set of p-blocks of a finite group G, and that if n is a positive integer then n_p denotes the p-part of n.

Lemma 2.11 (Brauer's formula). *Suppose that $x_1, \dots, x_k$ are representatives of the noncentral conjugacy classes of p-elements in G. Let B be a p-block of G. Then*

$$k(B) = |\mathbf{Z}(G)|_p \, l(B) + \sum_{i=1}^{k} \sum_{\substack{b \in \mathrm{Bl}(\mathbf{C}_G(x_i)) \\ b^G = B}} l(b).$$

Proof. See [**Nav98**, Theorem 5.12]. □

Before proceeding with the main result, we recall two standard facts about the Frattini subgroup.

Lemma 2.12. *Let G be a finite group and let $\Phi(G)$ be its Frattini subgroup. If $G/\Phi(G)$ is cyclic then G is cyclic.*

Proof. Write $\langle x\Phi(G)\rangle = G/\Phi(G)$, and assume $\langle x \rangle < G$. Therefore $\langle x \rangle \subseteq M$ for some maximal subgroup M of G. Now $\Phi(G) \subseteq M$ by definition of the Frattini subgroup, so $\langle x \rangle \Phi(G) \subseteq M < G$, and therefore $\langle x\Phi(G)\rangle \subseteq M/\Phi(G) < G/\Phi(G)$, a contradiction. □

Lemma 2.13. *Let G be a finite p-group and let $\Phi(G)$ be its Frattini subgroup. Then $G/\Phi(G)$ is p-elementary abelian.*

Proof. Let $\{N_1, \dots, N_k\}$ denote the set of maximal subgroups of G. Since G is nilpotent, $N_i \lhd G$ and G/N_i is cyclic of order p. Further, the map

$$g \mapsto (N_1 g, \dots, N_k g)$$

is a group homomorphism $G \to G/N_1 \times \cdots \times G/N_k$ with kernel $\Phi(G) = N_1 \cap \cdots \cap N_k$, and the result follows. □

2.3. Theorem A

Now we are ready to prove the main result of this chapter, which will follow from Theorem 2.14. The following proof appeared in [**MRS23**].

Theorem 2.14. *Let G be a finite group, B a p-block of G with $k(B) = 4$. Let D be a defect group of B and assume $D \lhd G$. Then D is isomorphic to $\mathsf{C}_4, \mathsf{C}_2 \times \mathsf{C}_2$ or C_5.*

Proof. We divide the proof in steps.

Step 0. If there exists $1 < N < D$ *with* $N \lhd G$ *then* $p = 2, 3$*, and if* $\overline{B}$ *is a block of* G/N *dominated by* B *then* $k(\overline{B}) \in \{2, 3\}$*.*

Let $N \lhd G$, with $N < D$ and let $\overline{B}$ be a block of G/N dominated by B with defect group D/N (see Theorem 1.49). Then $k(\overline{B}) \leqslant 4$. If $k(\overline{B}) = 1$ then $N = D$ by Theorem 2.3. If $k(\overline{B}) = 4$ then every irreducible character in B contains N in its kernel, and Theorem 1.52 implies $N = 1$. Hence, if $1 < N < D$, it is necessary that $k(\overline{B}) \in \{2, 3\}$. By Theorem 2.3, this forces $p = 2, 3$.

Step 1. We may assume that $D \in \mathrm{Syl}_p(G)$ *and that* D *is* p*-elementary abelian. In particular,* G *is* p*-solvable. Furthermore we may assume* $p \neq 2, 3$*.*

The first part is Theorem 2.4. Let $\Phi(D) \lhd G$ be the Frattini subgroup of D and assume $\Phi(D) > 1$. Let $\overline{B}$ be the unique block of $G/\Phi(D)$ dominated by B. By Step 0 we have $p \in \{2, 3\}$ and $k(\overline{B}) \in \{2, 3\}$. By Theorem 2.3 we have that $|D/\Phi(D)| = p$ and hence D is cyclic by Lemma 2.12. Now the result follows easily by applying Theorem 2.1. Hence we may assume $\Phi(D) = 1$ so D is p-elementary abelian by Lemma 2.13.

Suppose that $p = 2$. Again by Theorem 2.3 we know that $|D| > 2$ and by Theorem 2.2 we obtain that $|D| = 4$ and then $D = \mathsf{C}_2 \times \mathsf{C}_2$, and we are done. Now, suppose that $p = 3$. In this case, by Theorem 2.2 we would have a contradiction.

Step 2. D *is a minimal normal subgroup of* G*.*

If there exists a minimal normal subgroup of G strictly contained in D, then by Step 0 we have that $p = 2, 3$, contradicting Step 1. Hence D is a minimal normal subgroup of G.

Step 3. We may assume that $\mathbf{Z}(G) = \mathbf{O}_{p'}(G)$*.*

Let $N = \mathbf{O}_{p'}(G)$ and let $\lambda \in \mathrm{Irr}(N)$ be such that if $b = \{\lambda\} \in \mathrm{Bl}(N)$ then B covers b. By the Fong–Reynolds correspondence (see Theorem 1.58) we may assume that b is G-invariant, and hence λ is G-invariant. Now (G, N, λ) is an ordinary-modular character triple (see Definition 1.68). By Theorem 1.69 we know that there exists an isomorphic ordinary-modular character triple (H, M, φ) with M a p'-group and φ linear and faithful (in particular, M is central). Moreover, since $G/N \cong H/M$, we have that $M = \mathbf{O}_{p'}(H)$ and that H is also p-solvable. Now let B_1 be a p-block of H covering $b_1 = \{\varphi\}$. By Theorem 1.66 we have that

$$|\mathrm{Irr}(B)| = |\mathrm{Irr}(G|\lambda)| = |\mathrm{Irr}(H|\varphi)| = |\mathrm{Irr}(B_1)|$$

and if D_1 is a defect group of B_1 then $D_1 \in \mathrm{Syl}_p(H)$. We claim that if D_1 is one of $\mathsf{C}_4, \mathsf{C}_5, \mathsf{C}_2 \times \mathsf{C}_2$, then so is D. Indeed, if $Q/M \in \mathrm{Syl}_p(H/M)$ then $Q = D_1 \times M$.

Since $G/N \cong H/M$, we have that

$$D \cong DN/N \cong D_1M/M \cong D_1$$

and D has one of the desired structures. Thus, by working with (H, M, φ), we may assume that λ is linear, faithful, and N is central.

Write $Z = \mathbf{Z}(G)$. Next we prove that $N = Z$. To do so, since $N \subseteq Z$, we just need to show that $|Z|_p = 1$. Assume by way of contradiction that $|Z|_p > 1$. It follows from Brauer's formula (see Lemma 2.11) that $k(B) \geqslant |Z|_p l(B)$. Since $k(B) = 4$ this forces $p = 2, 3$, contradicting Step 1. Thus $N = Z$.

Step 4. Let $\{x_1, \ldots, x_t\}$ be a set of representatives of the G-conjugacy classes of p-elements of G. Then $t = 2$ or $t = 3$.

We may assume that $x_1 = 1$, so $\{x_2, \ldots, x_t\}$ are a set of representatives of the nontrivial G-conjugacy classes of p-elements of G. By Brauer's formula (Lemma 2.11) and Step 3 we have

$$k(B) = l(B) + \sum_{i=2}^{t} \sum_{\substack{b \in \mathrm{Bl}(\mathbf{C}_G(x_i)) \\ b^G = B}} l(b). \tag{2.3.1}$$

The case where $l(B) = 1$ is done in [**Kül84**] (in a wider context), so we may assume $l(B) \geqslant 2$. Since $b^G = B$ for some $b \in \mathrm{Bl}(\mathbf{C}_G(x_i))$ (see Theorem 1.43) we have that either $t = 2$ or $t = 3$, as desired.

By Theorem 1.66 we have that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\lambda)$, where $\lambda \in \mathrm{Irr}(Z)$ is the character from Step 3. From now on we denote by b the unique block of ZD covered by B and notice that $1_D \times \lambda = \hat{\lambda} \in \mathrm{Irr}(b)$ is G-invariant (by Step 3) so b is G-invariant.

Step 5. Suppose that $|D| = p^2$. Then $l(B) = 2$ and G acts on $D \backslash \{1\}$ transitively.

Let $u \in D$, $u \neq 1$. Let $C = \mathbf{C}_G(u)$ and let $b_u \in \mathrm{Bl}(C|D)$ with $b^G = B$ (Theorem 1.43). Since $k(B) = 4$ and $1 < l(B) < 4$, Brauer's formula 2.3.1 forces either $l(b_u) = 1$ or $l(b_u) = 2$.

Suppose that $l(b_u) = 1$. Notice that since $D \lhd G$ and $D \subseteq C$ we have that D is a defect group of b_u by Theorem 1.42. By Theorem 1.49, b_u dominates a unique block $\bar{b}_u \in \mathrm{Bl}(C/\langle u \rangle)$ with defect group $D/\langle u \rangle$, which is cyclic since $|D| = p^2$. By Theorem 1.50 we have $l(\bar{b}_u) = l(b_u)$. By Theorem 2.1, $\bar{b}_u$ has inertial index $l(b_u)$ (see Definition 1.62), and so does b_u. Notice that $b \in \mathrm{Bl}(\mathbf{C}_G(D))$ is a root of b_u (see Definition 1.61), so that $b^C = b_u$. Recall that b is G-invariant because λ is G-invariant. In particular, b is C-invariant and we have $|C : \mathbf{C}_G(D)| = l(b_u) = 1$. Hence the action of G/ZD on D is Frobenius. This implies that the Sylow subgroups of G/ZD have trivial Schur multiplier by Lemma 2.6. By Lemma 2.5, $\hat{\lambda}$ extends to G and so does λ. By Gallagher's theorem (Theorem 1.11),

$|\operatorname{Irr}(G|\lambda)| = |\operatorname{Irr}(G/Z)|$ so G/Z has exactly four conjugacy classes. Using that $p \neq 2, 3$ by Step 1, we get that G/Z is isomorphic to the dihedral group of order 10, and $D \cong \mathsf{C}_5$ as desired.

Then we may assume that $l(b_u) = 2$ and hence, by Brauer's formula 2.3.1 we have that $t = 2$ and $l(B) = 2$.

Step 6. We have that G/Z is an affine primitive permutation group of rank 2 or 3.

By the Schur–Zassenhaus theorem there is $K \leqslant G$ such that $G = KD$ and $K \cap D = 1$. Write $\overline{G} = G/Z$, $\overline{D} = DZ/Z$ and $\overline{K} = K/Z$. We have that $\overline{G}$ acts on $\Omega = \{\overline{K}d \mid d \in D\}$ transitively via right multiplication. Notice that $\overline{K}$ is the stabilizer of the trivial coset in Ω. If $K < L < G$ then $L = KU$ where $U = L \cap D$. Since $D \lhd G$ we have $U \lhd L$ so K normalizes U and then $U \lhd G$, a contradiction with Step 2. Thus $\overline{K}$ is maximal in $\overline{G}$, we have that this action is primitive (see Theorem 1.72) and $\overline{G}$ is a primitive permutation group with socle $\overline{D}$.

Since the action of $\overline{K}$ on Ω has the same number of orbits as the action by conjugation of $\overline{G}$ on $\overline{D}$, by Step 4 it has 2 or 3 orbits. Hence the rank of $\overline{G}$ is 2 or 3, as wanted.

Step 7. We may assume the rank of $\overline{G}$ is 3.

Suppose that $\overline{G}$, and hence $\overline{K}$, has rank 2, so that $t = 2$. By Theorem 1.76 and using that $p \neq 2, 3$ by Step 1, either $\overline{K}$ is isomorphic to a subgroup of the semilinear group $\Gamma(p^d)$ or

$$p^d \in \{5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\},$$

where $|D| = p^d$.

We assume first that $\overline{K}$ is isomorphic to a subgroup of $\Gamma(p^d)$. In this case we know that there exists $Z \leqslant H \lhd K$ with H/Z cyclic (and hence, H abelian) with $|H : Z| = s \mid p^d - 1$ and index $|K : H| = t \mid d$. Since there are just two orbits of p-elements in G, we have that G/ZD acts transitively on $D \backslash \{1\}$ and $p^d - 1 \mid |G : ZD|$.

Recall that $\operatorname{IBr}(B) = \operatorname{IBr}(G|\lambda)$. Since $l(B) \leqslant 3$ we have that $\operatorname{IBr}(HD|\lambda)$ has at most three G-orbits, and each of the orbits is of size at most $|G : HD| = |K : H| = t \leqslant d$. Hence $|\operatorname{IBr}(HD|\lambda)| \leqslant 3d$. Notice that

$$\operatorname{IBr}(HD) = \operatorname{IBr}(HD/D) = \operatorname{Irr}(HD/D)$$

by Theorems 1.32 and 1.29, and hence by Gallagher's correspondence (see Theorem 1.11) we have $\operatorname{IBr}(HD|\lambda) = \operatorname{Irr}(HD|\hat{\lambda})$ where again $\hat{\lambda} = 1_D \times \lambda$ is the canonical extension of λ to ZD. Since $\hat{\lambda}$ is invariant and HD/ZD is cyclic, $\hat{\lambda}$ extends to HD and hence

$$|\mathrm{IBr}(HD|\lambda)| = |HD/ZD| = |H : Z| = s.$$

Then $s \leqslant 3d$. Now,

$$p^d - 1 \leqslant |G : ZD| = st \leqslant 3dt \leqslant 3d^2.$$

Notice that, if $d \in \{1, 2, 3\}$ we would have $p \in \{2, 3\}$ which is a contradiction. Hence we may assume $d > 3$ and $p > 3$. But then we have a contradiction since in this case $3d^2 < p^d - 1$.

Hence we may assume that we are in one of the exceptions listed above, so in particular $|D| = p^2$. By Step 5 we have that $l(B) = 2$ and G acts transitively on $D\backslash\{1\}$. Let $u \in D\backslash\{1\}$ and let $b_u \in \mathrm{Bl}(\mathbf{C}_G(u)|D)$ inducing B. By the argument in Step 5, b_u has inertial index 2, so $|\mathbf{C}_G(u) : \mathbf{C}_G(D)| = 2$, and $l(b_u) = 2$. Since G acts transitively on $D\backslash\{1\}$, we have that

$$|G : ZD| = |G : \mathbf{C}_G(D)| = 2|G : \mathbf{C}_G(u)| = 2(p^2 - 1).$$

Since we are dealing with the case that G/ZD is not a subgroup of the semilinear group, using Table 1.3.3 and the fact that $|G : ZD| = 2(p^2 - 1)$, we have that $G/Z = \mathrm{PrimitiveGroup}(r, i)$ with

$$(r, i) \in \{(5^2, 18), (11^2, 42), (29^2, 110)\}$$

in [**GAP**]. These groups have a normal subgroup $N/Z \lhd G/Z$ of index 2 and such that for $q \neq p$, the Sylow q-subgroups of N/Z are either cyclic or quaternion groups (in any case they have trivial Schur multiplier). In particular, $\hat{\lambda}$ extends to N by Lemma 2.5. Now $|\mathrm{Irr}(G|\hat{\lambda})| = 4$ and the G-orbits of $\mathrm{Irr}(N|\hat{\lambda})$ have size at most $|G : N| = 2$, so we have $|\mathrm{Irr}(N|\hat{\lambda})| \leqslant 8$ but by Gallagher's theorem

$$|\mathrm{Irr}(N|\hat{\lambda})| = |\mathrm{Irr}(N/ZD)| = k(N/ZD).$$

If $p \neq 5$ we have $k(N/ZD) > 8$ so these cases are impossible. If $p = 5$ then $N/ZD \cong \mathrm{SL}(2, 3)$ and $\mathrm{Irr}(N|\hat{\lambda})$ contains three characters of degree 1, three characters of degree 2 and a character of degree 3 by Gallagher's theorem. Since $|G : N| = 2$, the G-orbits in $\mathrm{Irr}(N|\hat{\lambda})$ have size at most 2. This yields at least 5 G-orbits in $\mathrm{Irr}(N|\hat{\lambda})$, which is a contradiction.

Final Step

By Step 7 the rank of $\overline{G}$ is 3. Notice that in this case we have $|\mathrm{Irr}(K|\lambda)| = l(B) = 2$. By Theorem 2.7, K/Z is solvable, and so is G/Z. By Step 5 we may assume that $d \neq 2$. By Theorem 1.77 (and taking into account that $\overline{K} = K/Z$ is a p'-group, $p \neq 2, 3$ and $d \neq 2$) we are in one of the following situations:

Case (i): $\overline{K} \leqslant \Gamma(p^d)$.

In this case, we have a subgroup $Z \leqslant H \leqslant K$ with H/Z cyclic (and hence, H abelian) with $|H : Z| = s \mid p^d - 1$ and index $|K : H| = t \mid d$. Since $|\mathrm{Irr}(K|\lambda)| = 2$ and $|\mathrm{Irr}(H|\lambda)| = |H/Z| = s$ we obtain that

$$s = |\mathrm{Irr}(H|\lambda)| \leqslant 2t \leqslant 2d.$$

Now, G/ZD acts on D in two nontrivial conjugacy classes. Hence

$$p^d - 1 \leqslant 2|G : DZ| = 2st \leqslant 4d^2.$$

If $d = 1$ we obtain $p = 5$ and we are done. In the other case, we have $d > 3$ and $p > 3$, but this is a contradiction.

Case (ii): $\overline{K}$ imprimitive.

We have that K/Z is an imprimitive linear group with imprimitivity spaces V_1, V_2, where $D = V_1 \times V_2$ and $|V_1| = |V_2| = p^a$. As in the comment after Theorem 1.77, we may write $\overline{H} = \mathbf{N}_{\overline{K}}(V_1) = \mathbf{N}_{\overline{K}}(V_2)$. Then $\overline{K} \cong (\overline{A} \times \overline{B}) \rtimes \mathsf{C}_2$ where $\overline{H} \cong \overline{A} \times \overline{B}$ and $\overline{A} \cong \overline{H}/\mathbf{C}_{\overline{H}}(V_1) \cong \overline{B}$. By part (iii) of Theorem 1.77, we have that $\overline{A}$ is a solvable linear transitive group on $V_1 \backslash \{0\}$, so we are in cases (i) or (ii) of Theorem 1.76 and we have that either $\overline{A} \leqslant \Gamma(p^a)$ or

$$p^a \in \{5^2, 7^2, 11^2, 23^2\}$$

(recall that $p \neq 3$ by Step 1). Since $|\,\mathrm{Irr}(K|\lambda)| = 2$, by Lemmas 2.9 and 2.10 and Step 1 we are in the latter situation. Notice that $|\mathrm{Irr}(K|\lambda)| = 2$ forces $|\mathrm{Irr}(H|\lambda)| \in \{1, 4\}$.

Since we are in case (ii) of Theorem 1.76 we have that $\overline{A}$ contains a normal subgroup $\overline{N}$ isomorphic to $\mathrm{SL}(2, 3)$ (this can be checked in [**GAP**] using Table 1.3.3). Write $N/Z = \overline{N}$. Then λ extends to N by Lemma 2.5 and by Gallagher's theorem the degrees of the irreducible characters of $\mathrm{Irr}(N|\lambda)$ are $\{1, 2, 3\}$, where the degrees 1 and 2 appear three times each and the degree 3 appears once. Let $\xi, \gamma, \delta \in \mathrm{Irr}(N|\lambda)$ with $\xi(1) = 1$, $\gamma(1) = 2$ and $\delta(1) = 3$. Observe that δ is H-invariant.

Suppose first that $|\mathrm{Irr}(H|\delta)| = 1$. Then δ is fully ramified in H and hence, $|H/N|$ is a square, but this is not possible since in all the possible cases $|H/N|_3 \in \{3, 27\}$. Therefore $|\mathrm{Irr}(H|\delta)| > 1$. Then $|\mathrm{Irr}(H|\lambda)| = 4$ and by Clifford's theorem we deduce that ξ and γ lie under a unique irreducible character in $\mathrm{Irr}(H|\lambda)$, so they are fully ramified in their stabilizers H_ξ and H_γ. Again, by Clifford's theorem it follows that $|H : H_\xi| = |H : H_\gamma| = 3$ and hence $|H : N|_2 = |H_\xi : N|_2$. Now $|H : N| = |\overline{A}|^2/|\overline{N}|$, and since $|\overline{N}|_2 = 8$ it follows that $|H : N|_2$ is never a square. We conclude that $1 < |H_\xi : N|_2$ is not a square, so $|H_\xi : N|$ is not a square, yielding a contradiction with the fact that ξ is fully ramified in H_ξ.

