A note on solitary subgroups of finite groups

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Abstract

We say that a subgroup H of a finite group G is solitary (respectively, normal solitary) when it is a subgroup (respectively, normal subgroup) of G such that no other subgroup (respectively, normal subgroup) of G is isomorphic to H. A normal subgroup N of a group G is said to be quotient solitary when no other normal subgroup K of G gives a quotient isomorphic to G/N. We show some new results about lattice properties of these subgroups and their relation with classes of groups and present examples showing a negative answer to some questions about these subgroups.

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1 Introduction

All groups in this note will be finite. Let G be a group. Following Kaplan and Levy [6], we say that a subgroup H of a group G is *solitary* (respectively, *normal solitary*) if H is a subgroup (respectively, a normal subgroup) of G and whenever K is a subgroup (respectively, a normal subgroup) of G and H is isomorphic to K, then $H = K$. The notion of solitary subgroup had previously appeared in a paper of Thévenaz [9] under the name of

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strongly characteristic subgroup. Solitary subgroups have also been studied by Atanasov and Foguel [1]. All these subgroups are clearly characteristic subgroups.

It seems natural to consider the dual notion for quotients of solitary subgroup. According to [8] (see also [7]), a normal subgroup N of G is said to be *quotient solitary* if whenever K is a normal subgroup of G and G/K is isomorphic to G/N, then $K = N$.

The terminology of classes of groups has proved to be a useful tool to express propositions about group-theoretical properties. Recall that a *class of groups* is a class $\mathfrak{X}$ whose elements are groups such that if $G \in \mathfrak{X}$ and H is a group isomorphic to G, then $H \in \mathfrak{X}$. Among the classes of groups, formations and Fitting classes are especially relevant. A *formation* is a class of groups $\mathfrak{F}$ such that if N is a normal subgroup of a group $G \in \mathfrak{F}$, then $G/N \in \mathfrak{F}$, and if M and N are normal subgroups of G such that G/M, $G/N \in \mathfrak{F}$, then $G/(M \cap N) \in \mathfrak{F}$. Given a non-empty formation $\mathfrak{F}$, every group G possesses a normal subgroup $G^{\mathfrak{F}}$, called the $\mathfrak{F}$-*residual* of G, such that $G^{\mathfrak{F}}$ is the smallest normal subgroup N of G such that $G/N \in \mathfrak{F}$. A *Fitting class* is a class of groups $\mathfrak{F}$ such that if N is a normal subgroup of a group $G \in \mathfrak{F}$, then $N \in \mathfrak{F}$, and if M and N are normal subgroups of G and M, $N \in \mathfrak{F}$, then $MN \in \mathfrak{F}$. Given a non-empty Fitting class $\mathfrak{F}$, every group G possesses normal subgroup $G_{\mathfrak{F}}$, called the $\mathfrak{F}$-*radical* of G, that is the largest normal subgroup belonging to $\mathfrak{F}$. The basic concepts about classes of groups can be found in [2, 3].

The aim of this note is to study some natural problems about solitary, normal solitary, and quotient solitary subgroups. These problems will be related to lattice properties and the relation with classes of groups. We will also present some examples which give negative answers to some natural questions in the scope of these types of subgroups.

2 Lattice properties

Kaplan and Levy [6, Theorem 25] have shown that the set of all solitary subgroups of a group is a lattice, where the supremum of a set of two solitary subgroups $\{A, B\}$ is simply the product AB and the infimum of $\{A, B\}$ is the product of all solitary subgroups contained in $A \cap B$. Dually, Tărnăuceanu [8, Proposition 2.1] has shown that quotient solitary subgroups also form a lattice, where the infimum of $\{A, B\}$ is $A \cap B$ and the supremum of $\{A, B\}$ is the intersection of all quotient solitary subgroups of G containing the product AB. However, these lattices are not, in general, sublattices of the lattice of normal subgroups:

Example 2.1. Let G be a direct product of a symmetric group $\Sigma_3 = \langle a, b \mid a^2 = b^3 = 1, b^a = b^{-1} \rangle$ of degree 3 and a cyclic group $C_3 = \langle c \mid c^3 = 1 \rangle$ of order 3. This group has two solitary subgroups, $\Sigma_3 = \langle a, b \rangle$ and the normal Sylow 3-subgroup $\langle b, c \rangle \cong C_3 \times C_3$, whose intersection is $\langle b \rangle$, a cyclic group of order 3, which is clearly not solitary in G. Hence the intersection of two solitary subgroups is not necessarily a solitary subgroup.

