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Corrigendum to “A Note on Finite $\mathcal{PST}$-Groups”*
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Theorem A states that the finite groups in which all subnormal subgroups are S-permutable ($\mathcal{PST}$-groups) are exactly those groups in which subnormal subgroups of defect 2 are S-permutable ($\mathcal{T}^*$-groups). The proof of this result uses a characterization of finite $\mathcal{PST}$-groups in [4], namely the $\mathcal{PST}$-version of Theorem 3.1 (see Postscript). However, the hypothesis of this result was incorrectly stated: the inequality $1 \leq r < k$ should read $0 \leq r < k$. Unfortunately, with the weaker hypothesis the result becomes false, as the following example shows:

Example 1. Let A be a group such that $\mathrm{Z}(A) = \Phi(A) \cong C_3$ and $A/\Phi(A) \cong A_6$, the alternating group on 6 letters (such a group exists because the Schur multiplier of A_6 is isomorphic to C_6). Let $G = A \times \Sigma_3$.

We see that G satisfies the condition of the PST-version of Theorem 3.1 in [4, Postscript], but G is not a $\mathcal{PST}$-group. Consider $D = A$. Then D is a perfect normal subgroup of G. Moreover:

1. $G/D \cong \Sigma_3$ is a soluble $\mathcal{PST}$-group.
2. $D/\mathrm{Z}(D) = U_1/\mathrm{Z}(D) \cong A_6$ is simple and $U_1 = D$ is normal in G.

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3. If $\{i_1, i_2, \dots, i_r\} \subseteq \{1\}$, where $1 \leq r < 1$, then $G/U'_{i_1}U'_{i_2} \cdots U'_{i_r}$ satisfies $\mathbf{N}_p$ for all $p \in \pi\big(\mathrm{Z}(D)\big)$.

   This condition is trivially satisfied because there does not exist such a subset.

Consequently, G satisfies the conditions of the $\mathcal{PST}$-version of Theorem 3.1 in [4, Postscript], but G is not a $\mathcal{PST}$-group.

Careful examination the proof of Robinson's theorem shows that it uses not only that the quotients of G by the products $U'_{i_1} \cdots U'_{i_r}$ satisfy the condition $\mathbf{N}_p$, where $D/\mathrm{Z}(D) = U_1/\mathrm{Z}(D) \times \cdots \times U_n/\mathrm{Z}(D)$, but also G must satisfy the property $\mathbf{N}_p$, this being the case $r = 0$. With this change in the hypothesis, Theorem 3.1 (and Theorem 4.1) in [4] are correctly proved. In view of this situation, in order to complete the proof of our Theorem A we need only show that, with the hypothesis of that result, the group G satisfies the condition $\mathbf{N}_p$ for each $p \in \pi\big(\mathrm{Z}(D)\big)$, where D is the soluble residual of G.this statement.

First we prove a lemma.

Lemma 1. *Let N be a normal p-subgroup of a group G, p a prime. Then the p'-elements of G induce power automorphisms in N if and only if all chief factors of G below N are cyclic and G-isomorphic.*

Proof. Assume that all p'-elements of G induce power automorphisms in N. If a p'-element g of G does not centralize N, then N is abelian by [2, Hilfssatz 5]. In this event g induces a universal power automorphism in N by [3, 13.4.3]. It follows that p'-elements of G induce in N automorphisms that belong to the centre of $\mathrm{Aut}(N)$. Consequently the p-elements of $G/\mathrm{C}_G(N)$ form a normal subgroup and hence centralize each chief factor of G below N. It follows that all chief factors of G below N are cyclic and G-isomorphic.

Assume now that all chief factors of G below N are cyclic and G-isomorphic. Let q be a prime different from p and let Q be a Sylow q-subgroup of G. Then $X = NQ$ is a subgroup of G. Since G-chief factors of N have order p, all p-chief factors of X are cyclic and X-isomorphic. Therefore X satisfies $\mathcal{U}_p^*$, that is, all chief factors of X of order divisible by p are cyclic and isomorphic when regarded as X-modules. Since every subgroup H of N is subnormal and p'-perfect, it follows that H permutes with Q by [1]. Hence Q normalizes H and consequently all p'-elements of G induce power automorphisms in N. □

Proof of Theorem A completed. It suffices to show that if G is a $\mathcal{T}^*$-group, then it is a $\mathcal{PST}$-group. Suppose that G is a counterexample of minimum order; then G is an $\mathcal{SC}$-group and all its proper quotients are $\mathcal{PST}$-groups. To complete the proof in the insoluble case, it is enough to show that G satisfies $\mathbf{N}_p$ for each $p \in \pi\big(\mathrm{Z}(G)\big)$. Since every quotient of G by every non-trivial soluble normal subgroup of G is a $\mathcal{PST}$-group, it is enough to show that for every prime $p \in \pi\big(\mathrm{Z}(D)\big)$ the p'-elements of G induce power automorphisms in $\mathrm{O}_p(G)$. The result is clear if $|\mathrm{O}_p(G)| \leq p$. Assume $|\mathrm{O}_p(G)| > p$. Let N be a minimal normal subgroup of G contained in $\mathrm{O}_p(G)$. Then all p'-elements of G/N induce power automorphisms in $\mathrm{O}_p(G/N) = \mathrm{O}_p(G)/N$. Lemma 1 implies that all chief factors of G/N below $\mathrm{O}_p(G)/N$ are cyclic and G-isomorphic. Consider now a minimal normal subgroup A/N of G/N contained in $\mathrm{O}_p(G)/N$. Since G is an $\mathcal{SC}$-group, it follows that A is an abelian normal subgroup of G of order p^2. If H is a subgroup of A, then H is a subnormal subgroup of G of defect at most 2. Since G is a $\mathcal{T}^*$-group, H is S-permutable in G. It follows that every subgroup of A is normalized by all Sylow q-subgroups of G with $q \neq p$. In particular, the p'-elements of G act as power automorphisms on A. By Lemma 1, all chief factors of G below A are cyclic and G-isomorphic. Consequently all chief factors of G below $\mathrm{O}_p(G)$ are cyclic and G-isomorphic and so the p'-elements of G induce power automorphisms on $\mathrm{O}_p(G)$ by Lemma 1. Hence we can assume that the group is soluble and then the proof is exactly the same to the one appearing in our paper. □

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