Case (iii): $p^d = 7^4$.

In this case G/Z is a subgroup of one of the groups of case (ii)(b) of Theorem 1.77, and has degree 7^4. The primitive permutation groups of degree 7^4 are classified (see the main result of [**Ron05**]) and this guarantees that the [**GAP**] library of such groups is complete. By using [**GAP**] we obtain that G/Z is one of the groups $G/Z = \mathrm{PrimitiveGroup}(7^4, i)$ where $i \in \{774, 775\}$.

In the case $i = 774$ we can find a normal subgroup $Z \subseteq N \lhd G$ with $|G : N| = 4$ and all Sylow q-subgroups of N, for $q \neq p$, are cyclic, so they have trivial Schur multiplier. In this case we have that λ extends to N by Lemma 2.5. Then λ extends to $N \cap K$ and

$$|\mathrm{Irr}(N \cap K|\lambda)| = |\mathrm{Irr}(N \cap K/Z)| = |\mathrm{Irr}(N/DZ)| = 192.$$

However, since $|\mathrm{Irr}(K|\lambda)| = l(B) = 2$ and using $|K : K \cap N| = |G : N| = 4$ we have that $|\mathrm{Irr}(N \cap K|\lambda)| \leqslant 8$, a contradiction.

In the case $i = 775$ we can find a normal subgroup N/Z of order $3 \cdot 7^4$. Again, λ extends to N and if c is the unique block of N covered by B we have that $\mathrm{Irr}(c) = \mathrm{Irr}(N|\lambda)$. Then there are at most 4 orbits in $\mathrm{Irr}(N|\lambda)$ of size dividing 640 and the sum of the sizes is 803. By an easy counting argument we exclude the possibility that there are either 2 or 3 orbits, and if there are 4 orbits, then they have sizes $\{1, 2, 160, 640\}$, in particular there is an orbit of size 1. Let θ be the G-invariant irreducible character of c. Now, since $\mathbf{C}_G(D) \subseteq N$ we have that B is the unique block of G covering c (see Theorem 1.59) and hence there is just one irreducible character in G lying over θ (the one lying in B). This means that there exists $\theta \in \mathrm{Irr}(N|\lambda)$ that is fully ramified in G and by Lemma 1.26 we have that there is no self-centralizing cyclic subgroup in G/N. However if $P/N \in \mathrm{Syl}_5(G/N)$ and $C/N = \mathbf{C}_{G/N}(P/N)$ then C/N is cyclic of order 10 and $\mathbf{C}_{G/N}(C/N) = C/N$. □

Corollary 2.15. *Let G be a finite group, B a p-block of G with $k(B) = 4$ and D a defect group of B. Assume $k_0(B) = k_0(b)$ where $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ is the Brauer correspondent of B in $\mathbf{N}_G(D)$. Then D is isomorphic to one of $\mathsf{C}_4, \mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_5$.*

Proof. By Theorem 1.49, b dominates some block

$$\bar{b} \in \mathrm{Bl}(\mathbf{N}_G(D)/\Phi(D))$$

with defect group $D/\Phi(D)$. We have

$$k(\bar{b}) = k_0(\bar{b}) \leqslant k_0(b)$$

by Theorem 2.4. We explore all the possibilities for $k(\bar{b})$ using Theorem 2.3. If $k(\bar{b}) = 1$ then $D/\Phi(D) = 1$, which is impossible. If $k(\bar{b}) = 2$ then we have $|D/\Phi(D)| = 2$ and then D is cyclic and $p = 2$. Using Theorem 2.1 we conclude that $|D| = 4$ and we are done. If $k(\bar{b}) = 3$ then we have $D/\Phi(D)$ is cyclic of order 3, so D is a cyclic 3-group. Then using Theorem 2.1 we get a contradiction. Finally if $k(\bar{b}) = 4$ then $D/\Phi(D)$ is one of the groups from Theorem 2.14, and it must be elementary abelian. If $|D/\Phi(D)| = 5$ then D is cyclic and then by

Theorem 2.1 we have $D \cong \mathsf{C}_5$. Otherwise, $D/\Phi(D) \cong \mathsf{C}_2 \times \mathsf{C}_2$ and $p = 2$. Since $k(\bar{b}) \leqslant k_0(B)$ we have $k(B) = k_0(B)$ and we apply Theorem 2.2 to conclude that $|D| = 4$. $\square$

CHAPTER 3

Character degrees in blocks and defect groups

3.1. Introduction

One of the main themes in the representation theory of finite groups is to relate global and local invariants. For example, for a prime p, Brauer's Problem 12 asks what can be said about the Sylow p-subgroups of G from the makeup of the set of irreducible characters $\mathrm{Irr}(G)$. More generally, the problem of determining structural properties of a defect group of a p-block B of G based on the set $\mathrm{Irr}(B)$ of irreducible characters of B has been an important topic of study in the area. The proof of one direction of the Brauer Height Zero Conjecture in [**KM13**] and the recent proof of the remaining direction [**MNST22**] have been major breakthroughs in this line of investigation. Other recent results studying the p-structural properties of the defect groups in terms of properties of the irreducible characters in the block can be found in [**FLLMZ19**], [**GRSS20**] and [**NRSV21**]. The results in this chapter are a contribution to this lively area of research.

Let B be a Brauer p-block of a finite group G, with defect group D. We write $\mathrm{cd}(B) = \{\chi(1) \mid \chi \in \mathrm{Irr}(B)\}$ for the set of the degrees of the complex irreducible characters in B and $\mathrm{dl}(D)$ for the derived length of the solvable group D. As mentioned in the introduction, this chapter's main motivation is the following question.

Question B (Navarro). *Is it true that* $\mathrm{dl}(D) \leqslant |\mathrm{cd}(B)|$*?*

Question B has a positive answer if $|\mathrm{cd}(B)| = 1$. In fact, a theorem of T. Okuyama and Y. Tsushima [**OT83**] shows that in this case the defect group D is abelian. Similarly, Navarro's prediction is confirmed whenever D is normal in G by [**Rey63**, Theorem 8] and whenever the block B is nilpotent by [**BP80**, Theorem 1.2]. On the other hand, at the time of this writing, the case where $|\mathrm{cd}(B)| = 2$ remains an open problem, although it would follow from the recently proved Brauer Height Zero Conjecture, along with a conjecture of G. Malle and G. Navarro [**MN11**]. Specializing the question to the principal block $B = B_0(G)$, we obtain the following result.

Theorem C. *Let p be a prime, let G be a finite group with Sylow p-subgroup P, and let $B_0(G)$ be the principal p-block of G. If $|\mathrm{cd}(B_0(G))| \leqslant 2$, then G is p-solvable,* $\mathrm{dl}(P) \leqslant |\mathrm{cd}(B_0(G))|$ *and $G/\mathbf{O}_{p'}(G)$ has normal Sylow p-subgroups.*

The group $\mathfrak{A}_5$ has a principal 2-block with character degrees $\{1, 3, 5\}$, and this shows that when exploring this problem for principal blocks with more than 2 character degrees, p-solvability is no longer guaranteed. However, we can still prove that Navarro's question has a positive answer for principal blocks with three character degrees.

Theorem D. *Let p be a prime, let G be a finite group with Sylow p-subgroup P, and let $B_0(G)$ be the principal p-block of G. If $|\mathrm{cd}(B_0(G))| \leqslant 3$, then* $\mathrm{dl}(P) \leqslant |\mathrm{cd}(B_0(G))|$.

It was proved in [**Man00**] that if a p-group P has derived length $\mathrm{dl}(P) > 4$ then $|P| > p^{21}$. The following is an immediate corollary.

Corollary E. *Let p be a prime, let G be a finite group with Sylow p-subgroup P, and let $B_0(G)$ be the principal p-block of G. If* $\mathrm{dl}(P) \leqslant 4$ *then* $\mathrm{dl}(P) \leqslant |\mathrm{cd}(B_0(G))|$. *In particular, Question B has a positive answer for $B_0(G)$ if $|G|_p \leqslant p^{21}$.*

Both Theorems C and D are proved via a reduction to finite simple groups. At the end of this Chapter we prove the following block version of a well-known theorem of J. G. Thompson.

Theorem F. *Let G be a finite group, p and q distinct primes dividing $|G|$ and let $B_0(G)$ be the principal p-block of G. Then* $\mathrm{cd}(B_0(G))$ *consists only of q-powers if and only if $G/\mathbf{O}_{p'}(G)$ has an abelian normal q-complement.*

This result does not depend on the Classification of finite simple groups, and it has a more elementary nature.

Theorems C and F appeared in [**M21**] and Theorem D and Corollary E appeared in [**GMS22**].

3.2. The reduction of Theorem C

Let us record the following elementary well-known situation.

Lemma 3.1. *Let G be a finite group. Then all irreducible characters of $B_0(G)$ are linear if and only if $G/\mathbf{O}_{p'}(G)$ is abelian.*

Proof. This follows from Theorem 1.52 using that $G' \subseteq \ker(\lambda)$ for all linear characters λ of G. □

If S is a group, then $S/\mathbf{Z}(S)$ is naturally embedded as a normal subgroup in $A = \mathrm{Aut}(S)$. We will need the following result on almost simple groups, whose proof will be deduced from Propositions 3.15, 3.18 and 3.19.

Theorem 3.2. *Suppose that S is a nonabelian simple group of order divisible by p with abelian Sylow p-subgroups. Let $S \leqslant H \leqslant A = \mathrm{Aut}(S)$. Then there exist $\alpha, \beta \in \mathrm{Irr}(H) \backslash \mathrm{Irr}(H/S)$ in the principal block of H such that $\alpha(1) \neq \beta(1)$.*

Before proceeding with the proof of the main result of this section, we recall the Isaacs–Smith theorem.

Theorem 3.3. *Suppose that G is a finite group and that p divides the degree of every nonlinear irreducible character in $B_0(G)$. Then G has a normal p-complement.*

Proof. This is the main result of [**IS76**]. □

We will also make use of the recent proof of Brauer's height zero conjecture for principal blocks.

Theorem 3.4 (Malle–Navarro). *Assume every character in $B_0(G)$ has height zero and let $P \in \mathrm{Syl}_p(G)$. Then P is abelian.*

Proof. This is the main result of [**MN21**]. □

Assuming that we have proved Theorem 3.2, we can prove our main result.

Theorem 3.5. *Suppose that G is a finite group, and assume that $\mathrm{cd}(B_0(G)) = \{1, m\}$. Then G is p-solvable.*

Proof. By Lemma 3.1 we may assume $m > 1$, and by Theorem 3.3 we may assume that m is not divisible by p, and then all characters in $B_0(G)$ have height zero. Then, by Theorem 3.4, the Sylow p-subgroups of G are abelian. Arguing by induction and using that $B_0(G/M) \subseteq B_0(G)$ for any $M \lhd G$ (see Corollary 1.53), we may assume that G has a unique minimal normal subgroup N. Also N is nonabelian and has order divisible by p. Therefore there exists $S \subseteq N$ simple such that $N = S^{x_1} \times \cdots \times S^{x_t}$ for some $x_i \in G$, and if $H = \mathbf{N}_G(S)$, then

$$G = \coprod_{j=1}^{t} H x_j$$

is a disjoint union. Let $C = \mathbf{C}_G(S)$ and $S_i = S^{x_i}$. We set $S = S_1$.

Now, $S \cong SC/C \leqslant H/C \leqslant A = \mathrm{Aut}(S)$. By Theorem 3.2, there exist α, β in the principal block of H/C such that $\alpha(1) \neq \beta(1)$, and α and β do not lie over the trivial character of S. Let $1 \neq \nu \in \mathrm{Irr}(S)$ be under α. Now, let $\eta = \nu \times 1_{S_2} \times \cdots \times 1_{S_t} \in \mathrm{Irr}(N)$. Notice that $G_\eta = H_\nu \subseteq H$, using that ν is nontrivial. Since $S_2 \cdots S_t \subseteq C \subseteq \ker(\alpha)$, we have that $\alpha \in \mathrm{Irr}(H|\eta)$. By the Clifford correspondence (Theorem 1.10) there is some $\psi \in \mathrm{Irr}(H_\nu)$ such that $\psi^H = \alpha$. Using that $G_\eta = H_\nu$ and applying again the Clifford correspondence we have that $\psi^G = (\psi^H)^G = \alpha^G \in \mathrm{Irr}(G)$. With the same argument $\beta^G \in \mathrm{Irr}(G)$. Therefore α^G and β^G are irreducible, and have degrees $t\alpha(1)$ and $t\beta(1)$, which are different. By Theorem 1.46, we have that $B_0(H)^G$ is defined and contains α^G and β^G. By Brauer's third main theorem (see Theorem 1.47), $B_0(H)^G = B_0(G)$. Hence, $t\alpha(1) = t\beta(1)$, but this is impossible. $\square$

Notice that Theorem 3.5 can be seen as a principal block version of the Isaacs–Passman Theorem.

Theorem 3.6 (Isaacs–Passman). *Let G be a finite group and let m be a positive integer. Assume that the set of degrees of irreducible characters of G is contained in $\{1, m\}$. Then G is metabelian.*

Proof. See [**Isa06**, Corollary 12.6]. $\square$

Using the Isaacs–Passman theorem we obtain the following corollary.

Corollary 3.7. *Suppose that G is a finite group with $\mathrm{cd}(B_0(G)) = \{1, m\}$. Then $G/\mathbf{O}_{p'}(G)$ is metabelian and has a normal Sylow p-subgroup.*

Proof. By Theorem 3.5 we have that G is p-solvable. By Theorem 1.66, $\mathrm{Irr}(B_0(G)) = \mathrm{Irr}(G/\mathbf{O}_{p'}(G))$. Then if $\mathrm{cd}(B_0(G)) = \{1, m\}$ we have

$$\mathrm{cd}(G/\mathbf{O}_{p'}(G)) = \{1, m\}.$$

By Theorem 3.6 we have that $G/\mathbf{O}_{p'}(G)$ is metabelian, so there exists some $K/\mathbf{O}_{p'}(G) \lhd G/\mathbf{O}_{p'}(G)$ abelian such that G/K is also abelian. For a different prime $q \neq p$, if $Q/\mathbf{O}_{p'}(G) \in \mathrm{Syl}_q(K/\mathbf{O}_{p'}(G))$ we have $Q/\mathbf{O}_{p'}(G) \lhd K/\mathbf{O}_{p'}(G)$ and $Q/\mathbf{O}_{p'}(G) \lhd G/\mathbf{O}_{p'}(G)$, but then $Q \subseteq \mathbf{O}_{p'}(G)$ so $K/\mathbf{O}_{p'}(G)$ must be contained in some $P/\mathbf{O}_{p'}(G) \in \mathrm{Syl}_p(G/\mathbf{O}_{p'}(G))$. Since G/K is abelian we have that $P/\mathbf{O}_{p'}(G) \lhd G/\mathbf{O}_{p'}(G)$, as desired. $\square$

3.3. Almost simple groups for Theorem C

We begin this section with an easy result.

Lemma 3.8. *Let G be finite group, let p be a prime dividing the order of G and let $B_0(G)$ be the principal p-block of G. For any $m \in \mathbb{N}, m \neq 1$ there exists a nontrivial $\chi \in \mathrm{Irr}(B_0(G))$ such that $\chi(1)$ is not divisible by m.*

Proof. If G has a nontrivial linear character in the principal p-block then we choose χ to be this character. Otherwise, assume by way of contradiction that any $\chi \in \mathrm{Irr}(B_0(G))\backslash\{1_G\}$ has degree divisible by m, so $\frac{\chi(1)}{m}$ is an integer. Let $g \in G$ be p-singular. By weak block orthogonality (see Theorem 1.36) we have

$$0 = \sum_{\chi \in \mathrm{Irr}(B_0(G))} \chi(1)\chi(g) = 1 + \sum_{1_G \neq \chi \in \mathrm{Irr}(B_0(G))} \chi(1)\chi(g) =$$

$$= 1 + m \sum_{1_G \neq \chi \in \mathrm{Irr}(B_0(G))} \frac{\chi(1)}{m}\chi(g)$$

so

$$-\frac{1}{m} = \sum_{1_G \neq \chi \in \mathrm{Irr}(B_0(G))} \frac{\chi(1)}{m}\chi(g)$$

and by the previous comment, the right hand side is an algebraic integer, so $-\frac{1}{m}$ is an algebraic integer, a contradiction. □

In what follows we use a well-known fact about finite simple groups of Lie type in characteristic p. If S is such a group and $S \notin \{\mathrm{PSp}_4(2)', \mathrm{G}_2(2)', {}^2\mathrm{G}_2(3)', {}^2\mathrm{F}_4(2)'\}$ then there is a character $\mathsf{St}_S \in \mathrm{Irr}(S)$ known as the **Steinberg character**. It is a character of degree $\mathsf{St}_S(1) = |S|_p$, and it is the unique character of such a degree in $\mathrm{Irr}(S)$, so it is $\mathrm{Aut}(S)$-invariant. In [**Fei93**], Feit proved the following classical result.

Theorem 3.9 (Feit). *The character St_S extends to $\mathrm{Aut}(S)$.*

Proof. This is the main result of [**Fei93**]. □

The Steinberg character is what is known in the theory of representations of groups of Lie type as a **unipotent** character (see [**GM20**, Section 2.3]). These were proven to extend to their stabilizers in $\mathrm{Aut}(S)$ in [**Mal08**]. Notice that we can obtain Feit's theorem as a consequence of this result.

Theorem 3.10. *Let S be a simple group of Lie type defined over a field of characteristic p, and assume $S \notin \{\mathrm{PSp}_4(2)', \mathrm{G}_2(2)', {}^2\mathrm{G}_2(3)', {}^2\mathrm{F}_4(2)'\}$. Then the p-blocks of S are the principal block and the block that contains only the Steinberg character St_S.*

Proof. This is [**Cab18**, Theorem 3.3]. □

3.3.1. The case $p = 2$. In this section all blocks are 2-blocks and Brauer characters are 2-Brauer characters. We begin by recalling a classical theorem of Fong on the degree of real valued 2-Brauer characters.

Theorem 3.11 (Fong). *Suppose that $\varphi \in \mathrm{IBr}(G)$ is real valued. If $\varphi(1)$ is odd then $\varphi = 1_{G^0}$.*

Proof. This is [**Nav98**, Theorem 2.30]. □

The following result is part of [**B71**, Corollary 8B].

Lemma 3.12. *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$ be a real-valued character of odd degree. Then $\chi \in \mathrm{Irr}(B_0(G))$.*

Proof. Let χ^0 be the restriction of χ to the 2-regular elements of G. Notice that if $\phi \in \mathrm{IBr}(G)$ is a constituent of χ^0 and $\phi \neq \overline{\phi}$ then $\overline{\phi}$ is also a constituent of χ^0. Also, by Theorem 3.11 if $\phi \neq 1_G$ is a real-valued constituent of χ^0 then $\phi(1)$ is even. Since $\chi(1)$ is odd, χ^0 must contain the trivial character, so $\chi \in \mathrm{Irr}(B_0(G))$. □

Theorem 3.13. *Let G be a finite group with abelian Sylow 2-subgroups. Then $\mathbf{O}^{2'}(G/\mathbf{O}_{2'}(G))$ is the direct product of a 2-group and simple groups of one of the following types:*

(i) $\mathrm{PSL}_2(2^n), n > 1$,

(ii) $\mathrm{PSL}_2(q)$ $q \equiv 3$ *or* $5 \bmod 8$, $q > 3$,

(iii) *The Ree groups* ${}^2\mathrm{G}_2(3^{2n+1})$ *with* $n \geqslant 1$,

(iv) *The Janko group* J_1.