Example 2.2. Let $E = \langle a, b \rangle$ be an extraspecial group of order 27 and exponent 3. Let $c = [a, b]$, then E has an automorphism d of order 2 given by $a^d = a$ and $b^d = b^2$, hence $c^d = c^2$. Let $H = [E]\langle d \rangle$ be the corresponding semidirect product. Let $C = \langle e \rangle$ be a cyclic group of order 3. Then we can check (for instance, with the help of the computer algebra system GAP [5]) that the quotient solitary subgroups of $G = H \times \langle e \rangle$ are 1, $\langle c \rangle$, $\langle e \rangle$, $\langle a, c, e \rangle$, $\langle b, c \rangle$, $\langle a, b, e \rangle$, $\langle b, c, d \rangle$, and G. However, the product of $\langle c \rangle$ and $\langle e \rangle$ is not a quotient solitary subgroup of G. Therefore the product of two quotient solitary subgroups is not necessarily a quotient solitary subgroup.

The situation is even worse with normal solitary subgroups, since they do not form a lattice in general.

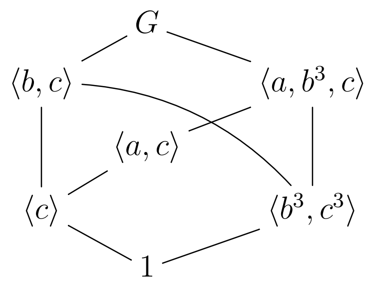

Figure 1: Partially ordered set of the normal solitary subgroups of the group of Example 2.3

Example 2.3. Let

$$G = \langle a, b, c \mid a^2 = b^9 = c^9, b^a = b, c^a = c^8, c^b = c^7 \rangle.$$

The normal solitary subgroups of G are 1, $\langle b^3, c^3 \rangle$, $\langle c \rangle$, $\langle a, c \rangle$, $\langle a, b^3, c \rangle$, $\langle b, c \rangle$, and G (they have been computed with GAP [5]). This partially ordered set is drawn in Figure 1. We see that the subset $\{\langle b^3, c^3 \rangle, \langle c \rangle\}$ has no supremum in the partially ordered set of normal solitary subgroups of G, and $\{\langle a, b^3, c \rangle, \langle b, c \rangle\}$ has no infimum.

It seems clear that one of the main objections for the partially ordered set of normal solitary subgroups to be a lattice is the fact that normality is not a transitive relation in general. The transitive closure of normality is subnormality: a subgroup H of a group G is said to be *subnormal* in G when there exists a series $H = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G$ such that H_{i-1} is a normal subgroup of H_i for $1 \leq i \leq n$.

Definition 2.4. A subgroup H of a group G is said to be a *subnormal solitary* subgroup of G when H is a subnormal subgroup of G and if K is another subnormal subgroup of G isomorphic to H, then $K = H$.

Obviously, subnormal solitary subgroups are characteristic. The following result is an immediate consequence of the definition.

Proposition 2.5. *Let H be a subgroup of a group G.*

1. *If H is solitary in G, then H is subnormal solitary in G.*
2. *If H is subnormal solitary in G, then H is normal solitary in G.*

Obviously, in nilpotent groups, the notion of solitary subgroup and subnormal solitary subgroup coincide, while in groups in which normality is a transitive relation, the so called T-groups, subnormal solitary subgroups and normal solitary subgroups coincide.

The converses of both implications of Proposition 2.5 are false:

Example 2.6. The dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1\rangle$ of order 8 has a normal solitary subgroup $\langle a^2\rangle = \Phi(D_8)$ that is not subnormal solitary in D_8, since this group has five subnormal cyclic subgroups of order 2.