Proof. This is the main result of [**Wal69**]. □

Lemma 3.14. *Let $S = \mathrm{PSL}_2(q) = \mathrm{SL}_2(q)$ for $q = 2^n$, $n \geqslant 2$. Then*

(i) *if $q - 1$ is divisible by 3, S has a unique rational character of degree $q + 1$,*

(ii) *if $q + 1$ is divisible by 3, S has a unique rational character of degree $q - 1$.*

Proof. With the notation of Theorem 2, Chapter 38 of [**Dor72**], in the first case this character is $\chi_{\frac{q-1}{3}}$ and in the second case it is $\theta_{\frac{q+1}{3}}$.

To see uniqueness, notice that in the first case, in the class denoted by (a), χ_j takes the value $\zeta^j + \zeta^{-j}$ for $\zeta = e^{\frac{2\pi i}{q-1}}$, and $j \in \{1, \dots, \frac{q-2}{2}\}$. It is easy to see that $j = \frac{q-1}{3}$ is the unique exponent such that $\zeta^j + \zeta^{-j}$ is an integer. The second case is done identically. □

Proposition 3.15. *Theorem 3.2 holds for $p = 2$.*

Proof. We have that S is isomorphic to one of the groups in Theorem 3.13. Let $A = \mathrm{Aut}(S)$ and $S \leqslant T \leqslant A$.

(i) If $S \cong \mathrm{SL}_2(q)$ for a power q of 2, then A/S is cyclic (see [**Ste60**, Section 3]). By Theorem 3.10, all characters of odd degree of S lie in $B_0(S)$. If $q-1$ is divisible by 3, by Lemma 3.14 there exists an A-invariant character $\alpha \in \mathrm{Irr}(B_0(S))$ of degree $q+1$. By Theorem 1.12 and the Gallagher correspondence (Theorem 1.11), all elements of $\mathrm{Irr}(T|\alpha)$ are extensions of α. By Theorem 1.56 there exists an extension $\hat{\alpha} \in \mathrm{Irr}(B_0(T))$. However, $B_0(S)$ also contains a character γ of degree $q-1$, and by Theorem 1.56 there exists some $\beta \in \mathrm{Irr}(B_0(T))$ over γ, so $\gamma(1)$ divides $\beta(1)$ by Clifford's theorem (Theorem 1.9). Then α, β are characters of different degrees that lie in $B_0(T)$ and which do not contain S in their kernel. The case where $q+1$ is divisible by 3 is done identically.

(ii) If $S \cong \mathrm{PSL}_2(q), q \equiv 3, 5 \bmod 8$, then A contains a subgroup $N \cong \mathrm{PGL}_2(q)$ such that $S \leqslant N$, and N/S and A/N are cyclic. The Steinberg character St_S and St_N is a real-valued character of degree q so by Lemma 3.12, it is in the principal block of S and N respectively. Now let $S \leqslant T \leqslant A$ and then $T \cap N \in \{S, T\}$. Then $\mathsf{St}_{T\cap N} \in \mathrm{Irr}(B_0(T \cap N))$. Furthermore $TN/N \cong N/T \cap N$ is cyclic, so by Theorem 1.12 and Theorem 1.56, there is an extension α of $\mathsf{St}_{T\cap N}$ in $B_0(T)$. By Lemma 3.8, $B_0(S)$ contains a character γ of q'-degree, and by Theorem 1.56 there exists some $\beta \in \mathrm{Irr}(B_0(T))$ over γ. By Clifford's theorem (Theorem 1.9) $\gamma(1)$ divides $\beta(1)$ so $\beta(1) \neq \alpha(1)$ and this case is done.

(iii) If $S \cong {}^2\mathrm{G}_2(q)$, for $q = 3^{2n+1}$ then A/S is cyclic (see [**GLS94**, Theorem 2.5.12]). By [**LM80**, Theorem 3.9], S has a unique character γ of degree q^3 in the principal block. Let $S \leqslant T \leqslant A$, we have that γ is T-invariant and by Theorem 1.12 and Theorem 1.56, γ extends to $\alpha \in \mathrm{Irr}(B_0(T))$. For the character β we argue identically as in the previous case.

(iv) If $S \cong \mathrm{J}_1$ then $A = S$ and we may choose the characters of degree 77 and 133 which lie in $B_0(S)$.

This ends the proof. □

3.3.2. The case $p = 3$. In this section all blocks are 3-blocks.

PROPOSITION 3.16. *The list of nonabelian simple groups with abelian Sylow 3-subgroups is the following:*

(i) $\mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7, \mathfrak{A}_8, \mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{HS}, \mathrm{O'N}, \mathrm{J}_1$,

(ii) $\mathrm{PSL}_2(q)$ *with* $3 \nmid q$,

(iii) $\mathrm{PSL}_3(q)$ *for* $3 \mid (q+1)$,

(iv) $\mathrm{PSU}_3(q)$ *for* $3 \mid (q-1)$,

(v) $\mathrm{PSL}_3(q)$ *for a power* q *of a prime with* $3 \mid (q-1)$ *and* $9 \nmid (q-1)$,

(vi) $\mathrm{PSU}_3(q)$ *for a power* q *of a prime with* $3 \mid (q+1)$ *and* $9 \nmid (q+1)$,

(vii) $\mathrm{PSp}_4(q)$ *for a power* q *of a prime with* $3 \mid (q-1)$,

(viii) $\mathrm{PSp}_4(q)$ *for a power* q *of a prime with* $q > 2$ *and* $3 \mid (q+1)$,

(ix) $\mathrm{PSL}_4(q)$ *for a power* q *of a prime with* $q > 2$ *and* $3 \mid (q+1)$,

(x) $\mathrm{PSU}_4(q)$ *for a power* q *of a prime with* $3 \mid (q-1)$,

(xi) $\mathrm{PSL}_5(q)$ *for a power* q *of a prime with* $3 \mid (q+1)$,

(xii) $\mathrm{PSU}_5(q)$ *for a power* q *of a prime with* $3 \mid (q-1)$,

(xiii) $\mathrm{PSL}_2(3^n)$, $n \geqslant 3$.

PROOF. This follows from the main result of [**SZ16**]. □

PROPOSITION 3.17. *Suppose that* S *is a simple group with abelian Sylow* 3*-subgroups,* $S \not\cong \mathrm{PSL}_2(3^n)$*. Then there exists* $1 \neq \alpha \in \mathrm{Irr}(B_0(S))$ *that has an extension* $\hat{\alpha} \in \mathrm{Irr}(B_0(T))$ *for all* $S \leqslant T \leqslant \mathrm{Aut}(S)$ *and* $\beta \in \mathrm{Irr}(B_0(S))$ *such that* $\beta(1)$ *does not divide* $\alpha(1)$*.*

PROOF. Write $A = \mathrm{Aut}(S)$. All groups of Proposition 3.16(i) excluding $\mathfrak{A}_6 \cong \mathrm{PSL}_2(3^2)$ can be checked with [**GAP**]. For the remaining cases we will prove the claim first for $T = A$.

If S is isomorphic to the groups of the form $\mathrm{PSL}_n(q), \mathrm{PSU}_n(q), \mathrm{PSp}_n(q)$, then A contains a subgroup N of inner-diagonal automorphisms, isomorphic to $\mathrm{PGL}_n(q)$, $\mathrm{PGU}_n(q)$ or $\mathrm{PCSp}_n(q)$ respectively, with $N \lhd A$ and A/N is abelian (see [**GLS94**, Theorem 2.5.12] and [**Ste60**, Section 3]), although the notation we use appears for example in [**LS13**].

By [**LS13**, Theorem 3.1] in cases (ii)-(xii) of Proposition 3.16, the Steinberg character is in the principal block of S and N. By Theorem 3.9 we have that St_N extends to A, so by Gallagher's Theorem (see Theorem 1.11) $\mathrm{Irr}(A|\mathsf{St}_N) = \{\mu\mathsf{St}_N \mid \mu \in \mathrm{Irr}(A/N)\}$. Hence, all characters in $\mathrm{Irr}(A|\mathsf{St}_N)$ are extensions of St_N, and at least one of these characters lies in $B_0(A)$ by Theorem 1.56. Finally, by Lemma 3.8, $B_0(S)$ contains a nonlinear character of q'-degree, so this case is done.

Finally we proceed to prove the claim for any arbitrary subgroup $S \leqslant T \leqslant \mathrm{Aut}(S)$. Then T contains a normal subgroup $M = T \cap N \lhd T$ where $S \lhd M$, M/S is cyclic and T/M is abelian. In any of the previous cases we found a character α that extended to $\hat{\alpha} \in \mathrm{Irr}(B_0(A))$ and $\hat{\alpha}_N \in \mathrm{Irr}(B_0(N))$. Now $M \lhd N$ so $B_0(N)$ covers $B_0(M)$, and then $\hat{\alpha}_M \in \mathrm{Irr}(B_0(M))$. Now $\hat{\alpha}_T \in \mathrm{Irr}(T)$ and then $\hat{\alpha}_M$ extends to T as well. By Gallagher's Theorem (Theorem 1.11) $\mathrm{Irr}(T|\hat{\alpha}_M) = \{\mu\hat{\alpha}_M \mid \mu \in \mathrm{Irr}(T/M)\}$ and since T/M is abelian, every element of $\mathrm{Irr}(T|\hat{\alpha}_M)$ is an extension of $\hat{\alpha}_M$. By Theorem 1.56 at least one of these characters lies in $B_0(T)$, so we are done. □

Proposition 3.18. *Theorem 3.2 holds for $p = 3$.*

Proof. Let S be a simple group of order divisible by 3 with abelian Sylow 3-subgroups. If $S = \mathfrak{A}_6$ this is easy to check with [**GAP**]. Otherwise, assume $S \not\cong \mathrm{PSL}_2(3^n)$, and let α, β be as in Proposition 3.17. Let $S \leqslant T \leqslant \mathrm{Aut}(S)$ and $\hat{\alpha} \in \mathrm{Irr}(B_0(T))$ be an extension of α. By Theorem 1.56 there exists $\chi \in \mathrm{Irr}(B_0(T))$ over β and by Clifford's theorem $\chi(1)$ is divisible by $\beta(1)$, which does not divide $\alpha(1)$. Then $\hat{\alpha}$ and χ are characters of different degrees that lie in $B_0(T)$ and which do not contain S in their kernel.

Now, assume that $S \cong \mathrm{PSL}_2(q)$ for $q = 3^n$, $n \geqslant 3$. Let $A = \mathrm{Aut}(S)$ and let $S \subseteq N \lhd A$ such that $N \cong \mathrm{PGL}_2(3^n)$. In this case, the unique character outside the principal block is the Steinberg character St_S by Theorem 3.10. Write $q \equiv \epsilon \bmod 4$, $\epsilon \in \{-1, 1\}$. The only characters of S that do not extend to N are the characters of degree $\frac{1}{2}(q+\epsilon)$.

If $\epsilon = 1$ then N has a family of characters ψ_k of degree $q+1$ whose only possibly nonrational values are of the form $\zeta^{jk} + \zeta^{-jk}$ for some integer j and $\zeta = e^{\frac{2\pi i}{q-1}}$, and ψ_k lie over characters of $B_0(S)$ of degree $\frac{1}{2}(q+1)$. Let σ be the generator of the cyclic group of field automorphisms in A. Then ψ_k^σ takes the values $(\zeta^{jk})^3 + (\zeta^{-jk})^3$. Since $q-1$ is divisible by 4 we can choose n such that ζ^n is a 4th root of unity, and then $(\zeta^{jn})^3 + (\zeta^{-jn})^3 = \zeta^{-jn} + \zeta^{jn}$, so ψ_n is $\langle\sigma\rangle$-invariant. By Theorem 1.12 and using that A/N is cyclic, ψ_n extends to $\varphi \in \mathrm{Irr}(B_0(A))$. Let $S \leqslant T \leqslant A$ and let α be a constituent of φ_T. Since α lies over a character of S of degree $\frac{1}{2}(q+1)$ then either $\alpha(1) = \frac{1}{2}(q+1)$ or $\alpha(1) = q+1$. Now S also has a character β in the principal block of degree $q-1$. By Theorem 1.56 there is some $\gamma \in \mathrm{Irr}(B_0(T))$ lying over β so that $\gamma(1)$ is a multiple of $\beta(1)$. Then both γ and α do not contain S in their kernel and $\alpha(1) \neq \gamma(1)$.

If $\epsilon = -1$ then an identical argument can be made to show there is a character $\chi \in \mathrm{Irr}(B_0(N))$ of degree $q-1$ that extends to $B_0(A)$, and since these groups also have characters of degree $q+1$ in the principal block, the same reasoning works. □

3.3.3. The case $p \geqslant 5$. In this case, Theorem 3.2 follows easily from the work on simple groups done in [**GRSS20**].

PROPOSITION 3.19. *Theorem 3.2 holds for $p \geqslant 5$.*

PROOF. Let S be a nonabelian simple group of order divisible by p, and let T be a subgroup $S \leqslant T \leqslant \mathrm{Aut}(S)$. If $S \not\cong \mathrm{P}\Omega_8^+(q)$ then we are in case (i) of [**GRSS20**, Proposition 2.1], and may choose the extensions of the nontrivial characters α, β to the principal p-block of T. Otherwise $S \cong \mathrm{P}\Omega_8^+(q)$ for some prime power q. Then we are in case (ii) of the same result, so there exist two nontrivial characters $\alpha, \beta \in \mathrm{Irr}(B_0(S))$ such that $\alpha(1) > 2\beta(1)$. The result produces characters $\hat{\alpha}, \hat{\beta} \in \mathrm{Irr}(B_0(T))$ such that $\hat{\alpha}_S \in \{\alpha, 2\alpha\}$ and $\hat{\beta}_S \in \{\beta, 2\beta\}$. In particular $\hat{\alpha}(1) \neq \hat{\beta}(1)$. Since $\hat{\alpha}$ and $\hat{\beta}$ lie over a nontrivial character of S, we are done. □

3.4. The reduction of Theorem D

3.4.1. Auxiliary lemmas.

LEMMA 3.20. *Let G be a finite group, $N \lhd G$, and $\theta \in \mathrm{Irr}(B_0(N))$. Assume θ extends to $\chi \in \mathrm{Irr}(B_0(G))$. Then $\{\mu\chi \mid \mu \in \mathrm{Irr}(B_0(G/N))\} \subseteq \mathrm{Irr}(G|\theta) \cap \mathrm{Irr}(B_0(G))$.*

PROOF. By Gallagher's correspondence (Theorem 1.11), $\mathrm{Irr}(G|\theta) = \{\mu\chi \mid \mu \in \mathrm{Irr}(G/N)\}$. We have $\chi = 1_{G/N}\chi$ and then by Lemma 1.51, if $\mu \in \mathrm{Irr}(B_0(G/N))$ then $\mu\chi \in \mathrm{Irr}(B_0(G))$. □

LEMMA 3.21. *Let G be a finite group and let $S_1 \times \cdots \times S_t \lhd G$, where the S_i's are subgroups permuted by G. Let $S = S_1$. Assume there exists $\alpha \in \mathrm{Irr}(\mathbf{N}_G(S)/\mathbf{C}_G(S))$ in the principal block and such that S is not contained in $\ker(\alpha)$. Then $\alpha^G \in \mathrm{Irr}(B_0(G))$.*

PROOF. Write $H = \mathbf{N}_G(S)$ and $C = \mathbf{C}_G(S)$. Let $\eta \in \mathrm{Irr}(S)$ be under α. Now let

$$\psi = \eta \times 1_{S_2} \times \cdots \times 1_{S_t} \in \mathrm{Irr}(S_1 \times \cdots \times S_t).$$

Notice that $G_\psi = H_\eta$. Since $S_2 \cdots S_t \subseteq C \subseteq \ker(\alpha)$ we have that $\alpha \in \mathrm{Irr}(H|\psi)$. By the Clifford correspondence (Theorem 1.10) there is some $\mu \in \mathrm{Irr}(H_\eta)$ such that $\mu^H = \alpha$. Using that $G_\psi = H_\eta$ and applying again the Clifford correspondence, $\mu^G = (\mu^H)^G = \alpha^G \in \mathrm{Irr}(G)$. Now by Theorem 1.46, $B_0(H)^G$ is defined and contains α^G, and by Brauer's third main theorem (Theorem 1.47), $B_0(H)^G = B_0(G)$ so we are done. □

LEMMA 3.22. *Let G be a finite group and let $S_1 \times \cdots \times S_t \lhd G$, where the S_i's are subgroups permuted transitively by G. Let $S = S_1$, consider $\{x_1, \ldots, x_t\} \subseteq G$ such that $S_i = S^{x_i}$, and let $\alpha \in \mathrm{Irr}(S)$. If α is $\mathbf{N}_G(S)$-invariant then $\eta = \alpha^{x_1} \times \cdots \times \alpha^{x_t}$ is G-invariant.*

PROOF. Let $s \in S$, $x \in G$ and $H = \mathbf{N}_G(S)$. Notice that $G = \coprod_{j=1}^{t} Hx_j$ is a disjoint union, so $x^{-1} = hx_j$ for some $j \in \{1, \ldots, t\}$ and some $h \in H$. Then

$$\eta^x(s) = \eta(s^{x^{-1}}) = \eta(s^{hx_j}) = \alpha^{x_j}(s^{hx_j}) \prod_{i \neq j} \alpha^{x_i}(1).$$

Now $s^h \in S$ and then $\alpha^{x_j}(s^{hx_j}) = \alpha(s^h)$. Since α is H-invariant we have

$$\eta^x(s) = \alpha(s^h) \prod_{i \neq j} \alpha^{x_i}(1) = \alpha(s) \prod_{i \neq 1} \alpha^{x_i}(1) = \eta(s),$$

so $\eta^x(s) = \eta(s)$.

Now

$$\eta^x(s^{x_j}) = \eta^{xx_j^{-1}}(s) = \eta(s) = \eta^{x_j^{-1}}(s) = \eta(s^{x_j})$$

applying the equality from the previous paragraph twice. In particular, for any $y \in S_i$, $\eta^x(y) = \eta(y)$.

Let $s_i \in S_i$ and notice that

$$\eta(s_1 \cdots s_t) = \frac{\prod_{i=1}^{t} \eta(s_i)}{\prod_{i=1}^{t} \left(\prod_{j \neq i} \alpha^{x_i}(1) \right)} = \frac{\prod_{i=1}^{t} \eta(s_i)}{\alpha(1)^{t^2 - t}};$$

then if $x \in G$ we have

$$\eta^x(s_1 \cdots s_t) = \eta(s_1^{x^{-1}} \cdots s_t^{x^{-1}}) = \frac{\prod_{i=1}^{t} \eta(s_i^{x^{-1}})}{\alpha(1)^{t^2 - t}} = \frac{\prod_{i=1}^{t} \eta(s_i)}{\alpha(1)^{t^2 - t}}$$

which equals $\eta(s_1 \cdots s_t)$, so we are done. □

We will need the following elementary lemma.

LEMMA 3.23. *Let $\alpha, \beta \in \mathrm{Irr}(B_0(G))$ be such that one of $\alpha(1)$ and $\beta(1)$ is not divisible by p and $\alpha\beta \in \mathrm{Irr}(G)$. Then $\alpha\beta \in \mathrm{Irr}(B_0(G))$.*

PROOF. See for instance [**NT12**, Lemma 3.5]. □

In the next results we use a construction known as the **tensor induced character**. We refer to [**Nav18**, Section 10.2] for details on this construction.

LEMMA 3.24. *Let G be a finite group and let $N = S_1 \times \cdots \times S_t \lhd G$, where the S_i's are subgroups transitively permuted by G by conjugation. Let $S = S_1$. Assume that there exists $\alpha \in \mathrm{Irr}(B_0(S))$ with $\alpha(1)$ not divisible by p, $\mathbf{Z}(S) \subseteq \ker(\alpha)$ and such that α extends to $\hat{\alpha} \in \mathrm{Irr}(B_0(\mathbf{N}_G(S)/\mathbf{C}_G(S)))$. Then the tensor induced character $\hat{\alpha}^{\otimes G}$ is in the principal block of G.*

PROOF. Write $H = \mathbf{N}_G(S)$. Let $\{x_1, \ldots, x_t\}$ be a transversal for H in G such that $S^{x_i} = S_i$ and notice that $\mathbf{N}_G(S_i) = H^{x_i}$. Also, write $\alpha_i = \alpha^{x_i} \in \mathrm{Irr}(S_i)$. By Lemma 3.22, $\alpha_1 \times \cdots \times \alpha_t \in \mathrm{Irr}(N)$ is G-invariant. Let $M = \bigcap_{i=1}^t H^{x_i} \lhd G$. Since $B_0(H)$ covers $B_0(M)$ then $\hat{\alpha}_M \in \mathrm{Irr}(B_0(M))$. As in the proof of [**Nav18**, Corollary 10.5], the tensor induced character $\chi = \hat{\alpha}^{\otimes G}$ extends $\alpha_1 \times \cdots \times \alpha_t \in \mathrm{Irr}(B_0(N))$. Also, by [**Nav18**, Lemma 10.4] we have that for $g \in M$,

$$\chi(g) = \prod_{i=1}^t \hat{\alpha}^{x_i}(g),$$

so $\chi_M = \prod_{i=1}^t \hat{\alpha}^{x_i} \in \mathrm{Irr}(M)$. Moreover, $\hat{\alpha}^{x_i}(1) = \alpha(1)$ is not divisible by p, so χ_M is in the principal block of M by Lemma 3.23.

Now let $Q \in \mathrm{Syl}_p(M)$. We have that $Q \cap S_i \in \mathrm{Syl}_p(S_i)$. Then $\mathbf{C}_G(Q) \subseteq \mathbf{C}_G(Q \cap S_i) \subseteq H^{x_i}$, and then $\mathbf{C}_G(Q) \subseteq \bigcap_{i=1}^t H^{x_i} = M$. By Theorem 1.59, $B_0(M)^G$ is defined and is the unique block of G covering $B_0(M)$. Now by Brauer's third main theorem (Theorem 1.47), $B_0(M)^G = B_0(G)$ and we conclude that $\chi \in \mathrm{Irr}(B_0(G))$, as desired. □

The following result is necessary in the case that the character α of the previous result has no extension to $\mathrm{Irr}(B_0(\mathbf{N}_G(S)/\mathbf{C}_G(S)))$.

LEMMA 3.25 (Navarro). *Let $N \lhd G$, $N = S_1 \times \cdots \times S_t$ where the S_i's are permuted transitively by G and have order divisible by p. Write $S = S_1$ and write $S_i = S^{x_i}$ for some $x_i \in G$. Let $\theta_1 \in \mathrm{Irr}(S)$ with $\mathbf{Z}(S) \subseteq \ker(\theta_1)$ and assume that there exists $\alpha \in \mathrm{Irr}(\mathbf{N}_G(S)/\mathbf{C}_G(S))$ of p'-degree, in the principal block, such that $\alpha_S = a\theta_1$ for some integer a. Let $\theta_i = (\theta_1)^{x_i}$, and let $\theta = \theta_1 \times \cdots \times \theta_t \in \mathrm{Irr}(N)$. Then there is $\chi \in \mathrm{Irr}(B_0(G))$ of p'-degree such that $\chi_N = e\theta$ for some integer $1 \leqslant e \leqslant a^t$.*

PROOF. This is [**M21**, Lemma 4.4]. □

We finish the preliminaries of this chapter with a well known result.

LEMMA 3.26. *Let G be a finite group and assume $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$. Then G has a unique p-block.*

PROOF. Let $P = \mathbf{O}_p(G)$. Let b be the unique block of P. Since $\mathbf{C}_G(P)P \subseteq P$, by Theorem 1.43 b^G is defined. By Theorem 1.42, P is contained in every defect group of every block of G. Applying the second part of Theorem 1.43, we conclude that b^G is the unique block of G. □

3.4.2. Components and the generalized Fitting subgroup. In the proof of Theorem 3.36 we make use of some notable subgroups of a finite group G. These are the **layer**, denoted by $\mathbf{E}(G)$, which is the subgroup generated by the **components** of G, and the generalized Fitting subgroup $\mathbf{F}^*(G) = \mathbf{F}(G)\mathbf{E}(G)$, where $\mathbf{F}(G)$ is the Fitting subgroup. For the definition and important properties of these subgroups see [**Isa08**, Section 9A], [**KS03**, Section 6.5] or [**A86**, Chapter 31]. We summarize in this section what we will need to know about these subgroups.

LEMMA 3.27. *Let K be a component of G suppose that the prime p divides $|K|$. Then p divides $|K/\mathbf{Z}(K)|$.*

PROOF. This is [**A86**, (33.12)]. □

LEMMA 3.28. *Let K be a component of G and let N be its normal closure in G. Then N is the central product of the distinct G-conjugates of K, and*

$$N/\mathbf{Z}(N) \cong K_1/\mathbf{Z}(K_1) \times \cdots \times K_t/\mathbf{Z}(K_t)$$

where $\{K_1, \ldots, K_t\}$ denotes the set of G-conjugates of K.

PROOF. This is [**KS03**, (6.5.3)] and [**KS03**, 1.6.7]. □

LEMMA 3.29. *Let G be a finite group and let $\mathbf{F}^*(G)$ be its generalized Fitting subgroup. Then $\mathbf{C}_G(\mathbf{F}^*(G)) \subseteq \mathbf{F}^*(G)$.*

PROOF. This is [**A86**, (31.13)]. □

3.4.3. Some results on simple groups. We shall need the following results on simple groups.

LEMMA 3.30. *Let $S = \mathrm{PSL}_2(3^k)$ for $k \geqslant 3$ and let $p = 3$. Then for any $S \leqslant T \leqslant \mathrm{Aut}(S)$ there is a character $\gamma \in \mathrm{Irr}(B_0(T))$ such that $\gamma_S = a\alpha$ for some $\alpha \in \mathrm{Irr}(B_0(S))$, $a \in \{1, 2\}$ and $\alpha(1) \geqslant 13$ is not divisible by 3.*

PROOF. Write $q = 3^k$ and $q \equiv \epsilon \bmod 4$ so that $\epsilon \in \{-1, 1\}$. In Proposition 3.18 we find a character $\gamma \in \mathrm{Irr}(B_0(T))$ of degree $\gamma(1) \in \{\frac{1}{2}(q+\epsilon), q+\epsilon\}$ that lies over a T-invariant character $\alpha \in \mathrm{Irr}(B_0(S))$ of degree $\alpha(1) = \frac{1}{2}(q+\epsilon)$. Since $k \geqslant 3$ the smallest possible value for $\alpha(1)$ is 13. □

The proof of the next three theorems is done in Section 3 of [**GMS22**] for groups of Lie type and in Section 3.5 for the alternating groups.

THEOREM 3.31. *Let $p \in \{2,3\}$ and let S be a finite nonabelian simple group of order divisible by p. Let G be an almost simple group with socle S and assume $S \notin \{\mathrm{PSL}_2(3^k) \mid k \geqslant 2\}$ if $p = 3$. Then there is some $\alpha \in \mathrm{Irr}(B_0(S))$ with $\alpha(1) > 2$ that extends to $\mathrm{Irr}(B_0(G))$ and some $\beta \in \mathrm{Irr}(B_0(G))$ with $\beta(1) \neq \alpha(1)$ and $S \nsubseteq \ker(\beta)$.*

PROOF. For alternating groups, this is Corollary 3.39. For sporadic groups and simple groups of Lie type with exceptional Schur multiplier this is [**GMS22**, Proposition 3.1]. For the remaining simple groups of Lie type in nondefining characteristic see [**GMS22**, Lemma 3.5]. For simple groups of Lie type in defining characteristic see [**GMS22**, Proposition 3.8]. □

THEOREM 3.32. *Let $p = 2$ and let G be almost simple with socle S. If $|\mathrm{cd}(B_0(G))| = 3$, then S is $\mathrm{PSL}_2(q)$ for some q. In particular, in such a case, if K is a perfect central extension of S, then the Sylow 2-subgroups of K are metabelian.*

PROOF. For alternating groups, this is Corollary 3.38. For sporadic groups and simple groups of Lie type with exceptional Schur multiplier this is [**GMS22**, Proposition 3.1]. For the remaining simple groups of Lie type in nondefining characteristic see [**GMS22**, Proposition 3.6]. For simple groups of Lie type in defining characteristic see [**GMS22**, Proposition 3.8]. □

THEOREM 3.33. *Let $p = 3$ and let S be a finite nonabelian simple group of order divisible by 3. Let G be an almost simple group with socle S. If G has nonabelian Sylow 3-subgroups, then $|\mathrm{cd}(B_0(G))| > 3$.*

PROOF. For alternating groups, this is Corollary 3.38. For sporadic groups and simple groups of Lie type with exceptional Schur multiplier this is [**GMS22**, Proposition 3.1]. For the remaining simple groups of Lie type in nondefining characteristic see [**GMS22**, Proposition 3.7]. For simple groups of Lie type in defining characteristic see [**GMS22**, Proposition 3.8]. □

3.4.4. The reduction theorem. For the main result of this section we make use of the following result.

THEOREM 3.34. *Let G be a finite group and let m, n be positive integers. Assume that the set of degrees of irreducible characters of G is contained in $\{1, m, n\}$. Then G is solvable and has derived length at most 3*

PROOF. See Theorem [**Isa06**, Theorem 12.15]. □

Before proceeding with the reduction, we state a related result, which generalizes Theorem 3.3.

THEOREM 3.35 (Giannelli–Rizo–Sambale–Schaeffer Fry). *Assume $p > 3$ and that*

$$|\{\chi(1) | \chi \in \mathrm{Irr}(B_0(G)), p \nmid \chi(1)\}| \leqslant 2.$$

Then G is p-solvable.

PROOF. This is [**GRSS20**, Theorem A]. □

Next we prove our main result using Theorems 3.31, 3.32, and 3.33. For this reduction theorem, we make use of the following fact: the unique integral solution for $xy = x^y$ with $x, y > 1$ is $x = y = 2$. Furthermore if a, x, y are positive integers and $a, x > 1$ then $xy < ax^y$.

THEOREM 3.36. *Let G be a finite group, p a prime and assume that*

$$\mathrm{cd}(B_0(G)) = \{1, m, n\}.$$

Then the Sylow p-subgroups of G have derived length at most 3.

PROOF. We proceed by induction on $|G|$.

Step 1: We may assume $p = 2, 3$, p divides m but not n and that $\mathbf{O}_{p'}(G) = 1$.

If p divides n and m then by Theorem 3.3 we have $G/\mathbf{O}_{p'}(G)$ is a p-group. By Theorem 1.66, $\mathrm{Irr}(B_0(G)) = \mathrm{Irr}(G/\mathbf{O}_{p'}(G))$, so $G/\mathbf{O}_{p'}(G)$ has 3 character degrees, and then the result holds by Theorem 3.34. If p does not divide n and m then the result follows by Theorem 3.4. Hence we assume p divides m but not n. If $p > 3$ then by Theorem 3.35 we have G is p-solvable. Applying again Theorem 1.66 we have $\mathrm{Irr}(G/\mathbf{O}_{p'}(G)) = \mathrm{Irr}(B_0(G))$ and then $G/\mathbf{O}_{p'}(G)$ has at most 3 degrees and we are done by Theorem 3.34. Hence we assume $p = 2, 3$. Also, using $\mathrm{Irr}(B_0(G/\mathbf{O}_{p'}(G))) = \mathrm{Irr}(B_0(G))$ (by Corollary 1.53) and arguing by induction we may assume $\mathbf{O}_{p'}(G) = 1$.

Step 2: G has a nontrivial component K with $|K/\mathbf{Z}(K)|$ divisible by p.

Let $\mathbf{E}(G)$ be the layer of G. If $\mathbf{E}(G) = 1$ then $\mathbf{F}^*(G) = \mathbf{F}(G) = \mathbf{O}_p(G)$ and then $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$ by Lemma 3.29. Then G has a unique block by Lemma 3.26, and we are done by Theorem 3.34. We may assume that $\mathbf{E}(G) \neq 1$. Let K be a component of G and let N be the normal closure of K in G. By Lemma 3.28, $\prod_{g \in G} K^g \leqslant G$, so

$$N = \prod_{g \in G} K^g.$$

If $|K|$ is not divisible by p then $|N|$ is not divisible by p, contradicting $\mathbf{O}_{p'}(G) = 1$ from Step 1, so K has order divisible by p. By Lemma 3.27 we have that $|K/\mathbf{Z}(K)|$ is also divisible by p.

Step 3: If N is the normal closure of K in G then $N/\mathbf{Z}(N)$ is a direct product of isomorphic copies of $K/\mathbf{Z}(K)$ and $\mathrm{cd}(B_0(G/\mathbf{Z}(N))) = \{1, m, n\}$.

By Lemma 3.28, N is a central product of G-conjugates of K. Write $\bar{G} = G/\mathbf{Z}(N)$, and again by Lemma 3.28

$$N/\mathbf{Z}(N) = \bar{N} = \prod \bar{K}^g = \bar{K}_1 \times \cdots \times \bar{K}_t$$

is a direct product of nonabelian simple groups $\bar{K}_i \cong K/\mathbf{Z}(K)$, and the $\bar{K}_i$'s are transitively permuted by $\bar{G}$. We write $\bar{K}_1 = \bar{K}$. If $|\mathrm{cd}(B_0(\bar{G}))| < 3$ then by Theorem 3.5 we have that $\bar{G}$ is p-solvable, which is impossible, so $\mathrm{cd}(B_0(\bar{G})) = \{1, n, m\}$.

Step 4: If $p = 3$ then $\bar{K} \ncong \mathrm{PSL}_2(9)$

Let $\bar{H}/\bar{C} = \mathbf{N}_{\bar{G}}(\bar{K})/\mathbf{C}_{\bar{G}}(\bar{K})$. There is a natural isomorphism $\bar{K} \cong \bar{K}\bar{C}/\bar{C} \subseteq \bar{H}/\bar{C}$ and we view $\bar{H}/\bar{C} \subseteq \mathrm{Aut}(\bar{K})$. If $\bar{K} \cong \mathrm{PSL}_2(9)$ and $p = 3$ then it is easy to check in [**GAP**] that for any $\bar{K} \leqslant \bar{T} \leqslant \mathrm{Aut}(\bar{K})$ there are two characters $\alpha, \beta \in \mathrm{Irr}(B_0(\bar{T}))$ that do not contain $\bar{K}$ in their kernel with $\alpha(1) \neq \beta(1)$ and both $\alpha(1)$ and $\beta(1)$ are not divisible by p. If $\bar{T} = \bar{H}/\bar{C}$, then by Lemma 3.21 we have that $\alpha^{\bar{G}}, \beta^{\bar{G}} \in \mathrm{Irr}(B_0(\bar{G}))$ and $\alpha^{\bar{G}}(1) = t\alpha(1)$, $\beta^{\bar{G}}(1) = t\beta(1)$, (both p' or both divisible by p, depending on t). This is a contradiction, so $\bar{K}$ can not be isomorphic to $\mathrm{PSL}_2(9)$.

Step 5: If $p = 3$ and $\bar{K} \cong \mathrm{PSL}_2(3^k)$ for $k \geqslant 3$ then we may assume $\bar{K} \lhd \bar{G}$.

Let α, γ and $a \in \{1, 2\}$ be as in Lemma 3.30. Since $\bar{K}$ has abelian Sylow 3-subgroups, we know that there exists $\beta \in \mathrm{Irr}(B_0(\bar{H}))$ with $\beta(1) \neq \gamma(1)$ and $\bar{K} \nsubseteq \ker(\beta)$ by Theorem 3.2. Then we can argue as before with Lemma 3.21, and we get that $\gamma^{\bar{G}}, \beta^{\bar{G}} \in \mathrm{Irr}(B_0(\bar{G}))$, so $t\gamma(1), t\beta(1) \in \mathrm{cd}(B_0(\bar{G}))$. Then necessarily

$$n = \gamma^{\bar{G}}(1) = t\gamma(1) = ta\alpha(1)$$

and $m = t\beta(1)$. Also by Lemma 3.25 there exists some $1 \leqslant e \leqslant a^t \leqslant 2^t$ not divisible by 3 such that $n = e\alpha(1)^t$. Thus $ta\alpha(1) = e\alpha(1)^t$. It is easy to see that the above equality has no integral solutions verifying $\alpha(1) \geqslant 13$ and $t > 1$.

Step 6: In the remaining cases, we may also assume $\bar{K} \lhd \bar{G}$.

Assume by way of contradiction that $t > 1$. Let $\alpha \in \mathrm{Irr}(B_0(\bar{K}))$ with $\alpha(1) > 2$, and $\beta \in \mathrm{Irr}(B_0(\bar{H}/\bar{C}))$ be as in Theorem 3.31. Let $\hat{\alpha} \in \mathrm{Irr}(B_0(\bar{H}/\bar{C}))$ be an extension of α. By Lemma 3.21, $\hat{\alpha}^{\bar{G}}, \hat{\beta}^{\bar{G}} \in \mathrm{Irr}(B_0(\bar{G}))$. Hence $t\alpha(1), t\beta(1) \in \mathrm{cd}(B_0(G))$. Since $\alpha(1) \neq \beta(1)$ we have that t is not divisible by p (because n is not divisible by p).

If $\alpha(1)$ is not divisible by p, then by Lemma 3.24, $\hat{\alpha}^{\otimes \bar{G}} \in \mathrm{Irr}(B_0(\bar{G}))$, so $\hat{\alpha}^{\otimes \bar{G}}(1) = \alpha(1)^t \in \mathrm{cd}(B_0(G))$. Since $\beta(1)$ is necessarily divisible by p, this forces $\alpha(1)^t =$

$t\alpha(1) = n$ and if $t > 1$ the only possibility is $\alpha(1) = 2 = t$, a contradiction with $\alpha(1) > 2$.

We are left with the case that p divides $\alpha(1)$, so $\beta(1)$ is p'. By [**Nav18**, Corollary 10.5] $\chi = \hat{\alpha}^{\otimes \bar{G}} \in \mathrm{Irr}(\bar{G})$ extends $\eta = \alpha^{x_1} \times \cdots \times \alpha^{x_t} \in \mathrm{Irr}(B_0(\bar{N}))$, where $\{x_1, \ldots, x_t\}$ is a transversal of $\bar{H}$ in $\bar{G}$ (recall that η is G-invariant by Lemma 3.22). By Theorem 1.56 there exists some $\psi \in \mathrm{Irr}(B_0(\bar{G}))$ over η. By Gallagher's theorem (Theorem 1.11), $\psi = \mu\chi$ for some $\mu \in \mathrm{Irr}(\bar{G}/\bar{N})$. Since p divides $\alpha(1)$ then p divides $\psi(1)$ and $\psi(1) = m$. Then

$$\psi(1) = \mu(1)\chi(1) = \mu(1)\alpha(1)^t = m = t\alpha(1)$$

but this forces $\mu(1) = 1$ and $\alpha(1) = 2 = t$, which contradicts the fact that $\alpha(1) > 2$. This proves the claim.

Step 7: $G/\mathbf{C}_G(K)$ is almost simple with socle $K\mathbf{C}_G(K)/\mathbf{C}_G(K) \cong K/\mathbf{Z}(K)$.

By Steps 5 and 6 any component K of G is normal in G. Write $C = \mathbf{C}_G(K)$ and $Z = \mathbf{Z}(K)$. Since $K/Z \cong KC/C \lhd G/C$, it suffices to prove that $\mathbf{C}_{G/Z}(K/Z) = C/Z$. Let $xZ \in \mathbf{C}_{G/Z}(K/Z)$. We have that for all $k \in K$ there exists a unique $z_k \in Z$ such that $k^x = kz_k$. Notice that the map $k \mapsto z_k$ is a homomorphism $K \to Z$. Since K is perfect and Z is abelian, $z_k = 1$ for all $k \in K$, so $x \in C$. The reverse inclusion is immediate.

Final Step.