Example 2.7. The symmetric group Σ_4 of degree 4 has a subnormal solitary subgroup $V_4 = \langle (1, 2)(3, 4), (1, 3)(2, 4)\rangle$ that is not solitary in Σ_4, because V_4 is isomorphic to the non-subnormal subgroup $\langle (1, 2), (3, 4)\rangle$.

However, the following result holds.

Theorem 2.8. *The partially ordered set of all subnormal solitary subgroups of a group with the inclusion is a lattice.*

Proof. Assume that S_1 and S_2 are subnormal solitary subgroups in G. Then S_1 and S_2 are normal subgroups of G. Suppose that T is a subnormal subgroup of G isomorphic to $S = S_1S_2$. Note that S_1 and S_2 are normal subgroups of S. Hence T contains normal subgroups T_1 and T_2 such that $S_1 \cong T_1$ and $S_2 \cong T_2$. Since T_1 and T_2 are normal in G and T is subnormal in G, we have that T_1 and T_2 are subnormal subgroups of G. Since S_1 and S_2

are subnormal solitary, we obtain that $S_1 = T_1$ and $S_2 = T_2$. In particular, $S = T$. This implies that S is subnormal solitary and, obviously, S is the supremum of $\{S_1, S_2\}$ in the partially ordered set of all subnormal solitary subgroups of G.

The argument to show that a set of two subnormal solitary subgroups possesses an infimum is the same as in [6, Theorem 25]. $\square$

Example 2.9. In the group of Example 2.3, the unique normal solitary subgroup which is not subnormal solitary is $\langle c \rangle$. The intersection $\langle b, c \rangle \cap \langle a, b^3, c \rangle = \langle b^3, c \rangle$ is not a subnormal solitary subgroup of G. This also shows that the lattice of all subnormal solitary subgroups is not, in general, a sublattice of the normal subgroup lattice.

3 Relation with classes of groups

Given a class of groups $\mathfrak{X}$, the subgroup generated by all subgroups of G in $\mathfrak{X}$ is solitary in G by [6, Lemma 3]. Let $S_{\mathfrak{X}}(G)$ denote the subgroup generated by all subnormal subgroups of G in $\mathfrak{X}$. If we consider the subnormal solitary subgroups introduced in the previous section, we obtain:

Theorem 3.1. *Let $\mathfrak{X}$ be a class of groups. The subgroup $S_{\mathfrak{X}}(G)$ is a subnormal solitary subgroup of G.*

Proof. Let $\mathcal{S} = \{S_1, \dots, S_k\}$ the set of all subnormal subgroups of G in $\mathfrak{X}$. Let H be a normal subgroup of G isomorphic to $S_{\mathfrak{X}}(G)$. Since $\mathcal{S}$ is invariant by conjugation, we have that $S_{\mathfrak{X}}(G)$ is a normal subgroup of G. Let H be a subnormal subgroup of G isomorphic to $S_{\mathfrak{X}}(G)$. Then H contains exactly k subnormal subgroups in $\mathfrak{X}$, that is, all subgroups in $\mathcal{S}$ are contained in H. It follows that $H = S_{\mathfrak{X}}(G)$. $\square$

If $\mathfrak{X}$ is a Fitting class, we obtain that the $\mathfrak{X}$-radical of a group G, that is, the subgroup generated by all subnormal subgroups of G in $\mathfrak{X}$, is a subnormal solitary subgroup of G, in particular, a normal solitary subgroup of G. This improves the result of [6, Lemma 15].