If $|\mathrm{cd}(B_0(G/C))| > 3$ then we are done using that $\mathrm{Irr}(B_0(G/C)) \subseteq \mathrm{Irr}(B_0(G))$. Hence we assume $\mathrm{cd}(B_0(G/C)) = \{1, m, n\}$ (if $|\mathrm{cd}(B_0(G/C))| < 3$ then G/C is p-solvable by Theorem 3.5). Since m is divisible by p, by the main result of [**KM13**] the Sylow p-subgroups of G/C are nonabelian. Then by Theorems 3.32 and 3.33 we have $p = 2$ and that the Sylow 2-subgroups of K are metabelian. By Theorem 3.31 there is some nontrivial $\eta \in \mathrm{Irr}(B_0(KC/C))$ that extends to $\hat{\eta} \in \mathrm{Irr}(B_0(G/C))$. By Theorem 1.5, $\eta_K \in \mathrm{Irr}(K)$. Since $B_0(G)$ covers $B_0(K)$ we have $\eta_K \in \mathrm{Irr}(B_0(K))$ by Theorem 1.55, and η_K extends to $B_0(G)$. By Lemma 3.20, $\mathrm{cd}(B_0(G/K)) \subseteq \{1, m/n\}$ and since $m/n \notin \mathrm{cd}(B_0(G))$ we have $\mathrm{cd}(B_0(G/K)) = \{1\}$. In particular, if $P \in \mathrm{Syl}_2(G)$ then PK/K is abelian by Theorem 3.4, and $P \cap K$ is metabelian, so $\mathrm{dl}(P) \leqslant 3$. □

3.5. Alternating groups for Theorem D

We devote this section to the proofs of Theorems 3.31, 3.32 and 3.33 for the alternating groups. It is well known that if $n \geqslant 3$, $n \neq 6$, the symmetric group $\mathfrak{S}_n$ is the automorphism group of the alternating group $\mathfrak{A}_n$. The irreducible characters of the symmetric group $\mathfrak{S}_n$ are parametrised by partitions $\lambda = (\lambda_1, \ldots, \lambda_t)$ of n (see [**JK81**, Theorem 2.3.15]). We will write $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ for the character labelled by λ. Its degree is given by the hook-length formula [**JK81**, Theorem

2.3.21]. Recall that the values of all irreducible characters of $\mathfrak{S}_n$ are integers (see [**JK81**, Theorem 1.2.17]).

PROPOSITION 3.37. *Let $p = 2, 3$ and assume $n > 10$. Then both $B_0(\mathfrak{S}_n)$ and $B_0(\mathfrak{A}_n)$ contain at least three nontrivial character degrees. Furthermore, if $p = 2$ then at least two of those degrees are odd.*

PROOF. Write $S = \mathfrak{A}_n$ and $G = \mathfrak{S}_n$. Since the Sylow p-subgroups of S and G are nonabelian, by Theorem 3.4 we know that both $B_0(S)$ and $B_0(G)$ contain some character of degree divisible by p. Hence, it suffices to find two characters of degree not divisible by p in $B_0(G)$ that restrict irreducibly to a character in $B_0(S)$.

Assume first that $p = 2$. Consider the partitions, $\lambda_1 = (n-2, 1, 1)$, $\lambda_2 = (n-2, 2)$. Their corresponding characters of G have degrees $\chi_{\lambda_1}(1) = (n-1)(n-2)/2$ and $\chi_{\lambda_2}(1) = n(n-3)/2$. Notice that exactly one of the degrees of χ_{λ_1} and χ_{λ_2} is odd. Define the irreducible character

$$\psi = \begin{cases} \chi_{\lambda_1} & \text{if } (n-1)(n-2)/2 \text{ is odd} \\ \chi_{\lambda_2} & \text{otherwise} \end{cases}$$

and notice that $\psi(1)$ is odd. If n is even consider the partition $\lambda_3 = (n-1, 1)$, and we have $\chi_{\lambda_3}(1) = n - 1$. By Lemma 3.12, $\psi, \chi_{\lambda_3} \in \mathrm{Irr}(B_0(G))$ and $\psi(1) \neq \chi_{\lambda_3}(1)$. Furthermore, since both degrees are odd and $|G : S| = 2$, both restrict irreducibly to $\mathrm{Irr}(B_0(S))$ by Theorem 1.21 and we are done. If n is odd we argue identically with $\lambda_3 = (n-3, 2, 1)$, for which $\chi_{\lambda_3}(1) = n(n-2)(n-4)/3$.

If $p = 3$ then [**RSV20**, Table 1] contains two characters in $\mathrm{Irr}(B_0(G))$ of degree coprime to 3 that restrict irreducibly to S, as desired. □

COROLLARY 3.38. *Theorems 3.32 and 3.33 hold for $S = \mathfrak{A}_n$.*

PROOF. The cases where $n \leqslant 10$ can be checked in [**GAP**]. The remaining cases are done in Proposition 3.37. □

COROLLARY 3.39. *Theorem 3.31 holds for $S = \mathfrak{A}_n$.*

PROOF. Write $G = \mathfrak{S}_n$ and $S = \mathfrak{A}_n$. For $n \leqslant 10$ this can be checked with [**GAP**]. We may assume that $n > 10$.

We first deal with the case $p = 2$. Since $|G : S| = 2$, by Lemma 3.12 and Theorem 1.21, any irreducible character of G of odd degree belongs to $B_0(G)$ and restricts irreducibly to $B_0(S)$. Then we are done using Proposition 3.37.

If $p = 3$ the result again follows from [**RSV20**, Table 1]. □

3.6. More on Question B

In this section, we discuss certain cases of Question B and related questions. We denote by $\mathrm{ht}(B)$ the set of heights of characters in a block B.

3.6.1. Blocks of the general linear group in defining characteristic. Using results of A. Moretó, we show that Question B holds for blocks of the general linear group in defining characteristic.

Lemma 3.40. *If B is a p-block of positive defect of $\mathrm{GL}_n(q)$, where q is a power of p, then $|\mathrm{ht}(B)| \geqslant n-1$.*

Proof. This is [**Mor04**, Lemma 3.1]. □

Lemma 3.41. *The Sylow p-subgroups of $\mathrm{GL}_n(q)$, where q is a power of p, have nilpotency class $n-1$.*

Proof. This is [**Hup67**, Satz III.16.3]. □

Corollary 3.42. *Question B has an affirmative answer for the blocks of $\mathrm{GL}_n(q)$ in defining characteristic.*

Proof. The result follows from Lemmas 3.40 and 3.41 using that the derived length is bounded by the nilpotency class and $|\mathrm{ht}(B)| \leqslant |\mathrm{cd}(B)|$. □

Corollary 3.43. *Let $N \lhd \mathrm{GL}_n(q)$ for a power q of a prime p, and assume $|\mathrm{GL}_n(q) : N|$ is not divisible by p. Then Question B has an affirmative answer for the p-blocks of N. In particular, Question B holds for the blocks of $\mathrm{SL}_n(q)$ in the defining characteristic.*

Proof. Let b be a block of N and let B be a block of $\mathrm{GL}_n(q)$ covering b. By Theorem 1.63 the defect groups of B are defect groups of b. Let $\psi \in \mathrm{Irr}(b)$ and let $\chi \in \mathrm{Irr}(B)$ be over ψ (see Theorem 1.56). It follows from Theorem 1.21 that $\psi(1)_p = \chi(1)_p$ so we see that $\mathrm{ht}(B) = \mathrm{ht}(b)$, so this result follows from the argument of the proof of Corollary 3.42. □

3.6.2. Checking GAP libraries. Let p be a prime and let D be a p-group of size p^k. Since groups of order p^2 are abelian, it is easy to see that $\mathrm{dl}(D) \leqslant \frac{k+1}{2}$. Using this bound, we have checked in [**GAP**] that Question B holds for all sporadic groups. Furthermore, Question B has been checked for all perfect groups and the primitive groups of degree up to 1500 and size up to 10^6. Whenever the bound above does not work, in most cases it suffices to check that $|\mathrm{cd}(B)| \leqslant \mathrm{dl}(P)$ for a Sylow p-subgroup P, since $\mathrm{dl}(D) \leqslant \mathrm{dl}(P)$ for any defect group D. Otherwise, the fact that if D is a defect group of some block, then $D = \mathbf{O}_p(\mathbf{N}_G(D))$ (see [**Nav98**, Corollary 4.18]) helps locating possible defect groups of a block if the defect is known. The bound used on $\mathrm{dl}(D)$ can be improved using the main results of [**Man00**] and [**Sch07**] (in the latter case, when $p \geqslant 5$).

3.6.3. Related questions. If D is a p-group, Taketa's Theorem [**Isa06**, Theorem 5.12] states that $|\mathrm{cd}(D)| \geqslant \mathrm{dl}(D)$, so questions regarding the derived length of D are closely related to its set of character degrees. The following conjecture was recently asked in [**FLZ20**].

CONJECTURE 3.44 (Feng–Liu–Zhang). *Let B be a p-block of a finite group G with defect group D. Let p^a be the maximal element in $\mathrm{cd}(D)$ and let b be the maximal height of the characters in B. Then $a \leqslant b$.*

Using [**GAP**] notation, let $G =$ `SmallGroup(729, 122)`. Let $A = \mathrm{Aut}(G)$ and $Q \in \mathrm{Syl}_{13}(A)$. Let $D \in \mathrm{Syl}_3(\mathbf{N}_A(Q))$ and $R = \langle D, Q\rangle \leqslant A$. We consider the semidirect product $H = G \rtimes R$. Then H is a solvable group of size $28431 = 3^7 \cdot 13$, with $\mathbf{O}_{3'}(H) = 1$. Then $\mathrm{Irr}(H) = \mathrm{Irr}(B_0(H))$ and $\mathrm{cd}(H) = \mathrm{cd}(B_0(H)) = \{1, 3, 13, 39\}$. However if $P \in \mathrm{Syl}_p(H)$ we have $\mathrm{cd}(P) = \{1, 3, 9\}$, so H is a counterexample to Conjecture 3.44. This example appeared in [**Isa71**, Example 6.1]. It belongs to a larger family of examples constructed by I. M. Isaacs in an unpublished note regarding a similar question. In fact, this provides a counterexample to Conjecture 3.44 for all odd primes, as asked by G. Malle after G. Navarro found the counterexample `SmallGroup(192,955)` for $p = 2$.

In Corollaries 3.42 and 3.43, we have used the bound $\mathrm{dl}(D) \leqslant |\mathrm{ht}(B)|$ taking advantage of the fact that $|\mathrm{ht}(B)| \leqslant |\mathrm{cd}(B)|$. It is a natural question to ask if we can improve the bound in Question B by $\mathrm{dl}(D) \leqslant |\mathrm{ht}(B)|$ in general. However this bound does not hold even in solvable groups; the group H constructed above is a counterexample, as its Sylow 3-subgroups have derived length 3. We mention that for $p = 2$, the group `PerfectGroup(17280,1)` is a counterexample for this bound. No solvable counterexamples have been found for $p = 2$.

Finally, we would like to remark that if G is solvable and B is the principal block, then an affirmative answer to Question B would follow from assuming the Isaacs–Seitz conjecture (see [**Nav10**, (8.2)]).

3.7. Theorem F

We begin with a well-known consequence of the Hall–Higman Lemma.

LEMMA 3.45 (Hall–Higman). *Let G be a p-solvable group and assume $\mathbf{O}_{p'}(G) = 1$. Then $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$.*

PROOF. See [**Isa08**, Theorem 3.21]. □

LEMMA 3.46. *Let G be p-solvable and assume $\mathbf{O}_{p'}(G) = 1$. Then the principal block is the unique p-block of G.*

PROOF. Let $P = \mathbf{O}_p(G)$. By Lemma 3.45, $\mathbf{C}_G(P) \subseteq P$. Now the result follows from Lemma 3.26 □

LEMMA 3.47. *Let G be a finite group, p a prime dividing $|G|$. If $\mathrm{cd}(B_0(G))$ are all q-powers for a prime $q \neq p$ then G is p-solvable.*

PROOF. Let N be a minimal normal subgroup of G. Recall that $B_0(G/N) \subseteq B_0(G)$. By inductive hypothesis, G/N is p-solvable. If N is a p'-group or an elementary abelian p-group we are done, so we assume that N is the direct product of isomorphic nonabelian simple groups of order divisible by p. Now by Lemma 3.8, N contains a nonlinear q'-degree character θ in the principal block. By Theorem 1.56 there exists $\chi \in \mathrm{Irr}(B_0(G))$ over θ but then by Clifford's theorem, $\theta(1)$ must divide $\chi(1)$, a contradiction. Hence G is p-solvable. □

We will use the following classical result.

THEOREM 3.48 (Itô). *Let G be a finite group and let $A \lhd G$ be an abelian subgroup. Then for all $\chi \in \mathrm{Irr}(G)$, $\chi(1)$ divides $|G : A|$.*

PROOF. See [**Nav18**, Corollary 1.21]. □

The following result is a principal block version of a classical result of J. G. Thompson, which we restate here.

THEOREM 3.49 (Thompson). *Let G be a finite group and let q be a prime. Assume that the degrees of all nonlinear irreducible characters are divisible by q. Then G has a normal q-complement.*

PROOF. See [**Nav18**, Theorem 7.5]. □

THEOREM 3.50. *Let G be a finite group, p a prime dividing $|G|$ and $B_0(G)$ its principal block. Then $\mathrm{cd}(B_0(G))$ are all q-powers for a prime $q \neq p$ if and only if $G/\mathbf{O}_{p'}(G)$ has an abelian normal q-complement.*

PROOF. Assume first that $\mathrm{cd}(B_0(G))$ are all q-powers, and we argue by induction on $|G|$. By Lemma 3.47, G is p-solvable. Write $\overline{G} = G/\mathbf{O}_{p'}(G)$. By Theorem 1.49, $B_0(\overline{G}) = B_0(G)$. If $\mathbf{O}_{p'}(G) > 1$ by inductive hypothesis $\overline{G}/\mathbf{O}_{p'}(\overline{G}) = \overline{G}$ has an abelian normal q-complement. If $\mathbf{O}_{p'}(G) = 1$, by Lemma 3.46, G has a unique p-block, so $\mathrm{Irr}(G) = \mathrm{Irr}(B_0(G))$. Hence, every character in the principal q-block of G has q-power degree, so by Theorem 3.49, G has a normal q-complement $N \lhd G$. If N is not abelian then there exists $\alpha \in \mathrm{Irr}(N)$ with $\alpha(1) > 1$, and $\alpha(1)$

is not divisible by q. Let $\psi \in \mathrm{Irr}(G|\alpha)$. By Clifford's theorem $\alpha(1)$ must divide $\psi(1)$ which is a q-power, a contradiction.

Assume now that $\overline{G}$ has an abelian normal q-complement. Then it follows from Theorem 3.48 that all characters of $\overline{G}$ have q-power degree. Again by Theorem 1.49, $B_0(G/\mathbf{O}_{p'}(G)) = B_0(G)$ so $\mathrm{cd}(B_0(G))$ are all q-powers and we are done. □

We should point out here that if the degrees of an arbitrary p-block B are all p-powers, then the block B is nilpotent by the main result of [**NR05**].

CHAPTER 4

Divisibility in the Brauer correspondence

4.1. Introduction

As we mentioned in the introduction, the McKay Conjecture states that if p is a prime number, G is a finite group and P is a Sylow p-subgroup of G, then

$$|\operatorname{Irr}_{p'}(G)| = |\operatorname{Irr}_{p'}(\mathbf{N}_G(P))|.$$

For a solvable group G, Turull showed that the character correspondence from the McKay conjecture can be chosen to be compatible with divisibility of character degrees. Namely, in [**Tur07**] he showed that there exists a bijection

$$\Omega : \operatorname{Irr}_{p'}(G) \to \operatorname{Irr}_{p'}(\mathbf{N}_G(P))$$

such that $\Omega(\chi)(1)$ divides $\chi(1)$ for every $\chi \in \operatorname{Irr}_{p'}(G)$. Subsequently, in [**Riz19**] Rizo extended Turull's result to p-solvable groups by assuming the fact that the Glauberman correspondence is compatible with divisibility of character degrees, a fact later proved by M. Geck [**Gec20**] and whose proof relies on [**HT94**] and deep results of Lusztig. Moreover, [**BNRS22**, Theorem 3.1] provides a version of these results for p-Brauer characters.

The Alperin–McKay Conjecture generalizes the McKay conjecture by taking into account Brauer blocks and height zero characters. If B is a p-block of a finite group G and b is its Brauer correspondent, then

$$k_0(B) = k_0(b)$$

where we denote by $k_0(B)$ the number of height zero irreducible characters in B, as in Chapter 2.

The first theorem of this chapter generalizes all the above mentioned results by showing that, for both ordinary and p-Brauer characters of p-solvable groups, there exists a bijection between the height zero characters of B and b which is compatible with divisibility of character degrees. We denote by $\operatorname{Irr}_0(B)$ and $\operatorname{IBr}_0(B)$ the sets of height zero ordinary and Brauer characters in a block B.

Theorem G. *Let G be a finite p-solvable group, B a p-block of G with defect group D, and consider its Brauer correspondent $b \in \operatorname{Bl}(\mathbf{N}_G(D))$. Then there exist bijections*

$$\Omega : \operatorname{Irr}_0(B) \to \operatorname{Irr}_0(b)$$

and

$$\Psi : \mathrm{IBr}_0(B) \to \mathrm{IBr}_0(b)$$

such that $\Omega(\chi)(1)$ *divides* $\chi(1)$ *and* $\Psi(\varphi)(1)$ *divides* $\varphi(1)$ *for every* $\chi \in \mathrm{Irr}_0(B)$ *and* $\varphi \in \mathrm{IBr}_0(B)$*.*

A key ingredient in the proof of the above theorem is Proposition 4.7, which is a generalisation of the main result of [**Nav03**] and was kindly communicated to us by Navarro. We also mention that work of Navarro, Späth and Tiep shows that, if G is p-solvable and the inductive McKay condition holds for the prime q, then there exists a bijection between p-Brauer characters of q'-degree of G and p-Brauer characters of q'-degree of a q-Sylow normalizer (see [**NST17**, Theorem C]). However, we do not consider this cross-characteristic case.

Next, we consider divisibility between the dimension of a p-block and that of its Brauer correspondent. For a p-block C, we define $\dim(C)$ to be the dimension of the corresponding block algebra. If B is a p-block with Brauer correspondent b, then a theorem of Brauer shows that $\dim(b)_p$ divides $\dim(B)_p$ [**Lin18**, Theorem 10.1.1]. In our next theorem, we show that divisibility holds for the full dimension whenever G is p-solvable.

Theorem H. *Let G be a finite p-solvable group, B a p-block of G with defect group D and $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ its Brauer correspondent. Then* $\dim(b)$ *divides* $\dim(B)$*.*

As mentioned in the introduction, if H is a subgroup of G and c is a p-block of H that induces the p-block C of G, then it is not clear even whether $\dim(c) \leqslant \dim(C)$. We care to mention that if H is normal in G, then R. Kessar and M. Linckelmann recently proved that divisibility holds with respect to covering of blocks, hence answering a question asked by Navarro (see [**MR22**, Proposition 3.1]).

The results of Theorem G and Theorem H cannot be extended to arbitrary finite groups. As a counterexample consider the principal 5-block and the principal 2-block of the alternating group $\mathfrak{A}_5$ respectively.

In Section 4.2 we introduce some preliminary results including an extension of Fong's theory of p-solvable groups and Navarro's theorem. Then, in Section 4.3 we prove Theorem G in a series of steps working by induction on the order of the group. Finally, in Section 4.4 we prove Theorem H.

The results in this chapter appeared in [**MR22**].

4.2. Preliminary results

Throughout this chapter, if $\chi \in \mathrm{Irr}(G)$ then we denote by $\mathrm{bl}(\chi)$ the block of G containing χ.

We now consider some preliminary results on the Glauberman correspondence (see Theorem 1.14). If Q is a group acting on L with $(|L|, |Q|) = 1$ and $\theta \in \mathrm{Irr}(L)$ is Q-invariant, then we denote by $\theta^* \in \mathrm{Irr}(\mathbf{C}_L(Q))$ its Q-Glauberman correspondent. In [**Tur08**], Turull proved the existence of bijections above the Glauberman correspondence by constructing certain isomorphisms of character triples. This important result has been extended in various context and provides a fundamental tool for the understanding of the local-global conjectures (see, for instance, [**NS14**, Theorem 5.13], [**Lad16**], [**NSV20**, Section 4] and [**Ros22**, Theorem 3.7]).

We begin by stating Turull's astonishing result on character triple isomorphisms above the Glauberman correspondence.

Theorem 4.1 (Turull). *Let K be a normal p'-subgroup of G, and let P be a p-subgroup of G such that $KP = G$. Let $\theta \in \mathrm{Irr}(K)$ be P-invariant, let $\theta^* \in \mathrm{Irr}(\mathbf{C}_K(P))$ be its Glauberman correspondent, and let $T = G_\theta$. Then the character triples (T, K, θ) and $(T \cap \mathbf{N}_G(P), \mathbf{C}_K(P), \theta^*)$ are isomorphic.*

Proof. This is [**Tur08**, Theorem 6.5] and [**Tur09**, Theorem 7.12]. □

The following result was proved by A. Laradji.

Lemma 4.2 (Laradji). *Assume that the character triples $(G, N, \theta), (M, L, \psi)$ are isomorphic, where G and M are p-solvable and both N and L are p'-groups. Then there is a bijection $f : \mathrm{IBr}(G|\theta) \to \mathrm{IBr}(M|\psi)$ such that*

$$\frac{\varphi(1)}{\theta(1)} = \frac{f(\varphi)(1)}{\psi(1)}$$

for all $\varphi \in \mathrm{IBr}(G|\theta)$.

Proof. See [**Lar14**, Lemma 2.3]. □

In the following proposition, we provide yet another consequence of Turull's result and show that these bijections above the Glauberman correspondence are also compatible with divisibility of character degrees. We obtain similar bijections for Brauer characters using Lemma 4.2

Proposition 4.3. *Let L be a normal p'-subgroup of G and consider a p-subgroup Q of G such that $N = LQ \lhd G$. Let θ be a G-invariant irreducible character* *of L and consider the Q-Glauberman correspondent $\theta^* \in \mathrm{Irr}(\mathbf{C}_L(Q))$ of θ. Then there exist bijections*

$$\Omega_Q : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(\mathbf{N}_G(Q)|\theta^*)$$

and

$$\Psi_Q : \mathrm{IBr}(G|\theta) \to \mathrm{IBr}(\mathbf{N}_G(Q)|\theta^*)$$

such that $\Omega_Q(\chi)(1)$ divides $\chi(1)$ and $\Psi_Q(\varphi)(1)$ divides $\varphi(1)$ for every $\chi \in \mathrm{Irr}(G|\theta)$ and $\varphi \in \mathrm{IBr}(G|\theta)$.

Proof. By Theorem 4.1 there is a character triple isomorphism between (G, L, θ) and $(\mathbf{N}_G(Q), \mathbf{C}_L(Q), \theta^*)$. Applying Lemma 1.19 and Lemma 4.2 we obtain bijections

$$\Omega_Q : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(\mathbf{N}_G(Q)|\theta^*)$$

and

$$\Psi_Q : \mathrm{IBr}(G|\theta) \to \mathrm{IBr}(\mathbf{N}_G(Q)|\theta^*)$$

such that

$$\Omega_Q(\chi)(1)/\theta^*(1) = \chi(1)/\theta(1) \text{ and } \Psi_Q(\varphi)(1)/\theta^*(1) = \varphi(1)/\theta(1)$$

for every $\chi \in \mathrm{Irr}(G|\theta)$ and $\varphi \in \mathrm{IBr}(G|\theta)$. Since $\theta^*(1)$ divides $\theta(1)$ by Theorem 1.15, we conclude that $\Omega_Q(\chi)(1)$ divides $\chi(1)$ and that $\Psi_Q(\varphi)(1)$ divides $\varphi(1)$. $\square$

Observe that by the proof of Proposition 4.3 it follows that the bijections Ω_Q and Ψ_Q restrict to the subsets of p'-degree characters.

If G is p-solvable and $\theta \in \mathrm{Irr}(\mathbf{O}_{p'}(G))$ is G-invariant, then by Fong's theorem (Theorem 1.66) there exists a block $B \in \mathrm{Bl}(G)$ whose defect groups are Sylow p-subgroups and such that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$. Our next result is used to replace Fong's theorem in the case where we consider a normal p'-subgroup L of G (but not necessarily $\mathbf{O}_{p'}(G)$) and an irreducible character $\theta \in \mathrm{Irr}(L)$ which is not necessarily G-invariant.

Before proceeding with the proof of the next proposition, we introduce some terminology. Let G be a finite group, B a block of G with defect group D and $L \lhd G$ a p'-subgroup of G. Then, we say that a character $\theta \in \mathrm{Irr}(L)$ is (B, D)-**good** if $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G|\theta)$ and D is a defect group of the Fong–Reynolds correspondent of B over $\mathrm{bl}(\theta)$. Notice that if $L \lhd G$ is a p'-subgroup, then such a character θ always exists and is determined up to $\mathbf{N}_G(D)$-conjugation. Recall that we denote by h_χ the height of an irreducible ordinary or Brauer character χ.

Proposition 4.4. *Let G be a finite p-solvable group, B a p-block of G with defect group D and consider its Brauer correspondent $b \in \mathrm{Bl}(\mathbf{N}_G(D))$. If $L \lhd G$ is a p'-subgroup and $\theta \in \mathrm{Irr}(L)$ is (B, D)-good, then there exists a subgroup $L \leqslant U \leqslant G_\theta$ and a U-invariant character $\nu \in \mathrm{Irr}(\mathbf{O}_{p'}(U))$ such that*

(i) *induction* $\psi \mapsto \psi^G$ *defines a bijection* $\mathrm{Irr}(U|\nu) \to \mathrm{Irr}(B)$,

(ii) *induction* $\varphi \mapsto \varphi^G$ *defines a bijection* $\mathrm{IBr}(U|\nu) \mapsto \mathrm{IBr}(B)$,

(iii) *in the above bijections,* $\mathrm{Irr}_{p'}(U|\nu)$ *and* $\mathrm{IBr}_{p'}(U|\nu)$ *maps onto* $\mathrm{Irr}_0(B)$ *and* $\mathrm{IBr}_0(B)$ *respectively,*

(iv) $D \in \mathrm{Syl}_p(U)$ *and* ν *lies over* θ,

(v) *induction* $\psi \mapsto \psi^{\mathbf{N}_G(D)}$ *defines a bijection* $\mathrm{Irr}(\mathbf{N}_U(D)|\nu^*) \to \mathrm{Irr}(b)$,

(vi) *induction* $\varphi \mapsto \varphi^{\mathbf{N}_G(D)}$ *defines a bijection* $\mathrm{IBr}(\mathbf{N}_U(D)|\nu^*) \to \mathrm{IBr}(b)$,

(vii) *in the above bijections,* $\mathrm{Irr}_{p'}(\mathbf{N}_U(D)|\nu^*)$ *and* $\mathrm{IBr}_{p'}(\mathbf{N}_U(D)|\nu^*)$ *maps onto* $\mathrm{Irr}_0(b)$ *and* $\mathrm{IBr}_0(b)$ *respectively,*

where ν^* *is the* D*-Glauberman correspondent of* ν.

Proof. We argue by induction on $|G : L|$. Let $B_\theta \in \mathrm{Bl}(G_\theta)$ be the Fong–Reynolds correspondent of B over $\mathrm{bl}(\theta)$ (see Theorem 1.58) and suppose that $G_\theta < G$. Since θ is (B_θ, D)-good, the inductive hypothesis yields a subgroup $U < G_\theta$ and a U-invariant character $\nu \in \mathrm{Irr}(\mathbf{O}_{p'}(U))$ satisfying the properties (i)–(vii) with respect to (B_θ, D). Now, induction of characters defines bijections $\mathrm{Irr}(U|\nu) \to \mathrm{Irr}(B_\theta)$ and $\mathrm{IBr}(U|\nu) \to \mathrm{IBr}(B_\theta)$. On the other hand, the Fong–Reynolds correspondence shows that induction of characters also defines bijections $\mathrm{Irr}(B_\theta) \to \mathrm{Irr}(B)$ and $\mathrm{IBr}(B_\theta) \to \mathrm{IBr}(B)$ and hence we obtain (i) and (ii).

Next, the inductive hypothesis implies that $\mathrm{Irr}_{p'}(U|\nu)$ maps onto $\mathrm{Irr}_0(B_\theta)$, which maps onto $\mathrm{Irr}_0(B)$ by part (ii) of Theorem 1.58. Let $\varphi \in \mathrm{IBr}_0(B)$ and $\beta \in \mathrm{IBr}(B_\theta)$ such that $\beta^G = \varphi$. By the Fong–Swan theorem (see Theorem 1.64) there is some $\psi \in \mathrm{Irr}(B_\theta)$ such that $\psi^0 = \beta$ and $h_\beta = h_\psi$, where ψ^0 denotes the restriction of ψ to p-regular elements. Observing that $\psi^G \in \mathrm{Irr}(B)$ and $\varphi = \beta^G = (\psi^0)^G = (\psi^G)^0$, we deduce that $\psi^G \in \mathrm{Irr}_0(B)$ and hence $\psi \in \mathrm{Irr}_0(B_\theta)$ by part (ii) of Theorem 1.58. It follows that $h_\beta = 0$ and thus (iii) holds.

Let $\theta^* \in \mathrm{Irr}(\mathbf{C}_L(D))$. By Lemma 1.16 we have $\mathbf{N}_G(D)_{\theta^*} = \mathbf{N}_G(D)_\theta = \mathbf{N}_{G_\theta}(D)$. Let $b_\theta \in \mathbf{N}_{G_\theta}(D)$ be the Fong–Reynolds correspondent of b over $\mathrm{bl}(\theta^*)$. According to Brauer's first main theorem (Theorem 1.44) the induced block $(b_\theta)^{G_\theta}$ is defined and therefore $(b_\theta)^G = ((b_\theta)^{\mathbf{N}_G(D)})^G = b^G = B$ by using Lemma 1.45. Now, the Fong–Reynolds correspondence implies that $(b_\theta)^{G_\theta} = B_\theta$ and so b_θ is the Brauer correspondent of B_θ. By inductive hypothesis, induction of characters defines bijections $\mathrm{Irr}(\mathbf{N}_U(D)|\theta^*) \to \mathrm{Irr}(b_\theta)$ and $\mathrm{IBr}(\mathbf{N}_U(D)|\theta^*) \to \mathrm{IBr}(b_\theta)$. Arguing as in the previous paragraphs we conclude that (v), (vi) and (vii) are also satisfied. This shows that it is no loss of generality to assume $G_\theta = G$.

Finally, set $N = \mathbf{O}_{p'}(G)$. If $N = L$, then the result follows from Theorem 1.66 by choosing $U = G$ and $\nu = \theta$. Thus, we assume $L < N$. Let $\eta \in \mathrm{Irr}(N)$ be

(B, D)-good and notice that, since θ is G-invariant, Clifford's theorem implies that η lies over θ. Since $|G : N| < |G : L|$, the inductive hypothesis yields a subgroup $N \leqslant U \leqslant G_\eta$ and a U-invariant character $\nu \in \mathrm{Irr}(\mathbf{O}_{p'}(U))$ satisfying the properties (i)-(vii) with respect to N and η. Now, the result follows by noticing that $L \leqslant U \leqslant G_\theta$ and that ν covers θ. □

If $N \lhd G$ and $\chi \in \mathrm{Irr}(B)$ is such that $\chi_N = \theta \in \mathrm{Irr}(N)$, then by Gallagher's correspondence (Theorem 1.11), $\mathrm{Irr}(G|\theta) = \{\eta\chi \mid \eta \in \mathrm{Irr}(G/N)\}$, so it is a natural question to study the relation between B and the blocks $\overline{B}$ of G/N such that there is some $\eta \in \mathrm{Irr}(\overline{B})$ with $\eta\chi \in \mathrm{Irr}(B)$. M. Murai studied this topic in depth (see [**Mur96**] and [**Mur98**]). We now state a particular case of a result of Murai.

Theorem 4.5 (Murai). *Let $N \lhd G$, let $B \in \mathrm{Bl}(G)$ and $\chi \in \mathrm{Irr}(B)$ such that $\chi_N = \theta \in \mathrm{Irr}(N)$. Let D be a defect group of B, let $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ its Brauer correspondent and assume that $N \subseteq D$. Write $\mathrm{Bl}(B, \chi)$ for the set of blocks $\overline{B}$ of G/N containing a character η such that $\eta\chi \in \mathrm{Irr}(B)$, and $\mathrm{Bl}(b, \chi_{\mathbf{N}_G(D)})$ for the set of blocks $\overline{b}$ of $\mathbf{N}_G(D)/N$ containing a character σ such that $\sigma\chi_{\mathbf{N}_G(D)} \in \mathrm{Irr}(b)$. Then the correspondence from Brauer's first main theorem defines a bijection $\mathrm{Bl}(b, \chi_{\mathbf{N}_G(D)}) \to \mathrm{Bl}(B, \chi)$.*

Proof. This is proved in [**Mur98**, Corollary 2.5] in the more general case where N is not necessarily contained in D, although the fact that the bijection is defined by Brauer's first main theorem (Theorem 1.44) is only mentioned in the proof. □

The last result of this section is a generalization of the main result of [**Nav03**] whose proof has been kindly provided to us by Navarro. We will need a result on coprime action which also appeared in [**Nav03**].

Lemma 4.6. *Suppose that A is a finite group acting coprimely on a finite group G, and let H be an A-invariant subgroup of G. Let $C = \mathbf{C}_G(A)$. Then $|C : C \cap H|$ divides $|G : H|$.*

Proof. See [**Nav03**, Lemma 2.1]. □

Proposition 4.7 (Navarro). *Let G be a p-solvable group, $U \leqslant G$ and consider a p-subgroup P of G such that $P \cap U \in \mathrm{Syl}_p(U)$. Then $|U : \mathbf{N}_U(P \cap U)|$ divides $|G : \mathbf{N}_G(P)|$.*

Proof. We argue by induction, first on $|G : U|$, and then on $|G|$. Set $K = \mathbf{O}_p(G)$ and $Q = P \cap U$. Suppose that K is not contained in U and define $U_0 = UK$ and $P_0 = PK$. Since $U \cap P \leqslant U \cap P_0$ and $U \cap P_0$ is a p-subgroup of U, we deduce that $U \cap P_0 = U \cap P$. Then, by Dedekind's Lemma we conclude that $U_0 \cap P_0 \in \mathrm{Syl}_p(U_0)$. Noticing that $|G : U_0| < |G : U|$, the inductive hypothesis implies that $|U_0 : \mathbf{N}_{U_0}(P_0 \cap U_0)|$ divides $|G : \mathbf{N}_G(P_0)|$ which divides $|G : \mathbf{N}_G(P)|$. Now, the result follows since $|U_0 : \mathbf{N}_{U_0}(P_0 \cap U_0)| = |U : \mathbf{N}_U(Q)|$ and hence we may assume that $K \leqslant U$.

Now using bar notation, write $\overline{G} = G/K$. Since $P \cap U \in \mathrm{Syl}_p(G)$ it follows that $K \leqslant P \cap U$. Using that $K \leqslant U$ we have $|\overline{G} : \overline{U}| = |G : U|$ and $|\overline{G}| < |G|$, so by inductive hypothesis and using that $K \leqslant P \cap U$ we have $|U : \mathbf{N}_U(P \cap U)| = |\overline{U} : \mathbf{N}_{\overline{U}}(\overline{P \cap U})|$ divides $|\overline{G} : \mathbf{N}_{\overline{G}}(\overline{P})| = |G : \mathbf{N}_G(P)|$. We may assume that $K = 1$.

Since G is p-solvable, it follows that $L = \mathbf{O}_{p'}(G) > 1$. Then, by applying the inductive hypothesis to G/L, we deduce that $|UL : \mathbf{N}_{UL}(QL)|$ divides $|G : \mathbf{N}_G(P)L|$. Because $|UL : \mathbf{N}_{UL}(QL)| = |U : \mathbf{N}_U(Q)(L \cap U)|$, it suffices to show that $|\mathbf{N}_U(Q)(L \cap U) : \mathbf{N}_U(Q)|$ divides $|L\mathbf{N}_G(P) : \mathbf{N}_G(P)|$, or equivalently, that $|L \cap U : \mathbf{C}_{L\cap U}(Q)|$ divides $|L : \mathbf{C}_L(P)|$. By Lemma 4.6, we have that $|\mathbf{C}_L(Q) : \mathbf{C}_{L\cap U}(Q)|$ divides $|L : L \cap U|$, and it follows that $|L \cap U : \mathbf{C}_{L\cap U}(Q)|$ divides $|L : \mathbf{C}_L(Q)|$ which divides $|L : \mathbf{C}_L(P)|$. This concludes the proof. □

Notice that the above proposition immediately implies the main result of [**Nav03**]. In fact, if we choose $P \in \mathrm{Syl}_p(G)$ such that $P \cap U \in \mathrm{Syl}_p(U)$, then Proposition 4.7 shows that the number of Sylow p-subgroups of U, i.e. $|U : \mathbf{N}_U(U \cap P)|$, divides the number of Sylow p-subgroups of G, i.e. $|G : \mathbf{N}_G(P)|$.

Before proceeding further, we remark that the assumptions made in the above proposition are necessary. First, consider $G = \mathfrak{A}_5$, $U = \mathfrak{A}_4$ and $P \in \mathrm{Syl}_3(U)$. In this case, $|U : \mathbf{N}_U(P)| = 4$ does not divide $|G : \mathbf{N}_G(P)| = 10$. This shows that the result fails if G is not p-solvable. Next, let $G = \mathfrak{S}_4$, $U \in \mathrm{Syl}_2(G)$ and consider a subgroup P of order 2 in $\mathbf{O}_2(G)$. Now $|G : \mathbf{N}_G(P)| = 3$ might not be divisible by $|U : \mathbf{N}_U(P)|$ since P is not necessarily normal in U. Therefore, the hypothesis $P \cap U \in \mathrm{Syl}_p(U)$ is also necessary.

4.3. Theorem G

We now proceed to prove Theorem G by induction on the order of G. First, we show that without loss of generality we may assume B to be a block of maximal defect.

Lemma 4.8. *Let $L = \mathbf{O}_{p'}(G)$. In the situation of Theorem G, we may assume that B covers a G-invariant $\theta \in \mathrm{Irr}(L)$. In particular, we have $D \in \mathrm{Syl}_p(G)$, $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$ and $\mathrm{IBr}(B) = \mathrm{IBr}(G|\theta)$.*

Proof. Consider a character $\theta \in \mathrm{Irr}(L)$ whose block is covered by B. Without loss of generality we may assume that θ is (B, D)-good. If θ is G-invariant, then the result follows from Theorem 1.66. Thus, we assume $G_\theta < G$ and consider a subgroup $U \leqslant G_\theta$ and a U-invariant character $\nu \in \mathrm{Irr}(\mathbf{O}_{p'}(U))$ as in Proposition

4.4. By Theorem 1.66 there exists a unique block C of U covering $\mathrm{bl}(\nu)$ and we have $\mathrm{Irr}(C) = \mathrm{Irr}(U|\nu)$ and $\mathrm{IBr}(C) = \mathrm{IBr}(U|\nu)$. Furthermore, if c is the Brauer correspondent of C in $\mathbf{N}_U(D)$ and ν^* is the D-Glauberman correspondent of ν, then $\mathrm{Irr}(c) = \mathrm{Irr}(\mathbf{N}_U(D)|\nu^*)$ and $\mathrm{IBr}(c) = \mathrm{IBr}(\mathbf{N}_U(D)|\nu^*)$ according to Theorem 1.67. Since $U < G$, the inductive hypothesis implies that there exist bijections

$$\Omega_0 : \mathrm{Irr}_{p'}(U|\nu) \to \mathrm{Irr}_{p'}(\mathbf{N}_U(D)|\nu^*)$$

and

$$\Psi_0 : \mathrm{IBr}_{p'}(U|\nu) \to \mathrm{IBr}_{p'}(\mathbf{N}_U(D)|\nu^*)$$

such that $\Omega_0(\psi)(1)$ divides $\psi(1)$ and $\Psi_0(\eta)(1)$ divides $\eta(1)$ for every $\psi \in \mathrm{Irr}_{p'}(U|\nu)$ and every $\eta \in \mathrm{IBr}_{p'}(U|\nu)$. Next, by applying Proposition 4.4, we obtain bijections

$$\Omega : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b) \text{ and } \Psi : \mathrm{IBr}_0(B) \to \mathrm{IBr}_0(b)$$

by setting

$$\Omega(\psi^G) = \Omega_0(\psi)^{\mathbf{N}_G(D)} \text{ and } \Psi(\eta^G) = \Psi_0(\eta)^{\mathbf{N}_G(D)}$$

for every $\psi \in \mathrm{Irr}_{p'}(U|\nu)$ and every $\eta \in \mathrm{IBr}_{p'}(U|\nu)$. Furthermore, by Proposition 4.7 we conclude that

$$\Omega(\psi^G)(1) = |\mathbf{N}_G(D) : \mathbf{N}_U(D)|\Omega_0(\psi)(1)$$

divides

$$|G : U|\psi(1) = \psi^G(1)$$

and, similarly, that $\Psi(\eta^G)(1)$ divides $\eta^G(1)$. □

Next, we show that it is no loss of generality to assume $K = \mathbf{O}_p(G) = 1$.