Theorem 3.2. *Let $\mathfrak{F}$ be a Fitting class and let G be a group. Then the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G is a subnormal solitary subgroup of G.*

Quotient solitary subgroups satisfy a dual property:

Theorem 3.3. *Let $\mathfrak{X}$ be a class of groups. Then the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{X}$ is a quotient solitary subgroup of G.*

Proof. Let $\mathcal{N} = \{N_1, \dots, N_k\}$ be the set of all normal subgroups of G with quotient in $\mathfrak{X}$. Let H be the intersection of all these subgroups and assume that G/K is isomorphic to G/H. Then G/K possesses normal subgroups $K_1/K, \dots, K_k/K$ such that $K_1 \cap \cdots \cap K_k = K$ and $G/K_i \in \mathfrak{X}$. But these subgroups must be exactly the members of $\mathcal{N}$. Hence $K = H$. □

Since, for a formation $\mathfrak{F}$, $G^{\mathfrak{F}}$ is the intersection of all the normal subgroups of G with quotient in $\mathfrak{F}$, we have:

Corollary 3.4. *Let $\mathfrak{F}$ be a formation and let G be a group. Then the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is a quotient solitary subgroup of G.*

This result can be used to give a description of quotient solitary free groups, that is, groups G in which the unique quotient solitary subgroups are G and 1. It improves [8, Theorem 3.7].

Theorem 3.5. *The following statements are equivalent for a group G:*

1. *G is characteristically simple.*
2. *G is quotient solitary free.*
3. *G is a direct product of copies of a simple group S.*

Proof. The equivalence between the statements 1 and 3 is well known. Assume that G is quotient solitary free. Let M be a maximal normal subgroup of G, then $S = G/M$ is a simple group and we consider the class $\mathfrak{F} = \mathrm{D}_0(1, S)$ of all groups that can be expressed as a direct product of copies of S, together with the trivial group. If S is a non-abelian simple group, this class is a formation by [3, II, 2.13], and if $S \cong C_p$, p a prime, it is the class of all elementary abelian p-groups, which is also a formation. Since $G^{\mathfrak{F}} \leq M < G$, we have that $G^{\mathfrak{F}} = 1$, in other words, $G \in \mathfrak{F}$ and G is a direct product of copies of the simple group S. □

We will say that a class of groups $\mathfrak{X}$ is *closed under taking extensions* when if G is a group with a normal subgroup such that N and G/N belong to $\mathfrak{X}$, then $G \in \mathfrak{X}$. We also say that a class of groups $\mathfrak{X}$ is *closed under taking (normal) subgroups* when if G is a group in $\mathfrak{F}$ and H is a (normal) subgroup of G, then H belongs to $\mathfrak{F}$. A class of groups $\mathfrak{X}$ is said to be *closed under taking quotients* when if $G \in \mathfrak{F}$ and N is a normal subgroup of G, then $G/N \in \mathfrak{F}$. Kaplan and Levy proved in [6, Lemma 22] the following result:

Theorem 3.6. *Let $\mathfrak{F}$ be a formation of groups which is closed under taking extensions and (normal) subgroups. Then the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is (normal) solitary in G.*

We might wonder whether the condition of being closed under taking extensions and (normal) subgroups can be dispensed of. More precisely, what can be said about a formation in which, given a group G, the $\mathfrak{F}$-residual of G is always a (normal) solitary subgroup of G. We have obtained the following result for formations $\mathfrak{F}$ satisfying that $(G \times H)^{\mathfrak{F}} = G^{\mathfrak{F}} \times H^{\mathfrak{F}}$ for every two groups G and H. This condition is satisfied by all formations contained in the formation of soluble groups, as shown by Doerk and Hawkes [4] (see also [3, IV, 1.18]).

Theorem 3.7. *Assume that $\mathfrak{F}$ is a formation satisfying that $(G \times H)^{\mathfrak{F}} = G^{\mathfrak{F}} \times H^{\mathfrak{F}}$ for every two groups G and H. Assume, in addition, that, given a group G, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is a (normal) solitary subgroup of G. Then the formation $\mathfrak{F}$ is closed under taking extensions and (normal) subgroups.*

Proof. We will prove first that $\mathfrak{F}$ is closed under taking extensions. Let G be a group with a normal subgroup N such that G/N and N belong to $\mathfrak{F}$. Then $(G \times N)^{\mathfrak{F}} = G^{\mathfrak{F}} \times 1$ is a solitary subgroup of $G \times N$. But since $G/N \in \mathfrak{F}$, $G^{\mathfrak{F}} \leq N$ and so $1 \times G^{\mathfrak{F}}$ is a subgroup of $G \times N$ isomorphic to $G^{\mathfrak{F}} \times 1$. Since this is a solitary subgroup of G, we obtain that $G^{\mathfrak{F}} = 1$, that is, $G \in \mathfrak{F}$. Hence $\mathfrak{F}$ is closed under taking (normal) subgroups.