Lemma 4.9. *Let $\kappa \in \mathrm{Irr}(K)$ and suppose that $\mathrm{Irr}_0(B|\kappa)$ is nonempty. Then κ is linear and extends to its stabiliser in G.*

Proof. First, observe that B is a block of maximal defect by Lemma 4.8 and hence $\mathrm{Irr}_0(B) = \mathrm{Irr}_{p'}(B)$. In particular, the character κ must be linear. Next, let $\chi \in \mathrm{Irr}_{p'}(B|\kappa)$ and denote by $\psi \in \mathrm{Irr}(G_\kappa)$ its Clifford correspondent over κ (see Theorem 1.10). Since p does not divide the degree of χ, it follows that $\psi \in \mathrm{Irr}_{p'}(G_\kappa)$ and that G_κ contains some Sylow p-subgroup P of G. Then, the restriction ψ_P contains some linear constituent which must be an extension of κ. Furthermore, if $Q/K \in \mathrm{Syl}_q(G_\kappa/K)$ for a prime $p \neq q$ then κ extends to Q by Theorem 1.23. By Theorem 1.22 we conclude that κ extends to G_κ. □

We will make use of the following result.

Lemma 4.10. *Let $P \in \mathrm{Syl}_p(G)$, $L \lhd G$ and let $\chi \in \mathrm{Irr}_{p'}(G)$. Then χ_L has a P-invariant irreducible constituent and any two of them are $\mathbf{N}_G(P)$-conjugate.*

Proof. See [**Nav18**, Lemma 9.3]. □

We now construct a bijection for ordinary characters.

Proposition 4.11. *If $K > 1$, then there exists a bijection $\Omega : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ such that $\Omega(\chi)(1)$ divides $\chi(1)$ for every $\chi \in \mathrm{Irr}_0(B)$.*

Proof. Consider an $\mathbf{N}_G(D)$-transversal $\mathbb{T}$ in the set of D-invariant linear characters $\mathrm{Lin}(K)$. Using Lemma 4.10, we obtain

$$\mathrm{Irr}_0(B) = \coprod_{\kappa \in \mathbb{T}} \mathrm{Irr}_0(B|\kappa)$$

and

$$\mathrm{Irr}_0(b) = \coprod_{\kappa \in \mathbb{T}} \mathrm{Irr}_0(b|\kappa).$$

Therefore, it is enough to find a bijection $\mathrm{Irr}_0(B|\kappa) \to \mathrm{Irr}_0(b|\kappa)$ with the divisibility of character degrees for every $\kappa \in \mathbb{T}$.

Fix $\kappa \in \mathbb{T}$ and set $c = \mathrm{bl}(\kappa)$. Consider the set $\mathcal{B}$ of blocks $B'' \in \mathrm{Bl}(G_\kappa)$ that cover c and such that $(B'')^G = B$. Similarly, define $\mathcal{C}$ to be the set of blocks $b'' \in \mathrm{Bl}(\mathbf{N}_G(D)_\kappa)$ that cover c and such that $(b'')^{\mathbf{N}_G(D)} = b$. Notice that $\mathcal{B}$ coincides with the set of blocks containing the Clifford correspondent χ_κ of some $\chi \in \mathrm{Irr}(B)$ over κ. Moreover, since κ is D-invariant and $D \in \mathrm{Syl}_p(G)$, it follows that D is a defect group of every block $B'' \in \mathcal{B}$. Similarly, $\mathcal{C}$ coincides with the set of blocks containing the Clifford correspondent ψ_κ of some $\psi \in \mathrm{Irr}(b)$ over κ. As before, D is a defect group of every block $b'' \in \mathcal{C}$. We claim that induction of blocks defines a bijection

$$\begin{aligned} \mathcal{C} &\to \mathcal{B} \\ b'' &\mapsto (b'')^{G_\kappa}. \end{aligned}$$

First, observe that the map is well defined and, by Brauer's first main theorem (Theorem 1.44), injective. Then, let $B'' \in \mathcal{B}$ and consider its Brauer correspondent $b'' \in \mathrm{Bl}(\mathbf{N}_G(D)_\kappa)$. Since B'' covers $c = \mathrm{bl}(\kappa)$ and κ is G_κ-invariant, we deduce that b'' also covers c. Recalling that $(B'')^G = B$, by the Brauer correspondence we conclude that $(b'')^{\mathbf{N}_G(D)}$ coincides with b. Therefore $b'' \in \mathcal{C}$ and the map is surjective. Now, by the Clifford correspondence we obtain bijections

$$\coprod_{B'' \in \mathcal{B}} \mathrm{Irr}(B''|\kappa) \to \mathrm{Irr}(B|\kappa) \tag{4.3.1}$$

and

$$\coprod_{b'' \in \mathcal{C}} \mathrm{Irr}(b''|\kappa) \to \mathrm{Irr}(b|\kappa). \tag{4.3.2}$$

Since these bijections are given by induction of characters, and recalling that $D \leqslant G_\kappa$, it follows that (4.3.1) and (4.3.2) restrict to the sets of irreducible characters of height zero, or equivalently, of p'-degree. Now, for all $\chi \in \mathrm{Irr}_0(B)$ there is a unique $B'' \in \mathcal{B}$ and a unique $\psi \in \mathrm{Irr}_0(B'')$ such that $\psi^G = \chi$. If there exist bijections $\Omega_{B''} : \mathrm{Irr}_0(B'') \to \mathrm{Irr}_0(b'')$ such that $\Omega_{B''}(\psi)(1)$ divides $\psi(1)$, then we can define a bijection

$$\Omega : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$$

by setting

$$\Omega(\chi) = \Omega_{B''}(\psi)^{\mathbf{N}_G(D)}$$

and we have $\Omega(\chi)(1) = \Omega_{B''}(\psi)(1)|\mathbf{N}_G(D) : \mathbf{N}_{G_\kappa}(D)|$ which divides $\psi(1)|G : G_\kappa| = \chi(1)$ by Proposition 4.7. Therefore, in order to conclude, it is no loss of generality to assume that κ is G-invariant.

Notice that κ has an extension $\widehat{\kappa}$ to the inertial subgroup G_κ by Lemma 4.9. By Gallagher's theorem (Theorem 1.11), for every character $\chi \in \mathrm{Irr}(G|\kappa)$, there exists a unique $\eta \in \mathrm{Irr}(G/K)$ such that $\chi = \eta\widehat{\kappa}$. Furthermore, if η_1 and η_2 lie in the same block, then so do $\eta_1\widehat{\kappa}$ and $\eta_2\widehat{\kappa}$ (see Lemma 1.51). We denote by $\mathrm{Bl}(B, \widehat{\kappa})$ the set of blocks of G/K containing a character η such that $\eta\widehat{\kappa} \in \mathrm{Irr}(B)$. Then

$$\mathrm{Irr}_0\left(B|\kappa\right) = \coprod_{\overline{B} \in \mathrm{Bl}(B,\widehat{\kappa})} \left\{\eta\widehat{\kappa} \;\middle|\; \eta \in \mathrm{Irr}_0\left(\overline{B}\right)\right\}. \tag{4.3.3}$$

Similarly, if $\mathrm{Bl}\left(b, \widehat{\kappa}_{\mathbf{N}_G(D)}\right)$ is the set of blocks of $\mathbf{N}_G(D)/K$ containing a character η such that $\eta\widehat{\kappa}_{\mathbf{N}_G(D)} \in \mathrm{Irr}(b)$, then we obtain

$$\mathrm{Irr}_0\left(b|\kappa\right) = \coprod_{\overline{b} \in \mathrm{Bl}(b,\widehat{\kappa}_{\mathbf{N}_G(D)})} \left\{\sigma\widehat{\kappa}_{\mathbf{N}_G(D)} \;\middle|\; \sigma \in \mathrm{Irr}_0\left(\overline{b}\right)\right\}. \tag{4.3.4}$$

By Theorem 4.5, induction of blocks yields a bijection $\mathrm{Bl}(b, \widehat{\kappa}_{\mathbf{N}_G(D)}) \to \mathrm{Bl}(B, \widehat{\kappa})$ which maps a block to its Brauer correspondent. Now, as $K > 1$, the inductive hypothesis yields a bijection

$$\overline{\Omega}_{\overline{B}} : \mathrm{Irr}_0\left(\overline{B}\right) \to \mathrm{Irr}_0\left(\overline{b}\right)$$

such that $\overline{\Omega}_{\overline{B}}(\overline{\chi})(1)$ divides $\overline{\chi}(1)$ for every $\overline{B} \in \mathrm{Bl}(B, \widehat{\kappa})$ with Brauer correspondent $\overline{b} \in \mathrm{Bl}(b, \widehat{\kappa}_{\mathbf{N}_G(D)})$ and for every $\overline{\chi} \in \mathrm{Irr}_0(\overline{B})$. Finally, we define

$$\Omega\left(\chi\right) = \overline{\Omega}_{\overline{B}}\left(\eta\right)\widehat{\kappa}_{\mathbf{N}_G(D)}$$

for every $\chi \in \mathrm{Irr}(B|\kappa)$, $\eta \in \mathrm{Irr}(G/K)$ and $\overline{B}$ such that $\chi = \eta\widehat{\kappa}$ and $\mathrm{bl}(\eta) = \overline{B}$. Thanks to (4.3.3) and (4.3.4), this defines a bijection between $\mathrm{Irr}_0(B|\kappa)$ and $\mathrm{Irr}_0(b|\kappa)$. Furthermore, since $\overline{\Omega}_{\overline{B}}(\eta)(1)$ divides $\eta(1)$, it follows that $\Omega(\chi)(1)$ divides $\chi(1)$. This completes the proof. □

In the next results, if $K \lhd G$ we write $\mathrm{Bl}(G/K, B)$ for the set of blocks of G/K dominated by B.

LEMMA 4.12 (Navarro–Späth). *Let $K \lhd G$, let $K \leqslant H \leqslant G$ and write $\overline{G} = G/K$. Let $\overline{b} \in \mathrm{Bl}(\overline{H}, b)$ and let $\overline{B} \in \mathrm{Bl}(\overline{G}, B)$. If $\overline{b}^{\overline{G}}$ is defined and coincides with $\overline{B}$ then b^G is defined and coincides with B.*

Proof. See [**NS14**, Proposition 2.4(a)]. ☐

Lemma 4.13. *Let K be a normal p-subgroup of G, let B be a block of G with defect group D and Brauer correspondent $b \in \mathrm{Bl}(\mathbf{N}_G(D))$. Write $\overline{G} = G/K$. Then block induction defines a bijection* $\mathrm{Bl}(\overline{\mathbf{N}_G(D)}, b) \to \mathrm{Bl}(\overline{G}, B)$.

Proof. Since $K \subseteq D$ by Theorem 1.42, we have that $\overline{\mathbf{N}_G(D)} = \mathbf{N}_{\overline{G}}(\overline{D})$ and $\overline{D}$ is a defect group of every block in $\mathrm{Bl}(\overline{\mathbf{N}_G(D)}, b)$ by part (ii) of Theorem 1.49. By Brauer's first main theorem (Theorem 1.44) block induction defines an injection

$$\mathrm{Bl}(\overline{\mathbf{N}_G(D)}, b) \to \mathrm{Bl}(\overline{G}).$$

Now let $\overline{b} \in \mathrm{Bl}(\overline{\mathbf{N}_G(D)}, b)$ and assume $\overline{b}^{\overline{G}} = \overline{B'} \notin \mathrm{Bl}(\overline{G}, B)$ and let $B' \neq B$ be the block of G dominating $\overline{B'}$. Then by Lemma 4.12 we have $b^G = B' \neq B$, a contradiction. Conversely, if $\overline{B} \in \mathrm{Bl}(\overline{G}, B)$ then its Brauer correspondent must be dominated by b, again by Lemma 4.12. Thus block induction restricts to a bijection

$$\mathrm{Bl}(\overline{\mathbf{N}_G(D)}, b) \to \mathrm{Bl}(\overline{G}, B)$$

and we are done. ☐

Next, we consider the case of Brauer characters.

Proposition 4.14. *If $K > 1$, then there exists a bijection $\Psi : \mathrm{IBr}_0(B) \to \mathrm{IBr}_0(b)$ such that $\Psi(\varphi)(1)$ divides $\varphi(1)$ for every $\varphi \in \mathrm{IBr}_0(B)$.*

Proof. Define $\overline{G} = G/K$. By Lemma 4.13, block induction defines a bijection $\mathrm{Bl}(\overline{\mathbf{N}_G(D)}, b) \to \mathrm{Bl}(\overline{G}, B)$. Let $\overline{B} \in \mathrm{Bl}(\overline{G}, B)$ and consider its Brauer correspondent $\overline{b} \in \mathrm{Bl}(\overline{\mathbf{N}_G(D)})$. Noticing that $|\overline{G}| < |G|$, by inductive hypothesis we obtain a bijection $\overline{\Psi}_{\overline{B}} : \mathrm{IBr}_0(\overline{B}) \to \mathrm{IBr}_0(\overline{b})$ such that $\overline{\Psi}_{\overline{B}}(\overline{\varphi})(1)$ divides $\overline{\varphi}(1)$ for every $\overline{\varphi} \in \mathrm{IBr}_0(\overline{B})$ (see Theorem 1.32). Since $K \leqslant \ker(\varphi)$ for every $\varphi \in \mathrm{IBr}(G)$, we obtain a bijection

$$\Psi : \mathrm{IBr}_0(B) \to \mathrm{IBr}_0(b)$$

by setting

$$\Psi(\overline{\varphi}) = \overline{\Psi}_{\overline{B}}(\overline{\varphi})$$

for every $\overline{B} \in \mathrm{Bl}(\overline{G}, B)$, every $\overline{\varphi} \in \mathrm{IBr}_0(\overline{B})$ and where we identify $\overline{\varphi}$ and $\overline{\Psi}_{\overline{B}}(\varphi)$ via inflation with the corresponding Brauer characters of G and $\mathbf{N}_G(D)$ respectively. Since inflation of characters does not affect character degrees, we conclude that $\Psi(\varphi)(1)$ divides $\varphi(1)$ for every $\varphi \in \mathrm{IBr}_0(B)$. ☐

As an immediate consequence of Proposition 4.11 and Proposition 4.14 we deduce that it is no loss of generality to assume $K = 1$.

Corollary 4.15. *We may assume that $\mathbf{O}_p(G) = 1$.*

We are now ready to prove Theorem G.

Proof of Theorem G. Let $N = \mathbf{O}_{p',p}(G)$ and $L = \mathbf{O}_{p'}(G)$. By Lemma 4.8 we know that $D \in \mathrm{Syl}_p(G)$ and that there exists some G-invariant character θ of L whose block is covered by B. In particular, we deduce that $Q = N \cap D$ is a Sylow p-subgroup of N and $N = LQ$. If $Q \lhd G$, then Corollary 4.15 implies that $Q = 1$ and hence G is a p'-group, in which case the result follows trivially. Therefore, we may assume that $\mathbf{N}_G(Q) < G$.

Observe that $\mathbf{N}_G(D) \leqslant \mathbf{N}_G(Q)$ and consider the block $C = b^{\mathbf{N}_G(Q)}$. The inductive hypothesis yields bijections

$$\Omega_C : \mathrm{Irr}_0(C) \to \mathrm{Irr}_0(b)$$

and

$$\Psi_C : \mathrm{IBr}_0(C) \to \mathrm{IBr}_0(b)$$

such that $\Omega_C(\psi)(1)$ divides $\psi(1)$ and $\Psi_C(\varrho)(1)$ divides $\varrho(1)$ for every $\psi \in \mathrm{Irr}_0(C)$ and every $\varrho \in \mathrm{IBr}_0(C)$. Next, recall that $\mathrm{Irr}_0(B) = \mathrm{Irr}_{p'}(G|\theta)$ and that $\mathrm{IBr}_0(B) = \mathrm{IBr}_{p'}(G|\theta)$ according to Lemma 4.8. Furthermore, Theorem 1.67 implies that $\mathrm{Irr}_0(C) = \mathrm{Irr}_{p'}(\mathbf{N}_G(Q)|\theta^*)$ and that $\mathrm{IBr}_0(C) = \mathrm{Irr}_{p'}(\mathbf{N}_G(Q)|\theta^*)$, where θ^* is the Q-Glauberman correspondent of θ. Then, applying Proposition 4.3 we obtain bijections

$$\Omega_Q : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(C)$$

and

$$\Psi_Q : \mathrm{IBr}_0(B) \to \mathrm{IBr}_0(C)$$

such that $\Omega_Q(\chi)(1)$ divides $\chi(1)$ and $\Psi_Q(\varphi)(1)$ divides $\varphi(1)$ for every $\chi \in \mathrm{Irr}_0(B)$ and $\varphi \in \mathrm{IBr}_0(B)$. Now, the result follows by setting $\Omega = \Omega_C \circ \Omega_Q$ and $\Psi = \Psi_C \circ \Psi_Q$. □

4.4. Theorem H

By [**Nav98**, Theorem 3.14], if B is a block of a finite group G, then B as an FG-module affords the Brauer character

$$\rho_B = \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\chi^0.$$

Thus,

$$\dim(B) = \rho_B(1) = \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)^2.$$

Using this equality, together with Proposition 4.4 and Proposition 4.7, we can prove Theorem H.

Proof of Theorem H. We argue by induction on the order of G. Let U and $\nu \in \mathrm{Irr}(\mathbf{O}_{p'}(U))$ be as in Proposition 4.4 and observe that

$$\dim(B) = \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)^2 = \sum_{\psi \in \mathrm{Irr}(U|\nu)} \psi^G(1)^2 = \sum_{\psi \in \mathrm{Irr}(U|\nu)} |G:U|^2 \psi(1)^2. \tag{4.4.1}$$

Assume first that $U < G$. By the inductive hypothesis, and using the fact that $\mathrm{Irr}(U|\nu)$ is a full block $B' \in \mathrm{Bl}(U)$ by Theorem 1.66, we deduce that

$$\dim(B') = \sum_{\psi \in \mathrm{Irr}(U|\nu)} \psi(1)^2$$

is divisible by

$$\dim(b') = \sum_{\varphi \in \mathrm{Irr}(\mathbf{N}_U(D)|\nu^*)} \varphi(1)^2$$

where $b' \in \mathrm{Bl}(\mathbf{N}_U(D))$ is the Brauer correspondent of B' and $\mathrm{Irr}(b') = \mathrm{Irr}(\mathbf{N}_U(D)|\nu^*)$ according to Theorem 1.67. We conclude that

$$\begin{aligned}\dim(b) = \sum_{\varrho \in \mathrm{Irr}(b)} \varrho(1)^2 &= \sum_{\varphi \in \mathrm{Irr}(\mathbf{N}_G(D)|\nu^*)} \varphi^{\mathbf{N}_G(D)}(1)^2 \\ &= |\mathbf{N}_G(D) : \mathbf{N}_U(D)|^2 \sum_{\varphi \in \mathrm{Irr}(\mathbf{N}_U(D)|\nu^*)} \varphi(1)^2\end{aligned}$$

divides $\dim(B)$ by (4.4.1) and Proposition 4.7.

Thus, we may assume $U = G$. In this case, if $L = \mathbf{O}_{p'}(G)$, there exists some G-invariant character $\theta \in \mathrm{Irr}(L)$ such that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$ and $\mathrm{Irr}(b) = \mathrm{Irr}(\mathbf{N}_G(D)|\theta^*)$. Now

$$|G:L|\theta(1) = \theta^G(1) = \frac{1}{\theta(1)} \sum_{\chi \in \mathrm{Irr}(G|\theta)} \chi(1)^2$$

and

$$|\mathbf{N}_G(D) : \mathbf{C}_L(D)|\theta^*(1) = (\theta^*)^{\mathbf{N}_G(D)}(1) = \frac{1}{\theta^*(1)} \sum_{\varphi \in \mathrm{Irr}(\mathbf{N}_G(D)|\theta^*)} \varphi(1)^2$$

so $\dim(B) = |G:L|\theta(1)^2$ and $\dim(b) = |\mathbf{N}_G(D) : \mathbf{C}_L(D)|\theta^*(1)^2$. To conclude, observe that $|\mathbf{N}_G(D) : \mathbf{C}_L(D)|$ divides $|G:L|$ since $\mathbf{C}_L(D) = L \cap \mathbf{N}_G(D)$ and that $\theta^*(1)$ divides $\theta(1)$ by Theorem 1.15. □

4.5. Some consequences of Theorem G

4.5.1. Nilpotent blocks. Let B be a block of a p-solvable finite group G with defect group D and let b be its Brauer correspondent. Assume that $l(B) = 1$ and write $\{\varphi\} = \mathrm{IBr}(B)$. Then it follows from Theorem G that $l(b) = 1$ and if φ^* is the unique irreducible Brauer character of b then $\varphi^*(1)$ divides $\varphi(1)$.

If B is a nilpotent block, as defined in [**BP80**] then $l(B) = 1$ by [**BP80**, Theorem 1.2]. In such blocks, Broué and Puig also proved in [**BP80**, Theorem 1.2] that there is a bijection

$$\mathrm{Irr}(D) \to \mathrm{Irr}(B)$$
$$\eta \mapsto \chi \star \eta$$

where χ is any height zero character in $\mathrm{Irr}(B)$, and $(\chi \star \eta)(1) = \chi(1)\eta(1)$. Moreover, it follows from the definition of nilpotent block that if B is nilpotent then so is b.

Assuming that G is p-solvable, then fixing $\chi \in \mathrm{Irr}_0(B)$, our Theorem G combined with the bijection of Broué and Puig gives a correspondence defined by

$$\mathrm{Irr}(B) \to \mathrm{Irr}(b)$$
$$\chi \star \eta \mapsto \Omega(\chi) \star \eta$$

(here we abuse notation by using the star construction in B and in b). Under this correspondence, there is also degree divisibility. Also, $\chi^0 = \varphi$ is the unique Brauer character in B, whose degree must divide the degree of the unique Brauer character in b, which is $\Omega(\chi)^0$. Notice that this agrees with the phenomenon explained in the previous paragraph for blocks of p-solvable groups with one Brauer character.

For arbitrary groups, this does not hold, as shown by the following example due to D. A. Craven. Let $S = \mathfrak{S}_{23}$ and $p = 2$, and let χ be the character defined by the partition $(8, 5, 4, 3, 2, 1)$ has degree 21422145536 and lifts a Brauer character φ. Its correspondent φ^* has degree 1100742656.

4.5.2. The Dade–Glauberman–Nagao correspondence. Let $N \lhd G$ and assume that G/N is a p-group. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant and assume that $\theta(1)_p = |N|_p$ (we say that θ has defect zero). Then $\mathrm{Irr}(\mathrm{bl}(\theta)) = \{\theta\}$ and the defect groups of $\mathrm{bl}(\theta)$ are trivial. Now by Theorem 1.57 there is a unique block B of G covering $\mathrm{bl}(\theta)$. By Theorem 1.60, if D is a defect group of B then $DN = G$ and $D \cap N = 1$. Now $\mathbf{N}_G(D) = \mathbf{C}_N(D) \times D$. By Brauer's first main theorem (Theorem 1.44), there is a unique block $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ with defect group D inducing B. Again by Theorem 1.57, b covers a unique block $b^* \in \mathrm{Bl}(\mathbf{C}_N(D))$ with must have trivial defect group by Theorem 1.60. Thus $\mathrm{Irr}(b^*) = \{\theta^*\}$ for a unique $\theta^* \in \mathrm{Irr}(\mathbf{C}_N(D))$ and θ^* also has defect zero.

The map $\theta \mapsto \theta^*$ is known as the Dade–Glauberman–Nagao correspondence, and it has deep connections with the reduction theorems of the Alperin weight conjecture and its blockwise version (see [**NT11**, Section 4], [**Spa13a**, Section 3]). It generalizes the Glauberman correspondence, in the sense that if N is a p'-group, then θ^* is the D-Glauberman correspondent of θ.

Now, the blocks b^* and $\mathrm{bl}(\theta)$ described are nilpotent, and using that G/N is a p-group, we have that B and b are also nilpotent by [**KP90**, Proposition 6.5].

Furthermore, $(\theta)^0$ and $(\theta^*)^0$ extend to $\varphi \in \mathrm{IBr}(B)$ and $\varphi^* \in \mathrm{IBr}(b)$ respectively by Green's theorem [**Nav98**, Theorem 8.11].

By the previous section, if G (or equivalently N) is p-solvable then $\varphi^*(1)$ divides $\varphi(1)$ and so we have that, as a consequence of Theorem G, $\theta^*(1)$ divides $\theta(1)$, so there is also divisibility of degrees in the Dade–Glauberman–Nagao correspondence for p-solvable groups.

Again, this divisibility does not hold in general, as shown by the previous example of Craven. Indeed, the character χ restricts irreducibly to a 2-defect zero character of $\mathfrak{A}_{23}$.

Bibliography

[A86] M. Aschbacher, *Finite Group Theory*, Cambridge University Press, 1986.

[BNRS22] L. Bonazzi, G. Navarro, N. Rizo, L. Sanus, Brauer character degrees and Sylow normalizers, *Ann. Mat. Pura. Appl. (4)* **201**(6) (2022), 2575–2582.

[Bra82] J. Brandt, A lower bound for the number of irreducible characters in a block, *J. Algebra* **74**:2 (1982), 509–515.

[B71] R. Brauer, Some applications of the theory of blocks of characters of finite groups. IV. *J. Algebra* **17** (1971), 489–521.

[BF59] R. Brauer, W. Feit, On the number of irreducible characters of finite groups in a given block, *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959), 361–365.

[BP80] M. Broué, L. Puig, A Frobenius theorem for blocks, *Invent. Math.* **56** (1980), no.2, 117–128.

[Cab18] M. Cabanes, Local methods for blocks of finite simple groups. Local Representation Theory and Simple Groups, 179–265, *EMS Ser. Lect. Math., Eur. Math. Soc.*, Zürich, 2018.

[Cam81] P. J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), 1–22.

[CDPS16] C. Casolo, S. Dolfi, E. Pacifici, L. Sanus, Groups whose character degree graph has diameter three, *Israel J. Math* **215**:2 (2016), 523–558.

[Dad66] E. C. Dade, Blocks with cyclic defect groups, *Ann. of Math. (2)*, **84**, No. 1 (1966), 20–48.

[Dor72] L. Dornhoff, *Group Representation Theory, Part A*, Marcel Dekker, 1972.

[Fei93] W. Feit, Extending Steinberg characters, *Linear algebraic groups and their representations (Los Angeles, CA, 1992)*, Contemp. Math., **153** Amer. Math. Soc., Providence, RI, (1993), 1–9.

[FLLMZ19] Z. Feng, C. Li, Y. Liu, G. Malle, J. Zhang, Robinson's conjecture on character heights, *Compositio Math.* **155** (2019), 1098–1117.

[FLZ20] Z. Feng, Y. Liu and J. Zhang, On heights of characters of finite groups, *J. Algebra* **556** (2020), 106–135.

[Fou69] D. A. Foulser, Solvable primitive permutation groups of low rank, *Trans. Amer. Math. Soc.*, **143** (1969), 1–54.

[GAP] The GAP Group, '*GAP – Groups, Algorithms, and Programming,*', Version 4.11.1; 2021.

[Gec20] M. Geck, Green functions and Glauberman degree-divisibility, *Ann. of Math. (2)* **192** (2020), 229–249.

[GM20] M. Geck, G. Malle, *The character theory of finite groups of Lie type*, Cambridge University Press, 2020.

[GMS22] E. Giannelli, J. M. Martínez, A. A. Schaeffer Fry, Character degrees in blocks and defect groups, *J. Algebra* **594** (2022), 170–193.

[GRSS20] E. Giannelli, N. Rizo, B. Sambale, A. A. Schaeffer Fry, Groups with few p'-character degrees in the principal block, *Proc. Amer. Math. Soc.* **148** (2018), 4597–4614.

[GMRS04] D. Gluck, K. Magaard, U. Riese, P. Schmidt, The solution to the $k(GV)$-problem, *J. Algebra* **279** (2004), 694–719.

[GLS94] D. Gorenstein, R. Lyons, and R. Solomon, *The Classification of the Finite Simple Groups*, Math. Surveys Monogr., vol. 3, Amer. Math. Soc., Providence, 1994.

[HT94] B. Hartley, A. Turull, On characters of coprime operator groups and the Glauberman character correspondence, *J. Reine Angew. Math.* **451** (1994), 175–219.

[HK00] L. Héthelyi, B. Külshammer, On the number of conjugacy classes of a finite solvable group, *Bull. London Math. Soc.* **32** (2000), 668–672.

[Hig88] R. J. Higgs, Groups with two projective characters, *Math. Proc. Camb. Phil. Soc.*, **103** (1988), 5–14.

[HS22] N. N. Hung, A. A. Schaeffer Fry, On Héthelyi-Külshammer's conjecture for principal blocks, to appear in *Algebra Number Theory* (2022).

[Hup67] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.

[Isa71] I. M. Isaacs, The p-parts of character degrees in p-solvable groups, *Pacific J. Math.* **36** (1971), 677–691.

[Isa06] I. M. Isaacs, *Character theory of finite groups*, AMS-Chelsea, Providence, 2006.

[Isa08] I. M. Isaacs, *Finite Group Theory*, Grad. Stud. Math., vol.92, American Mathematical Society, Providence, 2008.

[IMN07] I. M. Isaacs, G. Malle, G. Navarro, A reduction theorem for the McKay conjecture, *Invent. Math* **170** (2007), 33–101.

[IS76] I. M. Isaacs, S. D. Smith, A note on groups of p-length 1, *J. Algebra* **38** (1976), 531–535.

[JK81] G. James, A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[KM13] R. Kessar, G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, *Ann. of Math. (2)* **178** (2013), 321–384.

[KS21] S. Koshitani, T. Sakurai, The principal p-blocks with four irreducible characters, *Bull. London Math. Soc.* **53**(4) (2021), 1124–1138.

[Kül84] B. Külshammer, Symmetric local algebras and small blocks of finite groups, *J. Algebra* **88** (1984), 190–195.

[Kül89] B. Külshammer, Blocks, solvable permutation groups and Landau's theorem, *J. Reine Angew. Math.* **398** (1989), 180–186.

[Kül90] B. Külshammer. Landau's theorem for p-blocks of p-solvable groups, *J. Reine Angew. Math.* **404** (1990), 189–191.

[KNST14] B. Külshammer, G. Navarro, B. Sambale, P. H. Tiep, Finite groups with two conjugacy classes of p-elements and related questions for p-blocks, *Bull. London Math. Soc.* **46**(2) (2014), 305–314.

[KP90] B. Külshammer, L. Puig, Extensions of nilpotent blocks, *Invent. Math* **102** (1990), 17–71.

[KR96] B. Külshammer, G. R. Robinson, Alperin-McKay implies Brauer's problem 21, *J. Algebra* **180** (1996), 208–210.

[KS03] H. Kurzweil, B. Stellmacher, *The theory of finite groups, an introduction*, Springer-Verlag, New York, 2003.

[Lad16] F. Ladisch. On Clifford theory with Galois action, *J. Algebra* **457** (2016), 45–72.

[Lan81] P. Landrock, On the number of irreducible characters in a 2-block, *J. Algebra* **68** (1981), 426–442.

[LM80] P. Landrock, G. O. Michler, Principal 2-blocks of the simple groups of Ree type, *Trans. Amer. Math. Soc* **260**(1) (1980), 83–111.

[Lar14] A. Laradji, On the Harris–Knörr correspondence for p-solvable groups, *J. Algebra* **405** (2014), 135–141.

[LPS88] M. W. Liebeck, C. E. Praeger, J. Saxl, On the O'Nan–Scott theorem for primitive permutation groups, *J. Austral. Math. Society* **44** (1988), 389–396.

[Lin18] M. Linckelmann, *The block theory of finite group algebras. Vol. II*, Cambridge University Press, 2018.

[LS13] Y. Liu, X. Song, The Steinberg character of finite classical groups, *Algebra Colloq.* **20**(1) (2013), pp. 163–168

[Mal08] G. Malle, Extensions of unipotent characters and the inductive McKay condition, *J. Algebra* **320** (2008), 2963–2980.

[MN11] G. Malle, G. Navarro, Blocks with equal height zero degrees, *Trans. Amer. Math. Soc.* **363** (2011), 6647–6670.

[MN21] G. Malle, G. Navarro, Brauer's height zero conjecture for principal blocks, *J. Reine Angew. Math.* **778** (2021), 119–125.

[MNST22] G. Malle, G. Navarro, A. A. Schaeffer-Fry, P. H. Tiep, Brauer's height zero conjecture, submitted (2022).

[Man00] A. Mann, The derived length of p-groups, *J. Algebra* **224** (2000),263–267.

[MW93] O. Manz, T. Wolf, *Representations of solvable groups*, Cambridge University Press, 1993.

[M21] J. M. Martínez, Degrees of characters in the principal block, *J. Algebra* **579** (2021), 195–209.

[MRS23] J. M. Martínez, N. Rizo, L. Sanus, The blocks with four irreducible characters, *Bull. London Math. Soc.* **55** (2023), 658–667.

[MR22] J. M. Martínez, D. Rossi, Degree divisibility in Alperin–McKay correspondences, submitted (2022) https://arxiv.org/pdf/2204.10124v3.

[Mar16] A. Maróti, A lower bound for the number of conjugacy classes of a finite group. *Adv. Math.* **290** (2016), 1062–1078.

[Mor04] A. Moretó, Heights of characters and defect groups, *Finite groups 2003*, Walter de Gruyter, Berlin, (2004), 267–273.

[Mur96] M. Murai, Normal subgroups and heights of characters, *J. Math. Kyoto. Univ.* **36**(1) (1996), 31–43.

[Mur98] M. Murai, Blocks of factor groups and heights of characters, *Osaka J. Math.* **35**(4) (1998), 835–854.

[Nav98] G. Navarro, *Characters and blocks of finite groups*, Cambridge University Press, 1998.

[Nav03] G. Navarro, Number of Sylow subgroups in p-solvable groups, *Proc. Amer. Math. Soc.* **131** (2003), 3019–3020.

[Nav10] G. Navarro, Problems in character theory, in: *Character theory of finite groups*, Contemp. Math. **524**, Amer. Math. Soc., Providence, RI, (2010) 97–125.

[Nav15] G. Navarro, The set of character degrees of a finite group does not determine its solvability. *Proc. Amer. Math. Soc.* **143** (2015), 989–990.

[Nav18] G. Navarro, *Character theory and the McKay conjecture*, Cambridge University Press, 2018.

[NR05] G. Navarro, G. R. Robinson, Blocks with p-power character degrees, *Proc. Amer. Math. Soc.* **133** (2005), 2845–2851.

[NRSV21] G. Navarro, N. Rizo, A. A. Schaeffer Fry, C. Vallejo, Characters and generation of Sylow 2-subgroups, *Represent. Theory* **25** (2021), 142–165.

[NS14] G. Navarro, B. Späth, On Brauer's Height Zero Conjecture, *J. Eur. Math. Soc. (JEMS)* **16** (2014), 695–747.

[NST17] G. Navarro, B. Späth, P. H. Tiep, Coprime actions and correspondences of Brauer characters, *Proc. London Math. Soc.* **114**(4) (2017), 589–613.

[NSV20] G. NAVARRO, B. SPÄTH, C. VALLEJO. A reduction theorem for the Galois–McKay conjecture. *Trans. Amer. Math. Soc.* **373**(9) (2020), 6157–6183.

[NT11] G. NAVARRO, P. H. TIEP, A reduction theorem for the Alperin weight conjecture, *Invent. Math.* **184** (2011), 529–565.

[NT12] G. NAVARRO, P. H. TIEP, Brauer's height zero conjecture for the 2-blocks of maximal defect, *J. Reine Angew. Math.* **669** (2012), 225–247.

[OT83] T. OKUYAMA AND Y. TSUSHIMA, Local properties of p-block algebras of finite groups, *Osaka J. Math.* **20** (1983), 33–41.

[Pas68] D. S. PASSMAN, p-solvable doubly transitive permutation groups, *Pacific J. Math.* **26** (1968), 555–577.

[Rey63] W. F. REYNOLDS, Blocks and normal subgroups of finite groups, *Nagoya Math. J.* **22** (1963), 15–32.

[Riz18] N. RIZO, p-Blocks relative to a character of a normal subgroup, *J. Algebra* **514** (2018), 254–272.

[Riz19] N. RIZO, Divisibility of degees in McKay correspondences, *Arch. Math. (Basel)* **112** (2019), 5–11.

[RSV20] N. RIZO, A. A. SCHAEFFER FRY, C. VALLEJO, Galois action on the principal block and cyclic Sylow subgroups, *Algebra Number Theory* **14**:7 (2020), 1953–1979.

[RSV21] N. RIZO, A. A. SCHAEFFER FRY, C. VALLEJO, Principal blocks with 5 irreducible characters, *J. Algebra* **585** (2021), 316–337.

[Ron05] C. M. RONEY-DOUGAL, The primitive permutation groups of degree less than 2500, *J. Algebra* **292** (2005), 154–183.

[Ros22] D. ROSSI, Character Triple Conjecture for p-solvable groups, *J. Algebra* **595** (2022), 165–193.

[Ros23] D. ROSSI, The McKay Conjecture and central isomorphic character triples, *J. Algebra* **618** (2023), 42–55.

[Sam14] B. SAMBALE, *Blocks of finite groups and their invariants*, Lecture Notes in Mathematics, Springer, 2014.

[Sch07] C. SCHNEIDER, The derived series of a finite p-group, *J. Algebra* **307** (2007) 136–152.

[SZ16] R. SHEN, Y. ZHOU, Finite simple groups with some abelian Sylow subgroups, *Kuwait J. Sci.* **43**(2) (2016), 1–12.

[Spa13a] B. SPÄTH, A reduction theorem for the blockwise Alperin weight conjecture, *J. Group Theory* **16** (2013), 159–220.

[Spa13b] B. SPÄTH, A reduction theorem for the Alperin–McKay conjecture, *J. Reine Angew. Math.* **680** (2013), 153–189.

[Ste60] R. STEINBERG, Automorphisms of finite linear groups. *Canadian J. Math.* **12** (1960), 606–615.

[Tur07] A. TURULL, Degree divisibility in character correspondences, *J. Algebra* **307** (2007), 300–305.

[Tur08] A. TURULL, Above the Glauberman correspondence, *Adv. Math.* **217** (2008), 2170–2205.

[Tur09] A. TURULL, The Brauer–Clifford group, *J. Algebra* **321** (2009), 3620–3642.

[Wal69] J. H. WALTER, The characterization of finite groups with abelian Sylow 2-subgroups, *Ann. of Math.* (2) **89** (1969), 405–514.