Now we prove that $\mathfrak{F}$ is closed under taking (normal) subgroups. Let H be a (normal) subgroup of $G \in \mathfrak{F}$. Then $(G \times H)^{\mathfrak{F}} = 1 \times H^{\mathfrak{F}}$, but $H^{\mathfrak{F}} \times 1$ is a (normal) subgroup of $G \times H$ isomorphic to the (normal) solitary subgroup $1 \times H^{\mathfrak{F}}$. This implies that $H^{\mathfrak{F}} = 1$, that is, $H \in \mathfrak{F}$. Consequently, $\mathfrak{F}$ is closed under taking (normal) subgroups. □

We can prove the dual result of Theorem 3.6 for quotient solitary subgroups.

Theorem 3.8. *Let $\mathfrak{F}$ be a Fitting class which is closed under taking extensions and quotients. Then the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ is quotient solitary in G.*

Proof. Suppose that N is a normal subgroup of G such that $G/G_{\mathfrak{F}}$ is isomorphic to G/N. Since $\mathfrak{F}$ is closed under taking extensions, we have that $(G/G_{\mathfrak{F}})_{\mathfrak{F}} = 1$. On the other hand, $G_{\mathfrak{F}}N/N \cong G_{\mathfrak{F}}/(N \cap G_{\mathfrak{F}}) \in \mathfrak{F}$ because $\mathfrak{F}$ is closed under taking extensions. But since G/N is isomorphic to $G/G_{\mathfrak{F}}$, G/N cannot have non-trivial normal subgroups in $\mathfrak{F}$. It follows that $G_{\mathfrak{F}} = G_{\mathfrak{F}} \cap N$, that is, $G_{\mathfrak{F}} \leq N$ and, by order considerations, we conclude that $G_{\mathfrak{F}} = N$. □

The dual version of Theorem 3.7 for quotient solitary subgroups holds for Fitting classes $\mathfrak{F}$ satisfying that $(G \times H)_{\mathfrak{F}} = G_{\mathfrak{F}} \times H_{\mathfrak{F}}$ for every two groups G and H. These Fitting classes are known as *Lockett classes*. A detailed study of Lockett classes appears in [3, Chapter X, Section 1].

Theorem 3.9. *Suppose that $\mathfrak{F}$ is a Lockett class such that for every group G, the $\mathfrak{F}$-radical is a quotient solitary subgroup of G. Then $\mathfrak{F}$ is closed under taking extensions and quotients.*

Proof. We will prove first that $\mathfrak{F}$ is closed under taking extensions. Let G be a group with a normal subgroup N such that G/N and N belong to $\mathfrak{F}$. Then $\big(G \times (G/N)\big)_{\mathfrak{F}} = G_{\mathfrak{F}} \times (G/N)$ is a quotient solitary subgroup of G and $\big(G \times (G/N)\big)/\big(G_{\mathfrak{F}} \times (G/N)\big)$ is isomorphic to $G/G_{\mathfrak{F}}$. Since $N \in \mathfrak{F}$, $N \leq G_{\mathfrak{F}}$. Then the normal subgroup $G \times (G_{\mathfrak{F}}/N)$ satisfies that $\big(G \times (G/N)\big)/\big(G \times (G_{\mathfrak{F}}/N)\big)$ is isomorphic to $G_{\mathfrak{F}}$. Since $G_{\mathfrak{F}} \times (G/N)$ is quotient solitary, we obtain that $G = G_{\mathfrak{F}}$, that is, $G \in \mathfrak{F}$. We conclude $\mathfrak{F}$ is closed under extensions.

We will prove now that $\mathfrak{F}$ is closed under taking quotients. Let N be a normal subgroup of $G \in \mathfrak{F}$. Then $\big(G \times (G/N)\big)_{\mathfrak{F}} = G \times (G/N)_{\mathfrak{F}}$. Let $X/N = (G/N)_{\mathfrak{F}}$, we have that $\big(G \times (G/N)\big)/\big(G \times (G/N)_{\mathfrak{F}}\big) \cong G/X$. Then $X \times (G/N)$ is another normal subgroup of $G \times (G/N)$ giving a quotient isomorphic to $G/X \cong (G/N)\big/(G/N)_{\mathfrak{F}}$. Since $G \times (G/N)_{\mathfrak{F}}$ is quotient solitary, we obtain that $(G/N)_{\mathfrak{F}} = G/N$, that is, $G/N \in \mathfrak{F}$. Thus $\mathfrak{F}$ is closed under quotients. □

The fact that radicals for a Fitting class are subnormal solitary subgroups and the residuals for a formation are quotient solitary subgroups motivates the question of whether all subnormal solitary subgroups can be regarded as radicals for suitable Fitting classes or all quotient solitary subgroups can be regarded as residuals for suitable formations. In the case of abelian p-groups for a prime p, the quotient solitary subgroups are exactly the residuals for the formations $\mathfrak{F}^k$, where $\mathfrak{F}$ is the formation of all elementary abelian p-groups. This has been shown by Tărnăuceanu [8]. However, this is not true in general. The key to show this is to observe that the smallest formation (respectively, Fitting class) containing the dihedral group of order 8 contains the quaternion group of order 8 and the smallest formation (respectively, Fitting class) containing the dihedral group of order 8 contains the quaternion group of order 8. For completeness, we give proofs of these facts.

Lemma 3.10. *The smallest Fitting class containing the quaternion group Q_8 of order 8 coincides with the smallest Fitting class containing the dihedral group D_8 of order 8.*

Proof. This follows from the well-known fact that the extraspecial groups of order 32 which is a central product of two copies of D_8 is isomorphic to a central product of two copies of Q_8 (see [3, A, 20.4]). □

Lemma 3.11. *The smallest formation containing the quaternion group* Q_8 *of order* 8 *coincides with the smallest formation containing the dihedral group* D_8 *of order* 8.

Proof. Let $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, b^a = bc, c^a = c, c^b = c\rangle$. Then G possesses four normal subgroups $N_1 = \langle a^2, b^2\rangle$, $N_2 = \langle ca^2, b^2\rangle$, $N_3 = \langle cb^2, a^2\rangle$, and $N_4 = \langle ca^2, a^2b^2\rangle$ such that $N_1 \cap N_2 \cap N_3 = 1$, $G/N_i \cong D_8$ for $i \in \{1, 2, 3\}$ and $G/N_4 \cong Q_8$. This proves that Q_8 belongs to the smallest formation containing D_8.

Now let $H = \langle a, b, c \mid a^4 = c^4 = 1, a^2 = b^2, b^a = b^3, c^a = c^3, c^b = c\rangle$. Then H has three normal subgroups $T_1 = \langle c\rangle$, $T_2 = \langle ca^2, c^2\rangle$, $T_3 = \langle b^3, a^2c^2\rangle$ such that $G/T_1 \cong G/T_2 \cong G/T_3 \cong Q_8$ and a normal subgroup $T_4 = \langle b\rangle$ such that $G/T_4 \cong D_8$. This proves that Q_8 belongs to the smallest formation containing D_8. □

Example 3.12. The quasidihedral group $G = \langle a, b, c \mid a^4 = b^2 = 1, c^2 = a^2, b^a = ba^2, c^a = ca^2, c^b = ca^2\rangle$ of order 16 has two solitary subgroups $\langle a, c\rangle \cong Q_8$ and $\langle b, c\rangle \cong D_8$. Hence none of them can be the radical for a Fitting class.

Example 3.13. Let $G = \langle a, b \mid a^4 = b^4 = 1, b^a = b^3\rangle$. Then G has two quotient solitary subgroups $A = \langle a^2\rangle$ and $B = \langle b^2a^2\rangle$ such that $G/A \cong D_8$ and $G/B \cong Q_8$. Hence none of these subgroups can be the residual for a formation.

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